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Iterative algorithms for common elements in fixed point sets and zero point sets with applications

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Abstract

In this study, Mann-type iterative process is considered for finding a common element in the fixed point set of strict pseudocontractions and in the zero point set of the operator which is the sum of inverse strongly- monotone operators and maximal monotone operators. Weak convergence theorems of common elements are established in the framework of Hilbert spaces. Some applications of main results are also provided.

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1 Introduction and preliminaries

Throughout this article, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot\rangle$, and the norm $||\cdot||$ and that C is a nonempty closed convex subset of H.

Let $A : C \to H$ be a mapping. Recall that A is said to be *monotone* if

$$\langle Ax - Ay,\ x - y\rangle \geq 0,\ \forall x,\ y \in C.$$

A is said to be *inverse strongly-monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay,\ x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x,\ y \in C.$$

For such a case, A is also said to be *α-inverse strongly monotone*.

Let $M : H \to 2^H$ be a set-valued mapping. The set $D(M)$ defined by $D(M) = \{x \in H: Mx \neq \emptyset\}$ is said to be the *domain* of M. The set $R(M)$ defined by $R(M) = \bigcup_{x\in H} Mx$ is said to be the *range* of M. The set $G(M)$ defined by $G(M) = \{(x, y) \in H \times H: x \in D(M), y \in R(M)\}$ is said to be the *graph* of M.

Recall that M is said to be *monotone* if

$$\langle x - y,\ f - g\rangle \geq 0, \quad \forall (x,\ f),\ (y,\ g) \in G(M).$$

M is said to be *maximal monotone* if it is not properly contained in any other monotone operator. Equivalently, M is maximal monotone if $R(I + rM) = H$ for all $r > 0$. The class of monotone mappings is one of the most important classes of mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone mappings, see [1-15] and the references therein. For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r = (I + rM)^{-1} : H \to D(M)$. It is known that J_r is firmly nonexpansive and $M^{-1}(0) = F(J_r)$, where $F(J_r)$ denotes the fixed point set of J_r.

Let $S : C \to C$ be a nonlinear mapping. In this study, we use $F(S)$ to denote the fixed point set of S. Recall that the mapping S is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be *κ-strictly pseudocontractive* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was introduced by Browder and Petryshyn [16]. If $\kappa = 0$, the class of strictly pseudocontractive mappings is reduced to the class of nonexpansive mappings. In case that $\kappa = 1$, we call S a pseudocontractive mapping. Marino and Xu [17] proved that fixed point sets of strictly pseudocontractive mappings are closed and convex. They also proved that $I - S$ is demi-closed at zero. To be more precise, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.

Let $A : C \to H$ be an inverse strongly-monotone mapping. Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote by $VI(C, A)$ of the solution set of (1.1). It is known that $x \in C$ is a solution to (1.1) if and only if x is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping. In [3], Iiduka and Takahashi showed that if $\lambda \in [0, 2\alpha]$, then $I - \lambda A$ is nonexpansive.

Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \ \forall y \in C. \tag{1.2}$$

To study the equilibrium problems (1.2), we may assume that F satisfies the following conditions:

($A1$) $F(x, x) = 0$ for all $x \in C$;
($A2$) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
($A3$) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

($A4$) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Putting $F(x, y) = \langle Ax, y - x\rangle$ for every $x, y \in C$, we see that the equilibrium problem (1.2) is reduced to the variational inequality (1.1).

Recently, many authors considered the convergence of iterative sequences for the variational inequality (1.1), the equilibrium problem (1.2) and fixed point problems of nonlinear mappings; see, for example, [2,4-7,11-15,18-26].

In 2003, Takahashi and Toyoda [13] proved the following weak convergence theorem.

Theorem 1.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping from C into H and S be a non-expansive mapping from C into itself such that* $F(S) \cap VI(C, A) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$x_0 \in C,\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0,$$

where $\lambda_n \in [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *and* $\alpha_n \in [c, d]$ *for some* $c, d \in (0, 1)$. *Then,* $\{x_n\}$ *converges weakly to* $z \in F(S) \cap VI(C, A)$, *where* $z = \lim_{n\to\infty} P_{F(S)\cap VI(C, A)}x_n$.

