Dehghan and Shahzad *Fixed Point Theory and Applications* 2012, **2012**:208
http://www.fixedpointtheoryandapplications.com/content/2012/1/208

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong convergence of a CQ method for k-strictly asymptotically pseudocontractive mappings

Hossein Dehghan[1] and Naseer Shahzad[2*]

*Correspondence: nshahzad@kau.edu.sa
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Let E be a real q-uniformly smooth Banach space, which is also uniformly convex (for example, L_p or ℓ_p spaces, $1 < p < \infty$), and C be a nonempty bounded closed convex subset of E. Let $T: C \to C$ be a k-strictly asymptotically pseudocontractive map with a nonempty fixed point set. A hybrid algorithm is constructed to approximate fixed points of such maps. Furthermore, strong convergence of the proposed algorithm is established.

Keywords: strong convergence; CQ method; k-strictly asymptotically pseudocontractive mapping

1 Introduction

Let E be a real Banach space and E^* be the dual of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^* \rangle$. The normalized duality mapping J from E to 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. It is known that a Banach space E is smooth if and only if the normalized duality mapping J is single valued. Some properties of the duality mapping have been given in [1, 2].

Let C be a nonempty subset of E. The mapping $T: C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. Also, T is called *uniformly L-Lipschitz* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and each $n \geq 1$. The mapping $T: C \to C$ is called *k-strictly asymptotically pseudocontractive* if there exist a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and a constant $k \in [0, 1)$, and for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \frac{1}{2}(1 + k_n)\|x - y\|^2 - \frac{1}{2}(1 - k)\|x - T^n x - (y - T^n y)\|^2 \tag{1.1}$$

for each $n \geq 1$. If I denotes the identity operator, then (1.1) can be written in the form

$$\langle (I - T^n)x - (I - T^n)y, j(x-y)\rangle \geq \frac{1}{2}(1-k)\|(I-T^n)x - (I-T^n)y\|^2 - \frac{1}{2}(k_n - 1)\|x-y\|^2. \tag{1.2}$$

The class of k-strictly asymptotically pseudocontractive mappings was first introduced in Hilbert spaces by Qihou [3]. In Hilbert spaces, j is the identity and it is shown [4] that (1.1) (and hence (1.2)) is equivalent to the inequality

$$\|T^n x - T^n y\|^2 \leq k_n\|x-y\|^2 + k\|(I-T^n)x - (I-T^n)y\|^2$$

which is the inequality considered by Qihou [3]. In the same paper, the author proved strong convergence of the modified Mann iteration processes for k-strictly asymptotically pseudocontractive mappings in Hilbert spaces. The modified Mann iteration scheme was introduced by Schu [5, 6] and has been used by several authors (see, for example, [7–12]). In [13] Osilike extended Qihou's result from Hilbert spaces to much more general real q-uniformly smooth Banach spaces, $1 < q < \infty$.

The classes of nonexpansive and asymptotically nonexpansive mappings are important classes of mappings because they have applications to solutions of differential equations which have been studied by several authors (see, *e.g.*, [14–16] and references contained therein). It would be of interest to study the class of k-strictly asymptotically pseudocontractive mappings in view of the fact that it is closely related to the above two classes.

On the other hand, using the metric projection, Matsushita and Takahashi [17] introduced the following iterative algorithm for nonexpansive mappings: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{\text{co}}\{z \in C : \|z - Tz\| \leq t_n\|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}x, \quad n = 0, 1, 2, \dots, \end{cases} \tag{1.3}$$

where $\overline{\text{co}}D$ denotes the convex closure of the set D, J is the normalized duality mapping, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and $P_{C_n\cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. Then, they proved that $\{x_n\}$ generated by (1.3) converges strongly to a fixed point of the mapping T.

