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The fixed set of a derivation in lattices

Xiao Long Xin*

*Correspondence:
xlxin@nwu.edu.cn
Department of Mathematics,
Northwest University, Xi'an, 710069,
P.R. China

Abstract

The related properties of derivations in lattices are investigated. We show that the set of all isotone derivations in a distributive lattice can form a distributive lattice. Moreover, we introduce the fixed set of derivations in lattices and prove that the fixed set of a derivation is an ideal in lattices. Using the fixed sets of isotone derivations, we establish characterizations of a chain, a distributive lattice, a modular lattice and a relatively pseudo-complemented lattice, respectively. Furthermore, we discuss the relations among derivations, ideals and fixed sets in lattices.

MSC: 06B35; 06B99

Keywords: lattice; derivation; fixed set; ideal; standard ideal

1 Introduction

The system of lattice algebra plays a significant role in information theory [1], information retrieval [2], information access controls [3] and cryptanalysis [4]. In [1], Bell described the co-information lattice, used it to show how to express the probability density under a general hypergraphical model, and then used this to derive the lattice of dependent component analysis algorithms. In [2], Carpineto and Romano applied lattices to information retrieval. They introduced the bound facility and the integration of this and several other useful features, such as automatic indexing, fisheye view browser for lattice, and the use of thesaurus into a basic lattice framework. In [3], Sandhu showed that lattice-based mandatory access controls can be enforced by appropriate configuration of RBAC components. His constructions demonstrated that role hierarchies and constraints were required to effectively achieve this result. In [4], Durfee applied tools from the geometry of numbers to solve several problems in cryptanalysis. They used algebraic techniques to cryptanalyze several public key cryptosystems. They focused on RSA and RSA-like schemes and used tools from the theory of integer lattices to get some results.

The notion of derivation, introduced from the analytic theory, is helpful for the research of structure and property in an algebraic system. Recently, analytic and algebraic properties of lattices have been widely researched [5–7]. Several authors [8–12] studied derivations in rings and near-rings. Jun and Xin [13] applied the notion of derivation in ring and near-ring theory to BCI-algebras.

In [14], Xin *et al.* introduced the concept of derivation in a lattice and investigated some properties. They gave some equivalent conditions, under which a derivation is isotone for lattices with a greatest element, modular lattices and distributive lattices, respectively. They characterized modular lattices and distributive lattices by isotone derivations. But the relations among derivations, ideals and fixed sets were not investigated in that paper.

We will discuss when an ideal can appear as this 'fixed set' for a derivation in this paper. This paper is a continuation to the paper [14].

The remainder of this paper is organized as follows. In Section 2, we recall some definitions and some properties of lattice theory. In Section 3, we investigate further related properties of derivations in lattices and show a structural theorem of all isotone derivations in distributive lattices. In Section 4, we introduce the fixed set of derivations and get some interesting properties of them. Especially, using the fixed set of isotone derivations, we establish characterizations for some kinds of lattices. Furthermore, we discuss the relations among derivations, ideals and fixed sets in lattices. Finally, some concluding remarks are made in Section 5.

2 Preliminaries

Definition 2.1 [15] Let L be a nonempty set endowed with operations '$\wedge$' and '$\vee$'. If $(L, \wedge, \vee)$ satisfies the following conditions: for all $x, y, z \in L$,

(A) $x \wedge x = x$, $x \vee x = x$;
(B) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$;
(C) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$;
(D) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,

then L is called a lattice.

Definition 2.2 [15] A lattice L is distributive if the identity (E) or (F) holds.

(E) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(F) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (E) and (F) are equivalent.

Definition 2.3 [16] A lattice L is modular if the identity (M) holds.

(M) If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.4 [15] A relatively pseudo-complemented lattice (or Brouwerian lattice) is a lattice L in which, for any given elements $a, b \in L$, the set of all $x \in L$ such that $a \wedge x \leq b$ contains a greatest element $b : a$, the relative pseudo-complement of a in b.

Lemma 2.5 [15] *Any relatively pseudo-complemented lattice is distributive*.

Definition 2.6 [15] A Boolean algebra is an algebra $(B; \vee, \wedge, ', 0, 1)$ with two binary operations $\vee$, $\wedge$, one unary operation $'$, and two nullary operations 0, 1, such that the following conditions are satisfied:

(1) $(B; \vee, \wedge)$ is a distributive lattice;
(2) for all $a \in B$, $0 \vee a = a$, $a \wedge 1 = a$;
(3) for all $a \in B$, there is $a' \in B$ such that $a \vee a' = 1$, $a \wedge a' = 0$.