In 2007, Tada and Takahashi [11] obtained the following weak convergence theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and S be a nonexpansive mapping from C into H such that* $F(S) \cap EP(F) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 = x \in H$ *and let*

$$\begin{cases} u_n \in C \text{ such that } F(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n \end{cases}$$

for each $n \geq 1$, *where* $\{\alpha_n\} \subset [a, b]$ *for some* $a, b \in (0, 1)$ *and* $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then,* $\{x_n\}$ *converges weakly to* $w \in F(S)\cap EP(F)$ *where* $w = \lim_{n\to\infty} P_{F(S)\cap EP(F)}x_n$.

A very common problem in diverse areas of mathematics and physical sciences consists of trying to find a point in the intersection of convex sets. This problem is referred to as the convex feasibility problem; its precise mathematical formulation is as follows. Find an $x \in \bigcap_{m=1}^{N} C_r$, where $N \geq 1$ is an integer and each C_m is a nonempty closed convex subset of H. There is a considerable investigation on the convex feasibility problem in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning.

Let K be an integer, $S : C \to C$ a strict pseudocontraction, $A_m : C \to H$ be an α_m-inverse strongly-monotone mapping and $M_m : H \to 2^H$ be a maximal monotone operator such that $D(M_m) \subset C$, where $D(M_m)$ is the domain of M_m, where $m \in \{1, 2, ..., K\}$. In this article, motivated by Theorems 1.1 and 1.2, we consider the problem of finding a common element in the following set: $F(S) \cap \bigcap_{m=1}^{K} (A_m + M_m)^{-1}(0)$, where $F(S)$ is the fixed point set of S and $(A_m + M_m)^{-1}(0)$ is the zero point set of $A_m + M_m$. Weak convergence theorems of common elements are established in real Hilbert spaces. The results presented in this article improve and extend the corresponding results announced by Tada and Takahashi [11] and Takahshi and Toyoda [13].

In order to prove our main results, we also need the following lemmas.

Lemma 1.3. [16]*Let C be a nonempty closed convex subset of a real Hilbert space H and* $S : C \to C$ *be a κ-strict pseudo-contraction with a fixed point. Define* $S : C \to C$

by $S_a x = ax + (1 - a)Sx$ for each $x \in C$. If $a \in [\kappa, 1)$, then S_a is nonexpansive with $F(S_a) = F(S)$.

Lemma 1.4. [17]*Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a κ-strict pseudocontraction. Then*

(a) *S is $\frac{1+\kappa}{1-\kappa}$-Lipschitz;*

(b) *$I - S$ is demi-closed, this is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Sx_n \to 0$, then $x \in F(S)$.*

Lemma 1.5. [27]*Let H be a real Hilbert space and $0 < p \le t_n \le q < 1$ for all $n \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 1.6. [13]*Let C be a nonempty closed convex subset of a real Hilbert space H and P_C be the metric projection from H onto C. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $y \in C$*

$$\|x_{n+1} - y\| \le \|x_n - y\|, \quad \forall n \ge 1.$$

Then $\{P_C x_n\}$ converges strongly to some $z \in C$.

Lemma 1.7. [28]*Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ be a mapping and $M : H \to 2^H$ be a maximal monotone mapping. Then*

$$F(J_r(I - rA)) = (A + M)^{-1}(0), \quad \forall r > 0.$$

Lemma 1.8. [29]*Let H be a Hilbert space and suppose $\{x_n\}$ converges weakly to x. Then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in H$ with $x \ne y$.