In this paper, motivated by these facts, we introduce the following iterative algorithm for finding fixed points of a k-strictly asymptotically pseudocontractive mapping T in a uniformly convex and q-uniformly smooth Banach space: $x_1 = x \in C$, $C_0 = D_0 = C$ and

$$\begin{cases} C_n = \overline{\text{co}}\{z \in C_{n-1} : \|z - T^n z\| \leq t_n\|x_n - T^n x_n\|\}, \\ D_n = \{z \in D_{n-1} : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}x, \quad n = 1, 2, \dots, \end{cases} \tag{1.4}$$

where $\overline{\text{co}}D$ denotes the convex closure of the set D, J is the normalized duality mapping, $\{t_n\}$ is a sequence in $(0,1)$ with $t_n \to 0$, and $P_{C_n\cap D_n}$ is the metric projection from E onto $C_n \cap D_n$.

The purpose of this paper is to establish a strong convergence theorem of the iterative algorithm (1.4) for k-strictly asymptotically pseudocontractive mappings in a uniformly convex and q-uniformly smooth Banach space.

2 Preliminaries

The *modulus of smoothness* of a Banach space E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : \|x\| \le 1, \|y\| \le t\right\}.$$

E is *uniformly smooth* if and only if $\lim_{t\to 0^+} \rho_E(t)/t = 0$. Let $q > 1$. The Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \le ct^q$. Hilbert spaces, L_p (or ℓ_p) spaces, $1 < p < \infty$, and the Sobolev spaces, W_m^p, $1 < p < \infty$, are q-uniformly smooth.

When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. The Banach space E is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply that $x_n \to x$. Every uniformly convex Banach space has the Kadec-Klee property [1].

Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E. Then for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y\in C} \|y - x\|.$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the *metric projection* from E onto C. Let $x \in E$ and $u \in C$. Then it is known that $u = P_C x$ if and only if

$$\big\langle u - y, J(x - u)\big\rangle \ge 0 \tag{2.1}$$

for all $y \in C$ (see [1, 18]).

In the sequel, we need the following results.

Proposition 2.1 (See [19]) *Let C be a bounded closed convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ depending only on the diameter of C such that*

$$\gamma\left(\left\|\sum_{i=1}^n \lambda_i Tx_i - T\left(\sum_{i=1}^n \lambda_i x_i\right)\right\|\right) \le \max_{1\le i<j\le n}\big(\|x_i - x_j\| - \|Tx_i - Tx_j\|\big)$$

holds for any nonexpansive mapping $T : C \to E$, any elements $x_1, \ldots, x_n$ in C, and any numbers $\lambda_1, \ldots, \lambda_n \ge 0$ with $\lambda_1 + \cdots + \lambda_n = 1$.

Corollary 2.2 [20, Corollary 1.2] *Under the same suppositions as in Proposition 2.1, there exists a strictly increasing convex continuous function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ depending only on the diameter of C such that*

$$\gamma\left(\left\|\sum_{i=1}^n \lambda_i x_i - T\left(\sum_{i=1}^n \lambda_i x_i\right)\right\|\right) \le \max_{1\le i\le n}\big(\|x_i - Tx_i\|\big)$$

holds for any nonexpansive mapping $T : C \to E$, any elements $x_1, \dots, x_n$ in C, and any numbers $\lambda_1, \dots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$. (*Note that γ does not depend on T.*)

In order to utilize Corollary 2.2 for k-strictly asymptotically pseudocontractive mappings, we need the following lemmas.

Lemma 2.3 [4] *Let E be a real Banach space, C be a nonempty subset of E, and $T : C \to C$ be a k-strictly asymptotically pseudocontractive mapping. Then T is uniformly L-Lipschitzian.*

Lemma 2.4 [21, Lemma 3.1] *Let E be a real q-uniformly smooth Banach space and C be a nonempty convex subset of E. Let $T : C \to C$ be a k-strictly asymptotically pseudocontractive map, and let $\{\alpha_n\}$ be a real sequence in* $[0,1]$. *Define $S_n : C \to C$ by $S_n x := (1 - \alpha_n)x + \alpha_n T^n x$ for all $x \in C$. Then for all $x, y \in C$, we have*

$$\begin{aligned} \|S_n x - S_n y\|^q \leq{} & \left(1 + \frac{q}{2}\alpha_n(k_n - 1)\right)\|x - y\|^q \\ & - \alpha_n\left(\frac{q}{2}(1 - k)(1 + L)^{-(q-2)} - c_q\alpha_n^{q-1}\right)\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^q, \end{aligned}$$

where L is the uniformly Lipschitzian constant of T and $c_q > 0$ is the constant which appeared in [21, Theorem 2.1].