Definition 2.7 [15] Let $(L, \wedge, \vee)$ be a lattice. A binary relation '$\leq$' is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Lemma 2.8 [15] *Let* $(L, \wedge, \vee)$ *be a lattice. Define the binary relation* '$\leq$' *as in Definition* 2.7. *Then* $(L, \leq)$ *is a poset and for any* $x, y \in L$, $x \wedge y$ *is the g.l.b. of* $\{x, y\}$, *and* $x \vee y$ *is the l.u.b. of* $\{x, y\}$.

From Lemma 2.8, we can see that a lattice is not only an algebraic system, but also an order structure.

Definition 2.9 [15] Let $\theta : L \to M$ be a function from a lattice L to a lattice M. Then θ is a lattice-homomorphism (or homomorphism) when

$$\theta(x \wedge y) = \theta(x) \wedge \theta(y)$$

and

$$\theta(x \vee y) = \theta(x) \vee \theta(y)$$

for all $x, y \in L$.

As always, a homomorphism is called an isomorphism if it is a bijection, an epimorphism if onto, a monomorphism if one-to-one.

Definition 2.10 [15] An ideal is a non-void subset I of a lattice L with the properties

(1) $x \leq y, y \in I \Rightarrow x \in I$,
(2) $x, y \in I \Rightarrow x \vee y \in I$, for all $x, y \in L$. Moreover, an ideal I of a lattice L is called a prime ideal if I satisfies the following condition:
(3) $x \wedge y \in L$ implies $x \in L$ or $y \in L$ for all $x, y \in L$.

Note that if I_1 and I_2 are ideals of a lattice L, so is $I_1 \cap I_2$.

3 The derivations in lattices

In this section, we recall some definitions and results of the paper [14].

The following definition introduces the notion of derivation for a lattice, which comes in analogy with Leibniz's formula for derivations in a ring.

Definition 3.1 [14] Let L be a lattice and $d : L \to L$ be a function. We call d a derivation on L if it satisfies the condition $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$.

We often abbreviate $d(x)$ to dx.

Now we give some examples and present some properties for the derivations in lattices.

Example 3.2 Let L be the lattice of Figure 1, and define functions d_1 and d_2 on L by

$$d_1x = \begin{cases} x, & x = 0 \text{ or } 1, \\ b, & x = a, \\ a, & x = b, \end{cases} \qquad d_2x = \begin{cases} a, & x = a \text{ or } 1, \\ 0, & x = b, \\ 0, & x = 0. \end{cases}$$

Then we can see that d_1 is not a derivation but d_2 is a derivation on L.

Proposition 3.3 [14] *Let L be a lattice and d be a derivation on L. Then the following hold*:

(1) $dx \leq x$;

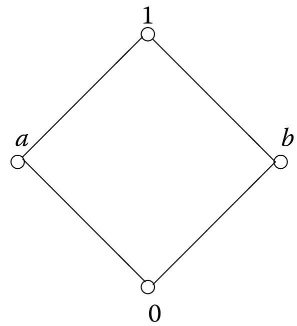

Figure 1 Diagram of Example 3.2.

(2) $dx \wedge dy \leq d(x \wedge y) \leq dx \vee dy$;
(3) *If I is an ideal of L, then $dI \subseteq I$, where $dI = \{dx \mid x \in I\}$*;
(4) *If L has a least element* 0, *then* $d0 = 0$.

Remark 3.4 In Proposition 3.3, we get an interesting property of derivation, *i.e.*, $dx \leq x$. This means that any derivation in lattices is a contraction mapping. By the principle of a contraction mapping, any derivation in lattices must have fixed points. We will discuss the structures and properties of the fixed point set of a derivation for a lattice later.

Definition 3.5 [14] Let L be a lattice and d be a derivation on L.
(1) If $x \leq y$ implies $dx \leq dy$, we call d an *isotone derivation*.
(2) If d is one-to-one, we call d a *monomorphic derivation*.
(3) If d is onto, we call d an *epic derivation*.

By analogy with principal ideals, we introduce a principal derivation in lattices as follows.

Definition 3.6 Let L be a lattice and $a \in L$. Define a function d_a on L by $d_a(x) = x \wedge a$ for all $x \in L$. Then we can see that d_a is a derivation on L. In the following, we refer to such *derivations* as *principal*.

Proposition 3.7 *Every principal derivation of a lattice L is an isotone derivation of L.*

Proof Let d_a be a principal derivation of a lattice L. Since for any $x, y \in L$ and $x \leq y$, we have $d_a(x) = x \wedge a \leq y \wedge a = d_a(y)$ and hence d_a is isotone. □

Proposition 3.8 [14] *Let L be a lattice and d be a derivation on L. If $y \leq x$ and $dx = x$, then $dy = y$.*

Proposition 3.9 [14] *Let L be a lattice and d be a derivation on L. Define $d^2x = d(dx)$ for all $x \in L$. Then we have $d^2 = d$.*

Theorem 3.10 *Let L be a lattice and $d : L \to L$ be a derivation. Then the following are equivalent*:
(1) *d is an isotone derivation*;
(2) $d(x \wedge y) = dx \wedge y$.