2 Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a κ-strict pseudocontraction, $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Let $M : H \to 2^H$ and $W : H \to 2^H$ be maximal monotone operators such that $D(M) \subset C$ and $D(W) \subset C$. Assume that $\mathcal{F} := F(S) \cap (A + M)^{-1}(0) \cap (B + W)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \gamma_n J_{r_n}(x_n - r_n A x_n) + (1 - \gamma_n) J_{s_n}(x_n - s_n B x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\left(\beta_n y_n + (1 - \beta_n) S y_n\right), \quad n \ge 0, \end{cases}$$

where $J_{r_n} = (I + r_n M)^{-1}$, $J_{s_n} = (I + s_n W)^{-1}$, $\{r_n\}$ *is a sequence in* $(0, 2\alpha)$, $\{s_n\}$ *is a sequence in* $(0, 2\beta)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$. *Assume that the following restrictions are satisfied*

(a) $0 < a \le r_n \le b < 2\alpha$ *and* $0 < c \le s_n \le d < 2\beta$;
(b) $0 \le \kappa \le \beta_n < e < 1$, $0 < h \le \alpha_n \le i < 1$ *and* $0 < j \le \gamma_n \le k < 1$,

where a, b, c, d, e, h, i, j, k are real numbers. Then the sequence $\{x_n\}$ *converges weakly to* $\bar{x} \in \mathcal{F}$, *where* $\bar{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Note that $(I - r_nA)$ and $(I - s_nB)$ are nonexpansive for each fixed $n \ge 0$.

Indeed, for any $x, y, \in C$, we see from the restriction (a) that

$$\begin{aligned}\|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\le \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2 \\ &\le \|x - y\|^2.\end{aligned}$$

This shows that $(I - r_nA)$ is nonexpansive for each fixed $n \ge 0$, so is $(I - s_nB)$.

Put

$$S_n x = \beta_n x + (1 - \beta_n)Sx, \quad \forall x \in C.$$

In the restriction (b), we obtain from Lemma 1.3 that S_n is nonexpansive for each fixed $n \ge 0$. Fixing $p \in \mathcal{F}$ and since J_{r_n}, J_{s_n}, $I - r_nA$, and $I - s_nB$ are nonexpansive, we see that

$$\begin{aligned}\|y_n - p\| &\le \gamma_n \|J_{r_n}(x_n - r_nAx_n) - p\| + (1 - \gamma_n)\|J_{s_n}(x_n - s_nBx_n) - p\| \\ &\le \|x_n - p\|.\end{aligned}$$

Since S_n is nonexpansive, we see that

$$\begin{aligned}\|x_{n+1} - p\| \,| &\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\|S_ny_n - p\| \\ &\le \alpha_n\|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{2.1}$$

Hence, the limit of the sequence $\{||x_n - p||\}$ exists. This shows that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Without loss of generality, we may assume that $\lim_{n\to\infty} ||x_n - p|| = d > 0$. Notice that

$$\begin{aligned}\|y_n - p\|^2 &\le \gamma_n\|J_{r_n}(x_n - r_nAx_n) - p\|^2 + (1 - \gamma_n)\|J_{s_n}(x_n - s_nBx_n) - p\|^2 \\ &\le \gamma_n\|(x_n - r_nAx_n) - p\|^2 + (1 - \gamma_n)\|(x_n - s_nBx_n) - p\|^2 \\ &\le \gamma_n\left(\|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2\right) \\ &\quad + (1 - \gamma_n)(\|x_n - p\|^2 - s_n(2\beta - s_n)\|Bx_n - Bp\|^2) \\ &\le \|x_n - p\|^2 - r_n\gamma_n(2\alpha - r_n)\|Ax_n - Ap\|^2 \\ &\quad - s_n(1 - \gamma_n)(2\beta - s_n)\|Bx_n - Bp\|^2.\end{aligned}$$

This in turn implies that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|S_ny_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\le \|x_n-p\|^2-(1-\alpha_n)r_n\gamma_n(2\alpha-r_n)\|Ax_n-Ap\|^2\\ &\quad -(1-\alpha_n)s_n(1-\gamma_n)(2\beta-s_n)\|Bx_n-Bp\|^2.\end{aligned} \tag{2.2}$$

It follows from the restrictions (a) and (b) that

$$(1-i)aj(2\alpha-b)\|Ax_n-Ap\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Since $\lim_{n\to\infty}||x_n\text{-}p||=d$, we see that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.3}$$