Let $\beta = \min\{1, [\frac{q}{2}(1-k)(1+L)^{-(q-2)}/c_q]^{1/(q-1)}\}$ and choose $\alpha \in (0, \beta)$. Set $\alpha_n = \alpha$ for all $n \geq 1$ in Lemma 2.4 and observe that $\|S_n x - S_n y\|^q \leq (1 + \frac{q}{2}\alpha(k_n - 1))\|x - y\|^q$. Thus,

$$\|S_n x - S_n y\| \leq \left(1 + \frac{q}{2}\alpha(k_n - 1)\right)^{1/q}\|x - y\| \tag{2.2}$$

for all $x, y \in C$ and each $n \geq 1$.

Theorem 2.5 [21, Theorem 3.1] *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let C be a nonempty closed convex subset of E and $T : C \to C$ be a k-strictly asymptotically pseudocontractive mapping with a nonempty fixed point set. Then $(I - T)$ is demiclosed at zero, i.e., if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$, where $F(T)$ is the set of all fixed points of T.*

3 Strong convergence theorem

In this section, we study the iterative algorithm (1.4) for finding fixed points of k-strictly asymptotically pseudocontractive mappings in a uniformly convex and q-uniformly smooth Banach space. We first prove that the sequence $\{x_n\}$ generated by (1.4) is well defined. Then, we prove that $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.

Lemma 3.1 *Let C be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space E, and let $T : C \to C$ be a mapping. If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ generated by* (1.4) *is well defined.*

Proof It is easy to check that $C_n \cap D_n$ is closed and convex and $F(T) \subset C_n$ for each $n \in \mathbb{N}$. Moreover, $D_1 = C$ and so $F(T) \subset C_1 \cap D_1$. Suppose $F(T) \subset C_k \cap D_k$ for $k \in \mathbb{N}$. Then there exists a unique element $x_{k+1} \in C_k \cap D_k$ such that $x_{k+1} = P_{C_k \cap D_k} x$. If $u \in F(T)$, then it follows from (2.1) that

$$\langle x_{k+1} - u, J(x - x_{k+1}) \rangle \geq 0,$$

which implies $u \in D_{k+1}$. Therefore, $F(T) \subset C_{k+1} \cap D_{k+1}$. By the mathematical induction, we obtain that $F(T) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well defined. □

In order to prove our main result, the following lemma is needed.

Lemma 3.2 *Let C be a nonempty bounded closed convex subset of a real q-uniformly smooth and uniformly convex Banach space E. Let $T : C \to C$ be a k-strictly asymptotically pseudocontractive mapping with $\{k_n\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by* (1.4), *then for any $j \in \mathbb{N}$*,

$$\lim_{n\to\infty} \| x_n - T^{n-j} x_n \| = 0.$$

Proof Fix $j \in \mathbb{N}$ and put $m = n - j$. Since $x_n = P_{C_{n-1} \cap D_{n-1}} x$, we have $x_n \in C_{n-1} \subseteq \cdots \subseteq C_m$. Since $t_m > 0$, there exist $y_1, \ldots, y_N \in C$ and $\lambda_1, \ldots, \lambda_N \geq 0$ with $\lambda_1 + \cdots + \lambda_N = 1$ such that