Proof (1) $\Rightarrow$ (2). Assume d is isotone. Then $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy) \geq dx \wedge y$. Conversely, since $x \wedge y \leq x$ and $x \wedge y \leq y$, we can get $d(x \wedge y) \leq dx$ and $d(x \wedge y) \leq dy$. Then $d(x \wedge y) \leq dx \wedge dy \leq dx \wedge y$. Therefore, $d(x \wedge y) = dx \wedge y$.

(2) $\Rightarrow$ (1). Assume $d(x \wedge y) = dx \wedge y$ for all x, y in L. Then $d(x \wedge y) = d(y \wedge x) = dy \wedge x$. Then $(x \wedge dy) \vee (dx \wedge y) = x \wedge dy = d(x \wedge y)$. Furthermore, if $x \leq y$, since $d(x \wedge y) = dx \wedge y = x \wedge dy$, then $dx = x \wedge dy$. Therefore, $dx \vee dy = (x \wedge dy) \vee dy = dy$. We can get $dx \leq dy$. □

Theorem 3.11 *Let L be a lattice and $d : L \to L$ be a derivation. Then the following are equivalent*:

(1) $d(x \wedge y) = dx \wedge y$;
(2) $d(x \wedge y) = dx \wedge dy$.

Proof (1) $\Rightarrow$ (2). Obversely, we have $(dx \wedge dy) \leq (dx \wedge y)$. By (1), $dx \wedge y = d(x \wedge y) = d(y \wedge x) = dy \wedge x$. Since $dx \wedge y \leq dx$ and $dy \wedge x \leq dy$, we can get $dx \wedge y = dy \wedge x \leq dx \wedge dy$.

(2) $\Rightarrow$ (1). Assume $d(x \wedge y) = dx \wedge dy$ for all x, y in L. If $x \leq y$, then $dx = d(x \wedge y) = dx \wedge dy$. We can get $dx \leq dy$. This shows that d is an isotone derivation. From Theorem 3.10, we know (1) holds. □

From the Theorem 3.10 and Theorem 3.11, we have the following theorem.

Theorem 3.12 *Let L be a lattice and $d : L \to L$ be a derivation. Then the following are equivalent*:

(1) *d is an isotone derivation*;
(2) $d(x \wedge y) = dx \wedge y$;
(3) $d(x \wedge y) = dx \wedge dy$.

However, derivations of distributive lattices have stronger properties.

Theorem 3.13 [14] *Let L be a distributive lattice and d be a derivation on L. Then the following are equivalent*:

(1) *d is isotone*;
(2) $d(x \wedge y) = dx \wedge dy$;
(3) $d(x \vee y) = dx \vee dy$.

Theorem 3.14 *Let L be a distributive lattice and d_1 and d_2 be two isotone derivations on L. Define*

$$(d_1 \wedge d_2)(x) = d_1x \wedge d_2x,$$

$$(d_1 \vee d_2)(x) = d_1x \vee d_2x.$$

Then $d_1 \wedge d_2$ and $d_1 \vee d_2$ are also isotone derivations on L.

Proof We first prove $d_1 \vee d_2$ is an isotone derivation on L.

By Theorem 3.13, we have

$$\begin{aligned}
&(d_1 \vee d_2)(x \wedge y) \\
&\quad = d_1(x \wedge y) \vee d_2(x \wedge y)
\end{aligned}$$

$$= (d_1x \wedge y) \vee (d_2x \wedge y)$$
$$= (d_1x \vee d_2x) \wedge y$$
$$= (d_1 \vee d_2)(x) \wedge y.$$

Similarly, we can get $(d_1 \vee d_2)(x \wedge y) = (d_1 \vee d_2)(y) \wedge x$.

Combining the above arguments, we have

$$(d_1 \vee d_2)(x \wedge y) = \big((d_1 \vee d_2)(x) \wedge y\big) \vee \big((d_1 \vee d_2)(y) \wedge x\big).$$

So, $d_1 \vee d_2$ is a derivation on L by Definition 3.1.

Moveover, $(d_1 \vee d_2)(x \vee y) = d_1(x \vee y) \vee d_2(x \vee y) = (d_1(x) \vee d_1(y)) \vee (d_2(x) \vee d_2(y)) = (d_1(x) \vee d_2(x)) \vee (d_1(y) \vee d_2(y)) = (d_1 \vee d_2)(x) \vee (d_1 \vee d_2)(y)$, so $d_1 \vee d_2$ is isotone by Theorem 3.13.