In view of (2.2), we see from the restrictions (a) and (b) that

$$(1-i)c(1-k)(2\beta-d)\|Bx_n-Bp\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Since $\lim_{n\to\infty}||x_n\text{-}p||=d$, we see that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{2.4}$$

Notice that J_{r_n} is firmly nonexpansive. Putting $u_n=J_{r_n}(x_n-r_nAx_n)$ and $v_n=J_{s_n}(x_n-s_nBx_n)$, we see that

$$\begin{aligned}\|u_n-p\|^2 &= \|J_{r_n}(x_n-r_nAx_n)-J_{r_n}(p-r_nAp)\|^2\\ &\le \langle u_n-p,(x_n-r_nAx_n)-(p-r_nAp)\rangle\\ &= \frac{1}{2}\Big(\|u_n-p\|^2+\|(x_n-r_nAx_n)-(p-r_nAp)\|^2\\ &\quad -\|(u_n-p)-((x_n-r_nAx_n)-(p-r_nAp))\|^2\Big)\\ &\le \frac{1}{2}\Big(\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n+r_n(Ax_n-Ap)\|^2\Big)\\ &= \frac{1}{2}\Big(\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2-r_n^2\|Ax_n-Ap\|^2\\ &\quad -2r_n\langle u_n-x_n,\ Ax_n-Ap\rangle\Big)\\ &\le \frac{1}{2}\Big(\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2+2r_n\|u_n-x_n\|\,\|Ax_n-Ap\|\Big).\end{aligned}$$

This in turn implies that

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|u_n-x_n\|^2+2r_n\|u_n-x_n\|\,\|Ax_n-Ap\|. \tag{2.5}$$

In a similar way, we can obtain that

$$\|v_n-p\|^2\le\|x_n-p\|^2-\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\,\|Bx_n-Bp\|. \tag{2.6}$$

Combining (2.5) with (2.6) yields that

$$\begin{aligned}\left\|x_{n+1}-p\right\|^2 &\le \alpha_n\left\|x_n-p\right\|^2+(1-\alpha_n)\left\|S_ny_n-p\right\|^2\\ &\le \alpha_n\left\|x_n-p\right\|^2+(1-\alpha_n)\left\|y_n-p\right\|^2\\ &\le \alpha_n\left\|x_n-p\right\|^2+(1-\alpha_n)(\gamma_n\left\|u_n-p\right\|^2+(1-\gamma_n)\left\|v_n-p\right\|^2) \\ &\le \left\|x_n-p\right\|^2-(1-\alpha_n)\gamma_n\|u_n-x_n\|^2+2r_n\|u_n-x_n\|\left\|Ax_n-Ap\right\|\\ &\quad -(1-\alpha_n)(1-\gamma_n)\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\left\|Bx_n-Bp\right\|.\end{aligned} \tag{2.7}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\gamma_n\|u_n-x_n\|^2 \le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2+2r_n\|u_n-x_n\|\left\|Ax_n-Ap\right\|\\ +\ 2s_n\|v_n-x_n\|\left\|Bx_n-Bp\right\|.\end{aligned}$$

In view of (2.3) and (2.4), we see from the restrictions (a) and (b) that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{2.8}$$

It also follows from (2.7) that

$$\begin{aligned}(1-\alpha_n)(1-\gamma_n)\|v_n-x_n\|^2 \le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2+2r_n\|u_n-x_n\|\left\|Ax_n-Ap\right\|\\ +2s_n\|v_n-x_n\|\left\|Bx_n-Bp\right\|.\end{aligned}$$

