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Fixed and coupled fixed points of a new type set-valued contractive mappings in complete metric spaces

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Abstract
In this paper, motivated by the recent work of Wardowski (Fixed Point Theory Appl. 2012:94, 2012), we introduce a new concept of set-valued contraction and prove a fixed point theorem which generalizes some well-known results in the literature. As an application, we derive a new coupled fixed point theorem. Some examples are also given to support our main results.
MSC: 47H10

Keywords: fixed point; coupled fixed point; set-valued contractions

1 Introduction

In the literature, there are plenty of extensions of the famous Banach contraction principle [1], which states that every self-mapping T defined on a complete metric space (X,d) satisfying

$$d(Tx,Ty) \le kd(x,y) \quad \text{for each } x,y \in X, \tag{1}$$

where $k \in [0;1)$, has a unique fixed point, and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to the fixed point. Some of the extensions weaken the right side of the inequality in the condition (1) by replacing k with a mapping; see, *e.g.*, [2–4]. In other results, the underlying space is more general; see, *e.g.*, [5–8]. In 1969, Nadler [9] extended the Banach contraction principle to set-valued mappings. For other extensions of the Banach contraction principle, see [10–21] and the references therein.

Recently, Wardowski [22] introduced a new concept of contraction and proved a fixed point theorem which generalizes the Banach contraction principle in a different way than in the known results from the literature. In this paper, we present an improvement and generalization of the main result of Wardowski [22]. To set up our results, in the next section, we introduce some definitions and facts.

Let (X,d) be a metric space and let $CB(X)$ denote the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\Big\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(v,A)\Big\}$$

for every $A,B \in CB(X)$, where $d(u,B) = \inf\{d(u,y) : y \in B\}$.

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Theorem 1.1 (Nadler [9]) *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be a set-valued map. Assume that there exists $k \in [0,1)$ such that*

$$H(Tx,Ty) \le kd(x,y) \quad \text{for each } x,y \in X. \tag{2}$$

Then T has a fixed point.

In 1989 Mizoguchi and Takahashi [13] proved the following generalization of Theorem 1.1.

Theorem 1.2 (Mizoguchi and Takahashi [13]) *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be a set-valued map satisfying*

$$H(Tx,Ty) \le \alpha\big(d(x,y)\big)d(x,y) \quad \text{for each } x,y \in X,$$

where $\alpha : [0,\infty) \to [0,1)$ satisfies $\limsup_{t\to r^+} \alpha(t) < 1$ for each $r \in [0,\infty)$. Then T has a fixed point.

2 Main results

Let $F : (0,\infty) \to \mathbb{R}$ and $\theta : (0,\infty) \to (0,\infty)$ be two mappings. Throughout the paper, let Δ be the set of all pairs (F,θ) satisfying the following:

(δ_1) $\theta(t_n) \not\to 0$ for each strictly decreasing sequence $\{t_n\}$;
(δ_2) F is strictly increasing;
(δ_3) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(δ_4) If $t_n \downarrow 0$ and $\theta(t_n) \le F(t_n) - F(t_{n+1})$ for each $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} t_n < \infty$.

Example 2.1 Let $\theta_1(t) = \tau$ for each $t \in (0,\infty)$, where $\tau > 0$ is a constant, and let $F_1 : (0,\infty) \to \mathbb{R}$ be a mapping satisfying $\lim_{x\to 0^+} x^k F(x) = 0$ for some $k \in (0,1)$ where $F : (0,\infty) \to \mathbb{R}$ is strictly increasing. Then the proof of the main result in [22] shows that $(F_1,\theta_1) \in \Delta$. We give the details for completeness. Using (δ_4), the following holds for every $n \in \mathbb{N}$:

$$F(t_n) \le F(t_{n-1}) - \tau \le F(t_{n-2}) - 2\tau \le \cdots \le F(t_0) - n\tau. \tag{3}$$

By (3), the following holds for every $n \in \mathbb{N}$:

$$t_n^k F(t_n) - t_n^k F(t_0) \le t_n^k\big(F(t_0) - n\tau\big) - t_n^k F(t_0) = -t_n^k n\tau \le 0. \tag{4}$$

Since $\lim_{n\to\infty} t_n^k F(t_n) = 0$, letting $n \to \infty$ in (4), we obtain $\lim_{n\to\infty} n t_n^k = 0$. Then there exists $n_1 \in \mathbb{N}$ such that $n t_n^k \le 1$ for $n \ge n_1$. Consequently, we have $t_n \le \frac{1}{n^{\frac{1}{k}}}$ for all $n \ge n_1$. Thus, $\sum_{n=1}^{\infty} t_n < \infty$ (note that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} < \infty$).