Similar to the above process, we can prove $d_1 \wedge d_2$ is an isotone derivation on L and we omit it. □

Theorem 3.15 *Let L be a distributive lattice and $\mathcal{D}(L)$ be a set of all isotone derivations on L. Then $\langle \mathcal{D}(L), \vee, \wedge\rangle$ is a distributive lattice.*

Proof From Theorem 3.14, $\vee$ and $\wedge$ are binary operators on $\mathcal{D}(L)$. Define a binary relation '$\leq$' on $\mathcal{D}(L)$ by $d_1 \leq d_2$ iff $d_1 \wedge d_2 = d_1$. Then '$\leq$' is a partial order relation on $\mathcal{D}(L)$ and $g.l.b.\{d_1, d_2\} = d_1 \wedge d_2$, $l.u.b.\{d_1, d_2\} = d_1 \vee d_2$. Therefore, $\langle \mathcal{D}(L), \vee, \wedge\rangle$ is a lattice.

In addition, for any $d_1, d_2, d_3 \in \mathcal{D}(L)$ and any $x \in L$,

$$\big(d_1 \wedge (d_2 \vee d_3)\big)(x)$$
$$= d_1x \wedge (d_2x \vee d_3x)$$
$$= (d_1x \wedge d_2x) \vee (d_1x \wedge d_3x)$$
$$= \big((d_1 \wedge d_2)x\big) \vee \big((d_1 \wedge d_3)x\big)$$
$$= \big((d_1 \wedge d_2) \vee (d_1 \wedge d_3)\big)(x).$$

Therefore, $d_1 \wedge (d_2 \vee d_3) = (d_1 \wedge d_2) \vee (d_1 \wedge d_3)$. This shows that $\langle \mathcal{D}(L), \vee, \wedge\rangle$ is a distributive lattice. □

4 The fixed set of a derivation in lattices

Theorem 4.1 *Let L be a lattice and d be an isotone derivation on L. Denote* $\mathrm{Fix}_d(L) = \{x \in L : dx = x\}$. *Then* $\mathrm{Fix}_d(L)$ *is an ideal of L.*

Proof By Proposition 3.8 we can see that $x \in \mathrm{Fix}_d(L)$ and $y \leq x$ imply $y \in \mathrm{Fix}_d(L)$. This means that $\mathrm{Fix}_d(L)$ satisfies the condition (1) of Definition 2.10. For the condition (2) of Definition 2.10, we consider $x, y \in \mathrm{Fix}_d(L)$. By the isotoneness of d, we have $x \vee y = dx \vee dy \leq d(x \vee y)$ and so $x \vee y = d(x \vee y)$. This means that $\mathrm{Fix}_d(L)$ satisfies Definition 2.10. It follows that $\mathrm{Fix}_d(L)$ is an ideal of L. □

In the following proposition, we can see that an isotone derivation d is determined by the ideal $\mathrm{Fix}_d(L)$.

Proposition 4.2 *Let L be a lattice and d_1 and d_2 be two isotone derivations on L. Then $d_1 = d_2$ if and only if $\mathrm{Fix}_{d_1}(L) = \mathrm{Fix}_{d_2}(L)$.*

Proof It is obvious that $d_1 = d_2$ implies $\mathrm{Fix}_{d_1}(L) = \mathrm{Fix}_{d_2}(L)$. Inversely, let $\mathrm{Fix}_{d_1}(L) = \mathrm{Fix}_{d_2}(L)$ and $x \in L$. By Proposition 3.9, $d_1x \in \mathrm{Fix}_{d_1}(L) = \mathrm{Fix}_{d_2}(L)$ and so $d_2(d_1x) = d_1x$. Similarly, we can get $d_1(d_2x) = d_2x$. Since d_1 and d_2 are isotone, we have $d_2(d_1x) \le d_2x = d_1(d_2x)$ and so $d_2(d_1x) \le d_1(d_2x)$. Symmetrically, we can also get $d_1(d_2x) \le d_2(d_1x)$, this shows that $d_1(d_2x) = d_2(d_1x)$. It follows that $d_1x = d_2(d_1x) = d_1(d_2x) = d_2x$, that is, $d_1 = d_2$. □

Theorem 4.3 *Let L be a lattice. Then the following are equivalent*:

(1) *L is a chain*;
(2) *For every isotone derivation d, $\mathrm{Fix}_d(L)$ is a prime ideal.*

Proof (1) $\Rightarrow$ (2). Let L be a chain and d be an isotone derivation on L. Then $\mathrm{Fix}_d(L)$ is an ideal of L by Theorem 4.1. Moreover, let $x \wedge y \in \mathrm{Fix}_d(L)$. Since L is a chain, then $x \le y$ or $y \le x$. Assume $x \le y$, then $dx \le dy$ and so $dx = dx \wedge dy = d(x \wedge y) = x \wedge y = x$. It follows that $x \in \mathrm{Fix}_d(L)$. This shows that $\mathrm{Fix}_d(L)$ is a prime ideal.