In view of (2.3) and (2.4), we see from the restrictions (a) and (b) that

$$\lim_{n\to\infty}\|v_n-x_n\|=0 \tag{2.9}$$

Notice that

$$\left\|y_n-x_n\right\|\le\|u_n-x_n\|+\|v_n-x_n\|.$$

It follows from (2.8) and (2.9) that

$$\lim_{n\to\infty}\left\|y_n-x_n\right\|=0. \tag{2.10}$$

On the other hand, we have

$$\lim_{n\to\infty}\left\|\alpha_n(x_n-p)+(1-\alpha_n)(S_ny_n-p)\right\|=d.$$

Notice that

$$\left\|S_ny_n-p\right\|\le\left\|y_n-p\right\|\le\left\|x_n-p\right\|.$$

This implies that

$$\limsup_{n\to\infty}\left\|S_ny_n-p\right\|\le d.$$

In view of Lemma 1.5, we arrive at

$$\lim_{n\to\infty}\left\|S_ny_n-x_n\right\|=0. \tag{2.11}$$

Note that

$$Sy_n - x_n = \frac{S_n y_n - x_n}{1-\beta_n} + \frac{\beta_n(x_n - y_n)}{1-\beta_n}.$$

From (2.10), (2.11) and the restriction (b), we get that

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0. \tag{2.12}$$

On the other hand, we see from Lemma 1.4 that

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - Sy_n\| + \|Sy_n - x_n\| \\ &\le \frac{1+\kappa}{1-\kappa}\|x_n - y_n\| + \|Sy_n - x_n\|.\end{aligned}$$

It follows from (2.10) and (2.12) that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0. \tag{2.13}$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\bar{x}$. By virtue of Lemma 1.4, we obtain that $\bar{x} \in F(S)$. Next, we show that $\bar{x} \in (A+M)^{-1}(0)$. Notice that

$$x_n - r_n A x_n \in u_n + r_n M u_n.$$

Let $\mu \in Mv$. Since M is monotone, we have

$$\left\langle \frac{x_n - u_n}{r_n} - Ax_n - \mu,\ u_n - v \right\rangle \ge 0.$$

In view of the restriction (a), we see from (2.8) that

$$\langle -A\bar{x} - \mu,\ \bar{x} - v\rangle \ge 0.$$

This implies that $-A\bar{x} \in M\bar{x}$, that is, $\bar{x} \in (A+M)^{-1}(0)$. In a similar way, we can obtain that $\bar{x} \in (B+W)^{-1}(0)$. This proves that $\bar{x} \in \mathcal{F}$.

Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x'. By the above proof, we also have $x' \in \mathcal{F}$. If $\bar{x} \neq x'$, we get from Lemma 1.8 that

$$\begin{aligned}\lim_{n\to\infty}\|x_n - \bar{x}\| &= \liminf_{i\to\infty}\|x_{n_i} - \bar{x}\| < \liminf_{i\to\infty}\|x_{n_i} - x'\| \\ &= \lim_{n\to\infty}\|x_n - x'\| = \liminf_{j\to\infty}\|x_{n_j} - x'\| \\ &< \liminf_{j\to\infty}\|x_{n_j} - \bar{x}\| = \lim_{n\to\infty}\|x_n - \bar{x}\|.\end{aligned}$$

This derives a contradiction. Hence, we have $\bar{x} = x'$. This implies that $x_n \rightharpoonup \bar{x} \in \mathcal{F}$. Let $e_n = P_{\mathcal{F}} x_n$. In view of (2.1), we obtain from Lemma 1.6 that $\{e_n\}$ converges strongly to some $e \in \mathcal{F}$. On the other hand, we see from $\bar{x} \in \mathcal{F}$ that $\langle x_n - e_n,\ e_n - \bar{x}\rangle \ge 0$. Note that $\{x_n\}$ converges weakly to $\bar{x}$. It follows that

$$\langle \bar{x} - e,\ e - \bar{x}\rangle \ge 0.$$

This implies that $\bar{x} = e = \lim_{n\to\infty} P_{\mathcal{F}} x_n$. The proof is completed. □