Example 2.2 Let $F_2(t) = \ln t$ and let $\theta_2(t) = -\ln(\alpha(t))$ for each $t \in (0,\infty)$, where $\alpha : (0,\infty) \to [0,1)$ satisfying

$$\limsup_{t\to r^+} \alpha(t) < 1 \quad \text{for each } r \in [0,\infty).$$

Now, we show that $(F_2,\theta_2)\in\Delta$. It is easy to see that F_2 and θ_2 satisfy (δ_1)-(δ_3). To show (δ_4), assume that $t_n\downarrow 0$ and

$$-\ln\bigl(\alpha(t_n)\bigr)\le \ln t_n-\ln t_{n+1}\quad \forall n\in\mathbb{N}.$$

Then $t_{n+1}\le\alpha(t_n)t_n$ for each $n\in\mathbb{N}$. Since $\limsup_{t\to 0^+}\alpha(t)<1$, then there exist $n_0\in\mathbb{N}$ and $0<r<1$ such that $\alpha(t_n)<r$ for $n\ge n_0$. Thus, $t_{n+1}\le rt_n$ for each $n\ge n_0$, and so $\sum_{n=1}^{\infty}t_n<\infty$.

Example 2.3 Let $F_3(t)=\ln t+t$ and let $\theta_3(t)=\tau$ for each $t\in(0,\infty)$, where $\tau>0$ is a constant. Now, we show that $(F_3,\theta_3)\in\Delta$. We only show (δ_4). Suppose that $t_n\to 0$ and

$$\tau\le(\ln t_n+t_n)-(\ln t_{n+1}+t_{n+1})\quad \forall n\in\mathbb{N}.$$

Then

$$s_{n+1}\le e^{-\tau}s_n\quad \forall n\in\mathbb{N},$$

where $s_n=t_ne^{t_n}$. Since $e^{-\tau}<1$, then from the above we get $\sum_{n=1}^{\infty}s_n<\infty$, and so $\sum_{n=1}^{\infty}t_n<\infty$ (note that $t_n\le s_n$ for each $n\in\mathbb{N}$).

Now, we state the main result of the paper.

Theorem 2.4 *Let (X,d) be a complete metric spaces, let $T:X\to CB(X)$ be a set-valued mapping and let $(F,\frac{\theta}{2})\in\Delta$. Assume that either T is compact valued or F is continuous from the right. Furthermore, assume that*

$$\theta\bigl(d(x,y)\bigr)+F\bigl(H(Tx,Ty)\bigr)\le F\bigl(d(x,y)\bigr)\quad \forall x,y\in X \text{ with } Tx\ne Ty. \tag{5}$$

Then T has a fixed point.

Proof Let $x_0\in X$ and $x_1\in Tx_0$. If $Tx_0=Tx_1$, then $x_1\in Tx_0=Tx_1$ and x_1 is a fixed point of T. So, we may assume that $Tx_0\ne Tx_1$. Since either T is compact valued or F is continuous from the right, $x_1\in Tx_0$ and $F(d(x_1,Tx_1))<F(H(Tx_0,Tx_1))+\frac{\theta(d(x_0,x_1))}{2}$ then there exists $x_2\in Tx_1$ such that (note that F is increasing)

$$F\bigl(d(x_1,x_2)\bigr)\le F\bigl(H(Tx_0,Tx_1)\bigr)+\frac{\theta(d(x_0,x_1))}{2}. \tag{6}$$

From (5) and (6), we have

$$\begin{aligned}&\theta\bigl(d(x_0,x_1)\bigr)+F\bigl(d(x_1,x_2)\bigr)\\&\quad\le\theta\bigl(d(x_0,x_1)\bigr)+F\bigl(H(Tx_0,Tx_1)\bigr)+\frac{\theta(d(x_0,x_1))}{2}\le F\bigl(d(x_0,x_1)\bigr)+\frac{\theta(d(x_0,x_1))}{2},\end{aligned}$$

and so

$$\frac{\theta(d(x_0,x_1))}{2}+F\bigl(d(x_1,x_2)\bigr)\le F\bigl(d(x_0,x_1)\bigr). \tag{7}$$