(2) $\Rightarrow$ (1). Let, for every isotone derivation d, $\mathrm{Fix}_d(L)$ be a prime ideal. For $x, y \in L$, consider the principal derivation $d_{x\wedge y}$, which is induced by $x \wedge y$. Then $\mathrm{Fix}_{d_{x\wedge y}}(L)$ is a prime ideal by hypothesis. Note that $x \wedge y \in \mathrm{Fix}_{d_{x\wedge y}}(L)$. Hence, $x \in \mathrm{Fix}_{d_{x\wedge y}}(L)$ or $y \in \mathrm{Fix}_{d_{x\wedge y}}(L)$. Assume $x \in \mathrm{Fix}_{d_{x\wedge y}}(L)$, then $x = d_{x\wedge y}x = x \wedge (x \wedge y) = x \wedge y$. So, $x \le y$. This means that L is a chain. □

To get a characterization of distributive lattices using the fixed set of a derivation, we introduce the following concept.

Let L be a lattice and I be an ideal of L. Define a relation '$\equiv$' in L by $x \equiv y (\mathrm{mod}\, I)$ if and only if $x \vee a = y \vee a$ and $x \wedge a' = y \wedge a'$ for some $a, a' \in I$. We can easily see that this relation is an equivalent relation.

Definition 4.4 [15] Let L be a lattice and I be an ideal of L. We call I a standard ideal if it satisfies the following condition: $x \equiv y (\mathrm{mod}\, I)$ implies $(x \vee z) \equiv (y \vee z)(\mathrm{mod}\, I)$ and $(x \wedge z) \equiv (y \wedge z)(\mathrm{mod}\, I)$ for all $z \in L$ or, equivalently, the relation '$\equiv$' is a congruence relation.

Theorem 4.5 *Let L be a lattice. Then the following are equivalent*:

(1) *L is distributive*;
(2) *For every isotone derivation d, $\mathrm{Fix}_d(L)$ is a standard ideal of L.*

Proof (1) $\Rightarrow$ (2). Let L be a distributive lattice and d be an isotone derivation. Now we claim that this relation is a congruence relation. In fact, let $c \in L$. If $x \equiv y(\mathrm{mod}\, I)$, then $x \vee a = y \vee a$ and $x \wedge a' = y \wedge a'$ for some $a, a' \in I$, and so $(x \vee c) \vee a = (y \vee c) \vee a$ and $(x \wedge c) \vee a = (x \vee a) \wedge (c \vee a) = (y \vee a) \wedge (c \vee a) = (y \wedge c) \vee a$. Similarly, we can get $(x \vee c) \wedge a' = (y \vee c) \wedge a'$ and $(x \wedge c) \wedge a' = (y \wedge c) \wedge a'$. This shows that $x \vee c \equiv y \vee c(\mathrm{mod}\, I)$ and $x \wedge c \equiv y \wedge c(\mathrm{mod}\, I)$. It follows that the relation is a congruence relation. Thus, $\mathrm{Fix}_d(L)$ is a standard ideal of L.

(2) $\Rightarrow$ (1). Assume that (2) holds. For any $a, b, c \in L$, consider the derivation d_a, which is induced by a, that is, $d_ax = x \wedge a$ for all $x \in L$. Note that $I = \mathrm{Fix}_{d_a}(L)$ is a standard ideal of

L and $a \in I$. Hence, the relation '$\equiv$', which is defined by $x \equiv y(\mathrm{mod}\, I)$ if and only if $x \vee u = y \vee u$ and $x \wedge u' = y \wedge u'$ for some $u, u' \in I$, is a congruence relation on L by hypothesis. Notice that $a, a \wedge b \in I$ and $(b \vee a) \vee a = b \vee a$, $(b \vee a) \wedge (b \wedge a) = b \wedge (b \wedge a)$, we have $b \vee a \equiv b(\mathrm{mod}\, I)$. Similarly, we can get $c \vee a \equiv c(\mathrm{mod}\, I)$. Moreover, $(b \vee a) \wedge (c \vee a) \equiv b \wedge c(\mathrm{mod}\, I)$. It follows that $((b \vee a) \wedge (c \vee a)) \vee a' = (b \wedge c) \vee a'$ for some $a' \in I$. From $a' \in I$, we have $a' = d_a(a') = a' \wedge a \leq a$, and then we get $((b \vee a) \wedge (c \vee a)) \vee a = (b \wedge c) \vee a$. Hence, $(b \vee a) \wedge (c \vee a) = (b \wedge c) \vee a$. It follows that L is distributive. □

In order to discuss the structural properties of the fixed set of isotone derivations in modular lattices, we introduce a semi-standard ideal in a lattice.