From Theorem 2.1, we can obtain the following immediately.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a κ-strict pseudocontraction, $A_m : C \to H$ be an α_m-inverse strongly monotone mapping and $M_m : H \to 2^H$ be a maximal monotone operator such that $D(M_m) \subset C$, where $m \in \{1, 2, ..., K\}$. Assume that $\mathcal{F} := F(S) \cap \bigcap_{m=1}^{N} (A_m + M_m)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \sum_{m=1}^{K} \gamma_{n,m} J_{r_n,m}(x_n - r_{n,m}A_m x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n y_n + (1-\beta_n)Sy_n),\ n \geq 0, \end{cases}$$

where $J_{r_n,m} = (I + r_{n,m}M_m)^{-1}$, $\{r_{n,m}\}$ is a sequence in $(0, 2\alpha_m)$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_{n,\mathrm{m}}\}$ are sequences in $(0, 1)$. Assume that the following restrictions are satisfied

(a) $0 < a_m \leq r_{n,m} \leq b_m < 2\alpha_m$ *for each* $m \in \{1, 2, ..., K\}$;

(b) $\sum_{m=1}^{K} \gamma_{n,m} = 1$;

(c) $0 \leq k \leq \beta_n < c < 1$, $0 < d \leq \alpha_n \leq e < 1$ *and* $0 < h_m \leq \gamma_{n,m} \leq i_m < 1$, *where* $a_1, a_2, ..., a_K, b_1, b_2, ..., b_K, c, d, e, h_1, h_2, ..., h_K, i_1, i_2, ..., i_K$ *are real numbers. Then the sequence* $\{x_n\}$ *converges weakly to* $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

If $S = I$, where I denotes the identity, then Theorem 2.2 is reduced to the following.

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_m : C \to H$ be an α_m-inverse strongly monotone mapping and $M_m : H \to 2^H$ be a maximal monotone operator such that $D(M_m) \subset C$, where $m \in \{1, 2, ..., K\}$. Assume that $\mathcal{F} := \bigcap_{m=1}^{N} (A_m + M_m)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^{K} \gamma_{n,m} J_{r_n,m}(x_n - r_{n,m}A_m x_n), \quad n \geq 0,$$

where $J_{r_n,m} = (I + r_{n,m}M_m)^{-1}$, $\{r_{n,m}\}$ is a sequence in $(0, 2\alpha_m)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_{n,m}\}$ are sequences in $(0, 1)$. Assume that the following restrictions are satisfied

(a) $0 < a_m \leq r_{n,m} \leq b_m < 2\alpha_m$ *for each* $m \in \{1, 2, ..., K\}$;

(b) $\sum_{m=1}^{K} \gamma_{n,m} = 1$;

(c) $0 < c \leq \alpha_n < d < 1$ *and* $0 < h_m \leq \gamma_{n,m} \leq i_m < 1$,

where $a_1, a_2, ..., a_K, b_1, b_2, ..., b_K, c, d, h_1, h_2, ..., h_K, i_1, i_2, ..., i_K$ are real numbers. Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

3 Applications

Let H be a Hilbert space and $f : H \to (-\infty, +\infty]$ a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y \rangle, \ z \in H\}, \ \forall x \in H.$$

From Rockafellar [9,30], we know that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$.

First, we consider the problem of finding common minimizers of proper convex lower semicontinuous functions.

Theorem 3.1. *Let H be a real Hilbert space. Let $f : H \to (-\infty, +\infty]$ and $g : H \to (-\infty, +\infty]$ be proper convex lower semi-continuous functions. Assume that $\mathcal{F} := (\partial f)^{-1}(0) \cap (\partial g)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H, \\ z_n = \arg\min_{z \in H}\{g(z) + \frac{\|z - x_n\|^2}{2s_n}\}, \\ y_n = \arg\min_{z \in H}\{f(z) + \frac{\|z - x_n\|^2}{2r_n}\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\gamma_n y_n + (1 - \gamma_n) z_n), \ n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0, 1)$. Assume that the following restrictions are satisfied

(a) $0 < a \leq r_n \leq b < \infty$ *and* $0 < c \leq s_n \leq d < \infty$;
(b) $0 < h \leq \alpha_n \leq i < 1$ *and* $0 < j \leq \gamma_n \leq k < 1$,

where a, b, c, d, h, i, j, k are real numbers. Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n \to \infty} P_{\mathcal{F}} x_n$.