We may also assume that $Tx_1 \neq Tx_2$ (otherwise, $x_2 \in Tx_2 = Tx_1$). Proceeding this manner, we can define a sequence $\{x_n\}$ in X satisfying

$$x_{n+1} \in Tx_n, \qquad \frac{\theta(t_n)}{2} \leq F(t_n) - F(t_{n+1}), \quad \text{for each } n \in \mathbb{N}, \tag{8}$$

where $t_n = d(x_n, x_{n+1})$. Since $\theta(t_n) > 0$ then from (8), we have $F(t_n) > F(t_{n+1})$ for each $n \in \mathbb{N}$. Since F is strictly increasing, then we deduce that $\{t_n\}$ is a nonnegative strictly decreasing sequence and so is convergent to some $r \geq 0$, $\lim_{n\to\infty} t_n = r$. Now we show that $r = 0$. On the contrary, assume that $r > 0$. From (8), we get

$$\frac{1}{2}\sum_{i=1}^{n} \theta(t_i) \leq F(t_1) - F(t_{n+1}) \quad \text{for each } n \in \mathbb{N}. \tag{9}$$

Since $\{t_n\}$ is strictly decreasing, then from (δ_1) we get $\theta(t_n) \not\to 0$. Thus, $\sum_{i=1}^{\infty} \theta(t_i) = \infty$, and then from (9), we have $\lim_{n\to\infty} F(t_n) = -\infty$. Then by (δ_3), $t_n \to 0$, a contradiction. Hence,

$$\lim_{n\to\infty} t_n = 0. \tag{10}$$

From (8), (10) and (δ_4), we have

$$\sum_{i=1}^{\infty} t_i = \sum_{i=1}^{\infty} d(x_i, x_{i+1}) < \infty.$$

Then, by the triangle inequality, $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we prove that x is a fixed point of T. To prove the claim, we may assume that $Tx_n \neq Tx$ for sufficiently large $n \in \mathbb{N}$. On the contrary, assume that $Tx_{n_i} = Tx$ for each $i \in \mathbb{N}$. Since Tx is closed, $x_{n_i+1} \in Tx_{n_i} = Tx$ and $x_{n_i+1} \to x$, then $x \in Tx$, and we are finished.

From (5), we have (note that $x_{n+1} \in Tx_n$ and $Tx_n \neq Tx$ for $n \geq N$)

$$\begin{aligned} F\bigl(d(x_{n+1}, Tx)\bigr) &\leq \theta\bigl(d(x_n, x)\bigr) + F\bigl(d(x_{n+1}, Tx)\bigr) \\ &\leq \theta\bigl(d(x_n, x)\bigr) + F\bigl(H(Tx_n, Tx)\bigr) \leq F\bigl(d(x_n, x)\bigr). \end{aligned} \tag{11}$$

Since $d(x_n, x) \to 0$, then (11) together with (δ_3) imply that

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = 0,$$

and so $d(x, Tx) = 0$. Hence, $x \in Tx$ (note that Tx is closed). □

Remark 2.5 By Example 2.1, Theorem 2.4 is an extension and improvement of Theorem 2.1 of Wardowski [22]. From Example 2.2, we infer that Theorem 2.4 is a generalization of the above mentioned Theorem 1.2 of Mizoguchi and Takahashi.

Now, we illustrate our main result by the following example.

Example 2.6 Consider the complete metric space ($X = \{0,1,2,3,\ldots\}, d$), where d is defined as

$$d(x,y) = \begin{cases} 0, & x = y, \\ x + y, & x \neq y. \end{cases}$$

Let $T : X \to CB(X)$ be defined as

$$Tx = \begin{cases} \{0,1,2,3,\ldots\}, & x = 0, \\ \{x-1, x, x+1, \ldots\}, & x > 0. \end{cases}$$