Let L be a lattice and I be a principal ideal of L generated by $a \in L$, that is, $I = \langle a \rangle$. Define a relation '$\sim$' in L by $x \sim y$ if and only if $x \wedge a = y \wedge a$ for all $x, y \in L$. Then we can see that the relation $\sim$ is an equivalent relation on L.

Definition 4.6 Let L be a lattice and $I = \langle a \rangle$ be a principal ideal of L. We call I a semi-standard ideal if it satisfies the following condition: $x \sim y$ implies $(x \vee b) \sim (y \vee b)$ for all $b \in I$.

In the following, we give a property of principal ideals in a modular lattice.

Proposition 4.7 *In a modular lattice, every principal ideal is a semi-standard ideal.*

Proof Let L be a modular lattice and $I = \langle a \rangle$ be a principal ideal of L. Assume $x, y \in L$ and $x \sim y$. Then $x \wedge a = y \wedge a$. Taking $b \in I$, then $b \leq a$. Notice that

$$(x \vee b) \wedge a = b \vee (x \wedge a) = y \wedge a = (y \vee b) \wedge a$$

since L is modular. It follows that $(x \vee b) \sim (y \vee b)$ and so I is a semi-standard ideal. □

Now, using fixed sets of derivations, we give a condition by which a lattice becomes a modular lattice.

Proposition 4.8 *Let L be a lattice. If d is a principal derivation of L, then* $\mathrm{Fix}_d(L) = I_d$ *is a principal ideal.*

Proof Assume that d is a principal derivation of L, that is, $dx = x \wedge a$ for some $a \in L$. We claim that $\mathrm{Fix}_d(L) = \langle a \rangle$. In fact, for any $x \in \mathrm{Fix}_d(L)$, we have $x = dx = x \wedge a$ and hence $x \leq a$. This means that $x \in \langle a \rangle$. Conversely, let $x \in \langle a \rangle$, that is, $x \leq a$. Then $dx = x \wedge a = x$ and hence $x \in \mathrm{Fix}_d(L)$. By the above arguments, we have $\mathrm{Fix}_d(L) = \langle a \rangle$, and so $\mathrm{Fix}_d(L)$ is a principal ideal. □

Proposition 4.9 *Let L be a lattice. If for every principal derivation d of L, the ideal* $\mathrm{Fix}_d(L)$ *is semi-standard, then L is modular.*

Proof Assume that for every principal derivation d of L, the ideal $\mathrm{Fix}_d(L)$ is semi-standard. Let $a, b, c \in L$ and $b \leq a$. Consider the derivation d_a induced by a, that is, $d_a(x) = x \wedge a$ for all $x \in L$. Since d_a is a principal derivation, then the fixed set $I = \mathrm{Fix}_{d_a}(L)$ is a principal ideal by Proposition 4.8 and hence it is semi-standard by Proposition 4.7. Notice that $a, b \in I$ and $(c \wedge a) \wedge a = c \wedge a$, we have $c \wedge a \sim c$. Moreover, $(c \wedge a) \vee b \sim c \vee b$ since I is semi-standard. This means that $((c \wedge a) \vee b) \wedge a = (c \vee b) \wedge a$. Since $(c \wedge a) \vee b \in I$, we have $((c \wedge a) \vee b) \wedge a = (c \wedge a) \vee b$. Hence, $(c \wedge a) \vee b = (c \vee b) \wedge a$ and so L is modular. □

Combining Proposition 4.7 and Proposition 4.9, we can get a characterization of a modular lattice by the fixed set of a derivation.

Theorem 4.10 *Let L be a lattice. Then the following are equivalent*:

(1) *L is modular*;
(2) *For every principal derivation d of L, the ideal* $\mathrm{Fix}_d(L)$ *is semi-standard.*

Now we discuss a characterization of relatively pseudo-complemented lattices by the fixed set of isotone derivations.

Theorem 4.11 *Let L be a lattice. Then the following are equivalent*:

(1) *L is a relatively pseudo-complemented lattice.*
(2) *Every principal derivation d of L satisfies that the set* $d^{-1}(b) = \{x|dx \leq b\}$ *has a greatest element for any* $b \in L$.
(3) *Every principal derivation d of L satisfies that the set* $d^{-1}(b) = \{x|dx \leq b\}$ *has a greatest element for any* $b \in \mathrm{Fix}_d(L)$.
(4) *Every principal derivation d of L satisfies that the set* $d^{-1}(b)$ *is a principal ideal of L for any* $b \in \mathrm{Fix}_d(L)$.