Proof. Putting $A = B = 0$ and $S = I$, the identity mapping, we can conclude from Theorem 2.1 the desired conclusion immediately. □

Let I_C be the indicator function of C, i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{3.1}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator.

Lemma 3.2. [12]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let P_C be the metric projection from H onto C, ∂I_C be the subdifferential of I_C, where I_C is as defined in* (3.1) *and $J_r = (I + r\partial I_C)^{-1}$. Then*

$$y = J_r x \ \Leftrightarrow \ y = P_C x, \ x \in H, \ y \in C.$$

Second, we consider the variation inequality (1.1).

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and P_C be the metric projection from H onto C. Let $S : C \to C$ be a κ-strict pseudocontraction, $A : C \to H$ be an α-inverse strongly monotone mapping and $B : C \to H$ be a β-inverse strongly monotone mapping. Assume that $\mathcal{F} := F(S) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \gamma_n P_C(x_n - r_n A x_n) + (1 - \gamma_n) P_C(x_n - s_n B x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\left(\beta_n y_n + (1 - \beta_n) S y_n\right), \quad n \geq 0, \end{cases}$$

where $\{r_n\}$ *is a sequence in* $(0, 2\alpha)$, $\{s_n\}$ *is a sequence in* $(0, 2\beta)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$. *Assume that the following restrictions are satisfied*

(a) $0 < a \leq r_n \leq b < 2\alpha$ *and* $0 < c \leq s_n \leq d < 2\beta$;
(b) $0 \leq \kappa \leq \beta_n < e < 1$, $0 < h \leq \alpha_n \leq i < 1$ *and* $0 < j \leq \gamma_n \leq k < 1$,

where a, b, c, d, e, h, i, j, k are real numbers. Then the sequence $\{x_n\}$ *converges weakly to* $\bar{x} \in \mathcal{F}$, *where* $\bar{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Put $M = W = \partial I_C$. Next, we show that $VI(C, A) = (A + \partial I_C)^{-1}(0)$ and $VI(C, B) = (B + \partial I_C)^{-1}(0)$, respectively. Notice that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}(0) &\Leftrightarrow 0 \in Ax + \partial I_C x \\ &\Leftrightarrow -Ax \in \partial I_C x \\ &\Leftrightarrow \langle Ax, y - x \rangle \geq 0 \\ &\Leftrightarrow x \in VI(C, A) \end{aligned}$$

In the same way, we can obtain that $x \in (B + \partial I_C)^{-1} \Leftrightarrow (0)\ x \in VI(C, B)$. From Lemma 3.2, we can conclude the desired conclusion immediately. □

Remark 3.1. Let S be a nonexpansive mapping, $A = B$, $M = W$ and $\beta_n = 0$ in Theorem 3.3. Then Theorem 3.3 is reduced to Theorem 1.1 in Section 1.

Third, we consider the problem of finding common fixed points of three strict pseudocontractions.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $S : C \to C$ *be a* κ*-strict pseudocontraction,* $T : C \to C$ *be an* α*-strict pseudocontraction and* $R : C \to C$ *be a* β*-strict pseudocontraction. Assume that* $\mathcal{F} := F(R) \cap F(S) \cap F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \gamma_n((1 - r_n)x_n + r_n T x_n) + (1 - \gamma_n)((1 - s_n)x_n + s_n R x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n y_n + (1 - \beta_n) S y_n), \ n \geq 0, \end{cases}$$

where $\{r_n\}$ *is a sequence in* $(0, 1 - \alpha)$, $\{s_n\}$ *is a sequence in* $(0, 1 - \beta)$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$. *Assume that the following restrictions are satisfied*