Let $f : X \to X$ be given by

$$fx = \begin{cases} 0, & x = 0, \\ x - 1, & x > 0. \end{cases}$$

Now, we show that T satisfies (5), where $\theta(t) = 1$ for each $t \in (0,\infty)$ and $F(x) = \ln x + x$ for each $x \in (0,\infty)$. To show the claim, notice first that $H(Tx_1, Tx_2) = d(fx_1, fx_2)$ for each $x_1, x_2 \in X$. Now let $x_1, x_2 \in X$ with $fx_1 \neq fx_2$. Since $d(fx_1, fx_2) - d(x_1, x_2) \leq -1$, then we have

$$\frac{d(fx_1, fx_2)}{d(x_1, x_2)} e^{d(fx_1, fx_2) - d(x_1, x_2)} \leq e^{-1}, \quad \text{for each } x_1, x_2 \in X \text{ with } fx_1 \neq fx_2.$$

Thus, from the above, we have

$$\begin{aligned} 1 &\leq \left[\ln d(x_1, x_2) + d(x_1, x_2)\right] - \left[\ln d(fx_1, fx_2) + d(fx_1, fx_2)\right] \\ &= F\big(d(x_1, x_2)\big) - F\big(d(fx_1, fx_2)\big). \end{aligned}$$

Therefore, (note that $H(Tx_1, Tx_2) = d(fx_1, fx_2)$)

$$1 \leq F\big(d(x_1, x_2)\big) - F\big(H(Tx_1, Tx_2)\big).$$

Then, by Theorem 2.4, T has a fixed point.

Now, we show that T does not satisfy the condition of Nadler's theorem. On the contrary, assume that there exists a function $k \in [0,1)$ such that

$$H(Tx_1, Tx_2) \leq k d(x_1, x_2)$$

for all $x_1, x_2 \in X$. Then

$$d(fx_1, fx_2) \leq k d(x_1, x_2).$$

Then, for each $x_1 > 1$ and $x_2 = x_1 + 1$, we have

$$2x_1 - 1 \leq k(2x_1 + 1), \quad \text{for each } x_1 > 1.$$

Hence,

$$1 = \lim_{x_1 \to \infty} \frac{2x_1 - 1}{2x_1 + 1} \le k,$$

a contradiction.

Example 2.7 For each $t \in (0, \infty)$, let $F_4(t) = -\frac{1}{t}$ and let

$$\theta_4(t) = \begin{cases} -\frac{\ln t}{t}, & 0 < t < 1, \\ 1, & 1 \le t. \end{cases}$$

Then it is easy to see that $(F_4, \theta_4) \in \Delta$, but F_4 does not satisfy the condition (F_3) of the definition of F-contraction in [22].

Now, by using the technique in [23], we present a new coupled fixed point result. For more details on coupled fixed point theory, see [23–25] and the references therein.

Corollary 2.8 *Let (M, ρ) be a complete metric space and let $(F, \frac{\theta}{2}) \in \Delta$. Let $f : M \times M \to M$ be a mapping satisfying*

$$\begin{aligned} &\theta\bigl(\rho(x,u) + \rho(y,v)\bigr) + F\bigl(\rho\bigl(f(x,y), f(u,v)\bigr) + \rho\bigl(f(y,x), f(v,u)\bigr)\bigr) \\ &\quad \le F\bigl(\rho(x,u) + \rho(y,v)\bigr) \end{aligned} \tag{12}$$

for each $x, y, u, v \in M$. Then f has a coupled fixed point (x_0, y_0), that is, $f(x_0, y_0) = x_0$ and $f(y_0, x_0) = y_0$.

Proof Let $X = M \times M$ and let d be the metric on M which is defined by

$$d\bigl((x,y),(u,v)\bigr) = \rho(x,u) + \rho(y,v).$$

Then it is straightforward to show that (X, d) is a complete metric space. Let $T : X \to X$ be defined by $T(x,y) = (f(x,y), f(y,x))$. From (12), we get

$$\theta\bigl(d\bigl((x,y),(u,v)\bigr)\bigr) + F\bigl(d\bigl(T(x,y), T(u,v)\bigr)\bigr) \le F\bigl(d\bigl((x,y),(u,v)\bigr)\bigr)$$

for each $(x,y), (u,v) \in X$. Then from Theorem 2.4 we deduce that T has a fixed point $u_0 = (x_0, y_0)$. Then (x_0, y_0) is a coupled fixed point of f. □

Competing interests
The author declares that they have no competing interests.

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