$$\left\| x_n - \sum_{i=1}^{N} \lambda_i y_i \right\| < t_m, \tag{3.1}$$

and $\|y_i - T^m y_i\| \leq t_m \|x_m - T^m x_m\|$ for all $i \in \{1, \ldots, N\}$. It follows from Lemma 2.3 that T is uniformly L-Lipschitzian. Put $M = \sup_{x \in C} \|x\|$, $u = P_{F(T)} x$ and $r_0 = \sup_{n \geq 1} (1 + L) \|x_n - u\|$. Thus,

$$\| y_i - T^m y_i \| \leq t_m \| x_m - T^m x_m \| \leq t_m (1 + L) \|x_m - u\| \leq r_0 t_m \tag{3.2}$$

for all $i \in \{1, \ldots, N\}$. Define $H_m : C \to E$ by

$$H_m x = \frac{1}{a_m} S_m x$$

for all $x \in C$, where $a_m = (1 + \frac{q}{2} \alpha (k_m - 1))^{1/q}$ and S_m is as in (2.2). It follows from (2.2) that H_m is nonexpansive. Using (3.2) and the fact that $\|y_i - S_m y_i\| = \alpha \|y_i - T^m y_i\|$, we have

$$\|y_i - H_m y_i\| \leq \left(1 - \frac{1}{a_m}\right) \|y_i\| + \frac{1}{a_m} \|y_i - S_m y_i\| \leq \left(1 - \frac{1}{a_m}\right) M + \alpha r_0 t_m \tag{3.3}$$

for all $i \in \{1, \ldots, N\}$. It follows from Corollary 2.2, (3.1), and (3.3) that

$$\begin{aligned} \|x_n - H_m x_n\| \leq {} & \left\| x_n - \sum_{i=1}^{N} \lambda_i y_i \right\| + \left\| \sum_{i=1}^{N} \lambda_i y_i - H_m \left(\sum_{i=1}^{N} \lambda_i y_i \right) \right\| \\ & + \left\| H_m \left(\sum_{i=1}^{N} \lambda_i y_i \right) - H_m x_n \right\| \end{aligned}$$

$$\leq 2t_m + \gamma^{-1}\Big(\max_{1\leq i\leq N}\|y_i - H_m y_i\|\Big)$$
$$\leq 2t_m + \gamma^{-1}\bigg(\Big(1-\frac{1}{a_m}\Big)M + \alpha r_0 t_m\bigg).$$

Since $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} t_n = 0$, it follows from the last inequality that $\lim_{n\to\infty}\|x_n - H_m x_n\| = 0$. Thus, $\lim_{n\to\infty}\|x_n - S_m x_n\| = 0$ and so $\lim_{n\to\infty}\|x_n - T^m x_n\| = 0$. This completes the proof. □

Theorem 3.3 *Let C be a nonempty bounded closed convex subset of a real q-uniformly smooth and uniformly convex Banach space E. Let $T : C \to C$ be a k-strictly asymptotically pseudocontractive mapping with $\{k_n\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by* (1.4). *Then $\{x_n\}$ converges strongly to the element $P_{F(T)}x$ of $F(T)$, where $P_{F(T)}$ is the metric projection from E onto $F(T)$.*

Proof Put $u = P_{F(T)}x$. Since $F(T) \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n\cap D_n}x$, we have that

$$\|x - x_{n+1}\| \leq \|x - u\| \tag{3.4}$$

for all $n \in \mathbb{N}$. By Lemma 3.2, we have

$$\|x_n - Tx_n\| \leq \big\|x_n - T^{n-1}x_n\big\| + \big\|T^{n-1}x_n - Tx_n\big\|$$
$$\leq \big\|x_n - T^{n-1}x_n\big\| + L\big\|T^{n-2}x_n - x_n\big\| \to 0 \quad \text{as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup v$. It follows from Theorem 2.5 (demiclosedness of T) that $v \in F(T)$. From the weakly lower semicontinuity of norm and (3.4), we obtain

$$\|x - u\| \leq \|x - v\| \leq \liminf_{i\to\infty}\|x - x_{n_i}\| \leq \limsup_{i\to\infty}\|x - x_{n_i}\| \leq \|x - u\|.$$