(a) $0 < a \leq r_n \leq b < 1 - \alpha$ *and* $0 < c \leq s_n \leq d < 1 - \beta$;
(b) $0 \leq \kappa \leq \beta_n < e < 1$, $0 < h \leq \alpha_n \leq i < 1$ *and* $0 < j \leq \gamma_n \leq k < 1$,

where a, b, c, d, e, h, i, j, k are real numbers. Then the sequence $\{x_n\}$ *converges weakly to* $\bar{x} \in \mathcal{F}$, *where* $\bar{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Putting $A = I - T$, we see that A is $\frac{1-\alpha}{2}$*-inverse-strongly monotone. We also have* $F(T) = VI(C, A)$ and $P_C(x_n - r_n A x_n) = (1 - r_n)x_n + r_n T x_n$. Putting $B = I - R$, we see that B is $\frac{1-\beta}{2}$-inverse-strongly monotone. We also have $F(R) = VI(C, B)$ and P_C

$(x_n - s_nBx_n) = (1 - s_n)x_n + s_nRu_n$. In view of Theorem 3.2, we can obtain the desired result immediately. □

The following lemma can be found in [31,32].

Lemma 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and let F be a bifunction from C × C to* $\mathbb{R}$ *which satisfies (A1)-(A4). Then, for any r >0 and x∠ H, there exists z∠ C such that*

$$F(z,\ y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_rx = \left\{z \in C\ :\ F(z,\ y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0,\ \forall y \in C\right\} \tag{3.2}$$

for all r >0 and x ∈ H. Then, the following hold:

(a) *T_r is single-valued;*
(b) *T_r is firmly nonexpansive, i.e., for any x, y ∈ H.,*

$$\left\|T_rx - T_ry\right\|^2 \leq \langle T_rx - T_ry,\ x - y\rangle;$$

(c) *$F(T_r) = EP(F)$;*
(d) *$EP(F)$ is closed and convex.*

Lemma 3.6. [12]*Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from C × C to* $\mathbb{R}$ *which satisfies (A1)-(A4) and A_F be a multivalued mapping from H into itself defined by*

$$A_Fx = \begin{cases} \{z \in H : F(x,\ y) \geq \langle y - x,\ z\rangle,\ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.3}$$

Then A_F is a maximal monotone operator with the domain $T_rx = (I + rA_F)^{-1}x,\ \forall x \in H,\ r > 0,$ *and*

$$T_rx = (I + rA_F)^{-1}x,\ \forall x \in H,\ r > 0,$$

where T_r is defined as in (3.2).

Finally, we consider the problem of finding common elements in solution set of equilibrium problems and in the fixed point set of strict pseudocontractions.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from C × C to* $\mathbb{R}$ *which satisfies (A1)-(A4), G be a bifunction from C × C to* $\mathbb{R}$ *which satisfies (A1)-(A4) and $S : C \to C$ be a κ-strict pseudocontraction. Assume that* $\mathcal{F} := F(S) \cap EP(F) \cap EP(G) \neq \emptyset$. *Let $\{r_n\}$ and $\{s_n\}$ be two positive sequences and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ sequences in (0, 1). Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \gamma_n u_n + (1 - \gamma_n) v_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \left(\beta_n y_n + (1 - \beta_n) S y_n \right), \; n \geq 0, \end{cases}$$

where u_n is such that

$$F(u_n,\ u) + \frac{1}{r_n} \langle u - u_n,\ u_n - x_n \rangle \geq 0, \quad \forall u \in C$$

and v_n is such that

$$G(v_n,\ v) + \frac{1}{s_n} \langle v - v_n,\ v_n - x_n \rangle \geq 0, \quad \forall v \in C.$$

Assume that the following restrictions are satisfied

(a) $0 < a \leq r_n \leq b < \infty$ *and* $0 < c \leq s_n \leq d < \infty$;
(b) $0 \leq \kappa \leq \beta_n < e < 1$, $0 < h \leq \alpha_n \leq i < 1$ *and* $0 < j \leq \gamma_n \leq k < 1$,

where a, b, c, d, e, h, i, j, k are real numbers. Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.

Proof. Putting $A = B = 0$, we can conclude from Lemma 3.6 the desired conclusion immediately. □

Remark 3.2. Let S be a nonexpansive mapping, $F = G$ and $\beta_n = 0$ in Theorem 3.7. Then Theorem 3.7 is reduced to Theorem 1.2 in Section 1.

Competing interests
The author declares that they have no competing interests.

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