Proof (1) $\Rightarrow$ (2). Let L be a relatively pseudo-complemented lattice and d be a principal derivation. Then there is $a \in L$ such that $d(x) = x \wedge a$. Assume that $b \in L$ and $x \in d^{-1}(b)$. Then $dx = x \wedge a \leq b$ and hence $x \leq b : a$ since L is a relatively pseudo-complemented lattice. On the other hand, $d(b : a) = (b : a) \wedge a \leq b$. It follows that $b : a \in d^{-1}(b)$. So, we have that $d^{-1}(b)$ has a greatest element $b : a$.

(2) $\Rightarrow$ (3). Straightforward.

(3) $\Rightarrow$ (4). Let (3) hold. Let b^* be the greatest element of $d^{-1}(b)$ for $b \in \mathrm{Fix}_d(L)$. Then $d^{-1}(b) = [b^*]$, where $[b^*]$ is the ideal generated by b^*. In fact, for $x \in d^{-1}(b)$, we have $x \leq b^*$ and so $x \in [b^*]$. Conversely, let $x \in [b^*]$, then $x \leq b^*$. It follows that $dx \leq db^* \leq b$, this means $x \in d^{-1}(b)$. So, $d^{-1}(b) = [b^*]$.

(4) $\Rightarrow$ (1). Let (4) hold and $a, b \in L$. Consider a principal derivation d_a, induced by a. By Proposition 3.7, d_a is isotone. Note that $d_a(a \wedge b) = a \wedge b$ and so $a \wedge b \in \mathrm{Fix}_{d_a}(L)$. By hypothesis, the set $d_a^{-1}(a \wedge b)$ is a principal ideal of L. Let $d_a^{-1}(a \wedge b) = [a^*]$, where $[a^*]$ is a principal ideal generated by a^*. Therefore, for any $x \in \{x \mid x \wedge a \leq b\}$, $x \wedge a \leq b \wedge a$. It follows that $d_a(x) \leq b \wedge a$ and hence $x \in d_a^{-1}(b \wedge a)$. So, $x \leq a^*$. On the other hand, from $a^* \in d_a^{-1}(a \wedge b)$, we have $d_a(a^*) = a^* \wedge a \leq a \wedge b \leq b$ and $a^* \in \{x \mid x \wedge a \leq b\}$. This shows that the set $\{x \mid x \wedge a \leq b\}$ has a greatest element a^*. It follows that $b : a$ exists. □

In the following, we discuss the relation between principal derivations and principal ideals in lattices.

Theorem 4.12 *Let L be a lattice.*

(1) *If d is a principal derivation of L, then* $\mathrm{Fix}_d(L) = I_d$ *is a principal ideal.*

(2) *If I is a principal ideal of L, then there exists a unique isotone derivation d such that* $\mathrm{Fix}_d(L) = I$.

Proof (1) It follows from Proposition 4.8.

(2) Let $I = [a]$ be a principal ideal of L. Consider the derivation d induced by a, that is, $dx = x \wedge a$ for all $x \in L$. Then $dx = x$ if and only if $x \leq a$. It follows that $\mathrm{Fix}_d(L) = I$. In order to prove the uniqueness, we assume that there exist two derivations d_1 and d_2, such that $\mathrm{Fix}_{d_1}(L) = I$ and $\mathrm{Fix}_{d_2}(L) = I$. So, $\mathrm{Fix}_{d_1}(L) = \mathrm{Fix}_{d_2}(L)$ and hence $d_1 = d_2$ by Proposition 4.2. □

Theorem 4.13 *Let L be a lattice and I be a non-void prime ideal of L. Then there exists a derivation d such that* $\mathrm{Fix}_d(L) = I$.