This together with the uniqueness of $P_{F(T)}x$ implies $u = v$, and hence $x_{n_i} \rightharpoonup u$. Therefore, we obtain $x_n \rightharpoonup u$. Furthermore, we have that

$$\lim_{n\to\infty}\|x - x_n\| = \|x - u\|.$$

Since E is uniformly convex, using the Kadec-Klee property, we have that $x - x_n \to x - u$. It follows that $x_n \to u$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Gava Zang, Zanjan, 45137-66731, Iran. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

Acknowledgements
The authors are grateful to the reviewers for their useful comments.

Received: 6 February 2012 Accepted: 26 October 2012 Published: 22 November 2012

References

1. Agarwal, RP, Regan, DO, Sahu, DR: Convexity, smoothness and duality mappings. In: Fixed Point Theory for Lipschitzian-type Mappings with Applications, pp. 49-115. Springer, New York (2009)
2. Cioranescu, I: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
3. Qihou, L: Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings. Nonlinear Anal. **26**(11), 1835-1842 (1996)
4. Osilike, MO, Aniagbosor, SC, Akuchu, BG: Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces. Panam. Math. J. **12**(2), 77-88 (2002)
5. Schu, J: Iterative construction of fixed point of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
6. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)
7. Ofoedu, EU: Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space. J. Math. Anal. Appl. **321**(2), 722-728 (2006)
8. Rafiq, A: On iterations for families of asymptotically pseudocontractive mappings. Appl. Math. Lett. **24**, 33-38 (2011)
9. Osilike, MO, Igbokwe, DI: Convergence theorems for asymptotically pseudocontractive maps. Bull. Korean Math. Soc. **39**(3), 389-399 (2002)
10. Tang, Y, Liu, L: Note on some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, Article ID 24978 (2006)
11. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
12. Dehghan, H: Approximating fixed points of asymptotically nonexpansive mappings in Banach spaces by metric projections. Appl. Math. Lett. **24**, 1584-1587 (2011)
13. Osilike, MO: Iterative approximations of fixed points of asymptotically demicontractive mappings. Indian J. Pure Appl. Math. **29**(12), 1291-1300 (1998)
14. Chidume, CE, Zegeye, H: Strong convergence theorems for common fixed points of uniformly L-Lipschitzian pseudocontractive semi-groups. Appl. Anal. **86**(3), 353-366 (2007)
15. Shioji, N, Takahashi, W: Strong convergence theorems for asymptotically nonexpansive semi-groups in Hilbert spaces. Nonlinear Anal. **34**, 87-99 (1998)
16. Suzuki, T: On strong convergence to common fixed points of nonexpansive semi-groups in Banach spaces. Proc. Am. Math. Soc. **131**, 2133-2136 (2003)
17. Matsushita, S, Takahashi, W: Approximating fixed points of nonexpansive mappings in a Banach space by metric projections. Appl. Math. Comput. **196**, 422-425 (2008)
18. Alber, YI: Metric and generalized projection operators in Banach spaces: properties and applications. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes in Pure and Applied Mathematics, pp. 15-50. Dekker, New York (1996)
19. Bruck, RE: On the convex approximation property and the asymptotic behaviour of nonlinear contractions in Banach spaces. Isr. J. Math. **38**, 304-314 (1981)
20. Kruppel, M: On an inequality for nonexpansive mappings in uniformly convex Banach spaces. Rostock. Math. Kolloqu. **51**, 25-32 (1997)
21. Osilike, MO, Udomene, A, Igbokwe, DI, Akuchu, BG: Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps. J. Math. Anal. Appl. **326**(2), 1334-1345 (2007)

doi:10.1186/1687-1812-2012-208
Cite this article as: Dehghan and Shahzad: **Strong convergence of a CQ method for k-strictly asymptotically pseudocontractive mappings.** *Fixed Point Theory and Applications* 2012 **2012**:208.