Proof Define a function d as follows:

$$dx = \begin{cases} x, & x \in I, \\ x \wedge a, & x \in L \setminus I, \end{cases}$$

where $a \in I$. We claim that d is a derivation. In fact, if $x, y \in I$, then we can see that $d(x \wedge y) = x \wedge y = (x \wedge y) \vee (x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$. If $x \in I$, $y \in L \setminus I$, then $x \wedge y \leq x$ and so $x \wedge y \in I$. Hence, $d(x \wedge y) = x \wedge y$, $(dx \wedge y) \vee (x \wedge dy) = (x \wedge y) \vee (x \wedge y \wedge a) = x \wedge y$. This shows that $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$. If $x, y \in L \setminus I$, then $x \wedge y \in L \setminus I$ since I is prime. Hence, $d(x \wedge y) = x \wedge y \wedge a$, $(dx \wedge y) \vee (x \wedge dy) = (x \wedge a \wedge y) \vee (x \wedge y \wedge a) = x \wedge y \wedge a$. By the above argument, we can get that d is a derivation. Clearly, $\mathrm{Fix}_d(L) = I$. □

Example 4.14 Let $L = (0,1]$ and $I = (0,1)$, then $(L, \leq)$ is a lattice and I is an ideal of L, where $\leq$ is the ordinary order. Moreover, we can see that there is not any isotone derivation d such that $\mathrm{Fix}_d(L) = I$.

We now determine some classes of lattices all of whose ideals are principle ideals.

Definition 4.15 A poset P is said to satisfy the ascending chain condition (A.C.C.) if every non-void subset of P has a maximal element. A poset P is said to satisfy the descending chain condition (D.C.C.) if every non-void subset of P has a minimal element.

Theorem 4.16 *Let L be a lattice. If L satisfies A.C.C., then every ideal of L is a principal ideal.*

Proof Let I be an ideal of L. By assumption, I has a maximal element a_0. Therefore, for any $x \in I$, $x \vee a_0 \in I$. Note that $a_0 \leq x \vee a_0$ and a_0 is a maximal element of I, we have $x \vee a_0 = a_0$. Hence, $x \leq a_0$. This shows that $I = [a_0]$. □

By Theorem 4.12 and Theorem 4.16, we have the following theorem.

Theorem 4.17 *Let L be a lattice satisfying A.C.C. Then for every ideal of L, there exists a unique isotone derivation d such that* $\mathrm{Fix}_d(L) = I$.

Finally, we can see that the set of fixed sets of isotone derivations has the same structure as the set of isotone derivations in distributive lattices.

Theorem 4.18 *Let L be a distributive lattice and $\mathcal{D}(L)$ be a set of isotone derivations on L. Denote $\mathcal{F} = \{\mathrm{Fix}_d(L) | d \in \mathcal{D}(L)\}$. Define*

$$\mathrm{Fix}_{d_1}(L) \vee \mathrm{Fix}_{d_2}(L) = \mathrm{Fix}_{d_1 \vee d_2}(L),$$
$$\mathrm{Fix}_{d_1}(L) \wedge \mathrm{Fix}_{d_2}(L) = \mathrm{Fix}_{d_1 \wedge d_2}(L).$$

Then $\langle \mathcal{F}, \vee, \wedge \rangle$ is a distributive lattice.

Proof By Theorem 3.13, for any $d_1, d_2 \in \mathcal{D}$, we have $d_1 \wedge d_2 \in \mathcal{D}$ and $d_1 \vee d_2 \in \mathcal{D}$. This shows that the operations '$\wedge$' and '$\vee$' are closed on $\mathcal{F}$. We can easily show that $\langle \mathcal{F}, \vee, \wedge \rangle$ is a lattice. Consider the function $f : \mathcal{D}(L) \longrightarrow \mathcal{F}$ defined by $f(d) = \mathrm{Fix}_d(L)$. Then we can see that f is an isomorphism from $\mathcal{D}(L)$ to $\mathcal{F}$. It follows from the distributivity of $(\mathcal{D}(L), \vee, \wedge)$ that $\langle \mathcal{F}, \vee, \wedge \rangle$ is a distributive lattice. □

From the proof of Theorem 4.18, we can get the following corollary.

Corollary 4.19 *Let L be a distributive lattice. Then the lattice $\langle \mathcal{D}(L), \vee, \wedge \rangle$ is isomorphic to the lattice $\langle \mathcal{F}, \vee, \wedge \rangle$.*

5 Conclusions

In this paper, we investigate further related properties of derivations in lattices. We show that the set of all isotone derivations in a distributive lattice forms a distributive lattice under suitable binary operations. Moreover, we introduce the fixed set of a derivation and prove that the fixed set of a derivation is an ideal in lattices. Using the fixed sets of isotone derivations, we establish characterizations of a chain, a distributive lattice, a modular lattice and a relatively pseudo-complemented lattice, respectively. Furthermore, we discuss the relation between ideals and fixed sets of derivations in lattices. We get that for every principal ideal I and every prime ideal I, there exists a derivation d such that the fixed set of d is I.

We have seen that in some situations like lattices satisfying A.C.C., for every ideal of L, there exists an isotone derivation d such that $\mathrm{Fix}_d(L) = I$. The question whether or not this property holds in general lattices remains unsolved. We will discuss this question on general ideals in further work.

Competing interests
The author declares that they have no competing interests.

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