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Strong convergence of a general algorithm for nonexpansive mappings in Banach spaces

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Abstract

In this work, we consider a general algorithm for a countable family of nonexpansive mappings in Banach spaces. We proved that the proposed algorithm converges strongly to a common fixed point of a countable family of nonexpansive mappings which solves uniquely the corresponding variational inequality. It is worth pointing out that our proofs contain some new techniques. Our results improve and extend the corresponding ones announced by many others.

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1 Introduction

Let X be a real Banach space and let C be a nonempty closed convex subset of X . Recall that a mapping $T : C \rightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. We denote by $\text{Fix}(T)$ the set of fixed points of T .

In 2009, Yao *et al.* [1] considered the following algorithm in a Hilbert space. For an arbitrary point $x_0 \in C$,

$$\begin{cases} y_n = P_C[(1 - \alpha_n)x_n], \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n Ty_n, \quad n \geq 0. \end{cases} \quad (1.1)$$

They proved if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions, the $\{x_n\}$ defined by (1.1) converges strongly to a fixed point of T .

Recently, motivated and inspired by the above results, Wang and Hu [2] introduced the following algorithm in a Hilbert space. For an arbitrary point $x_1 \in C$,

$$\begin{cases} y_n = P_C[(I - \alpha_n F)x_n], \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T_n y_n, \quad n \geq 1, \end{cases} \quad (1.2)$$

where $P_C : X \rightarrow C$ is a metric projection, $F : C \rightarrow X$ is a β -Lipschitzian and η -strongly monotone operators. They proved that the proposed algorithm converges strongly to $x^* \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$, which solves the variational inequality $\langle Fx^*, x^* - u \rangle \leq 0$, $u \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$.

On the other hand, Aoyama *et al.* [3] considered the following algorithm in a uniformly convex and 2-uniformly smooth Banach space. For $x_1 = x \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n), \tag{1.3}$$

where $Q_C : X \rightarrow C$ is a sunny nonexpansive retraction, and A is a β -Lipschitzian and η -inverse strongly accretive operator. They proved that $\{x_n\}$ generated by (1.3) converges weakly to a unique element z of $S(C, A)$, where $S(C, A) = \{u \in C : \langle Au, j(v - u) \rangle \geq 0, \forall v \in C\}$.

The results of Yao *et al.* [1] and Wang and Hu [2] both are obtained when the underlying space is a Hilbert space. Meanwhile, Aoyama *et al.* [3] just obtained a weak convergence theorem for strongly accretive and Lipschitzian operators. So, the above results bring us the following natural question.

Question 1.1 How to extend the results of Yao *et al.* [1] and Wang and Hu [2] to the more general uniformly convex and 2-uniformly smooth Banach space?

In this work, motivated and inspired by the above results, we introduce a general algorithm (3.1) (defined below) for a countable family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. We prove that the sequence $\{x_n\}$ defined by (3.1) converges strongly to $x^* \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$, which solves uniquely the variational inequality $\langle Fx^*, j(x^* - u) \rangle \leq 0, u \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$. Furthermore, we provide an affirmative answer to Question 1.1. It is worth pointing out that our proofs contain some new techniques.

2 Preliminaries

Let X be a real Banach space with the norm $\|\cdot\|$ and let X^* be its dual space. The value of $f \in X^*$ and $x \in X$ will be denoted by $\langle x, f \rangle$. For the sequence $\{x_n\}$ in X , we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x . $x_n \rightarrow x$ means that $\{x_n\}$ converges strongly to x .

Let $\eta > 0$. A mapping F from C into X is said to be η -strongly accretive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y) \rangle \geq \eta \|x - y\|^2$$

for all $x, y \in C$. A mapping F from C into X is said to be β -Lipschitzian if, for $\beta > 0$,

$$\|Fx - Fy\| \leq \beta \|x - y\|$$

for all $x, y \in C$. It is well known that the η -strongly accretive operators are the extension of the η -strongly monotone operators from Hilbert spaces to Banach spaces.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if, for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. Also, we define a function $\rho : [0, \infty) \rightarrow [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}.$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \rightarrow 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q -uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. One should note that no Banach space is q -uniformly smooth for $q > 2$; see [4] for more details. So, in this paper, we focus on a 2-uniformly smooth Banach space. It is well known that Hilbert spaces and Lebesgue L^p ($p \geq 2$) spaces are uniformly convex and 2-uniformly smooth.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([5]) *Let q be a given real number with $1 < q \leq 2$ and let X be a q -uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + 2\|Ky\|^q$$

for all $x, y \in X$, where K is the q -uniformly smooth constant of X and J_q is the generalized duality mapping from X into 2^{X^*} defined by

$$J_q(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in X$.

Let D be a subset of C and let Q be a mapping of C into D . Then Q is said to be sunny if

$$Q[Q(x) + t(x - Q(x))] = Q(x)$$

whenever $Q(x) + t(x - Q(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q . A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D . The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.2 ([6]) *Let C be a closed convex subset of a smooth Banach space X , let D be a nonempty subset of C and Q be a retraction from C onto D . Then Q is sunny and nonexpansive if and only if*

$$\langle u - Q(u), j(y - Q(u)) \rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Remark 2.3 It is well known that if X is a Hilbert space, then a sunny nonexpansive retraction $Q_C : X \rightarrow C$ is coincident with the metric projection P_C from X onto C .

Lemma 2.4 ([7]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and let T be a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightarrow x$ and $x_n - Tx_n \rightarrow 0$, then x is a fixed point of T .*

Lemma 2.5 ([8, 9]) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions: (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty} \lambda_n = \infty$, (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty} \lambda_n\delta_n < \infty$, (iii) $\gamma_n \geq 0$ ($n \geq 0$), $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.6 ([10, Lemma 3.2]) *Let C be a nonempty closed convex subset of a Banach space E . Suppose that*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C . Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n \rightarrow \infty} T_n y$ for all $y \in C$. Then $\lim_{n \rightarrow \infty} \sup\{\|Tz - T_nz\| : z \in C\} = 0$.

Furthermore, we need the following extension of Lemma 2.5 in Wang and Hu [2] in a 2-uniformly smooth Banach space.

Lemma 2.7 *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X . Let $F : C \rightarrow X$ be a β -Lipschitzian and η -strongly accretive operator with $0 < \eta \leq \sqrt{2}\beta K$ and $0 < t < \eta/2\beta^2 K^2$. Then $S = (I - tF) : C \rightarrow X$ is a contraction with a contraction coefficient $\tau_t = \sqrt{1 - 2t(\eta - t\beta^2 K^2)}$.*

Proof Using Lemma 2.1, we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \|(x - y) - t(Fx - Fy)\|^2 \\ &\leq \|x - y\|^2 - 2t\langle Fx - Fy, j(x - y) \rangle + 2t^2 K^2 \|Fx - Fy\|^2 \\ &\leq \|x - y\|^2 - 2t\eta \|x - y\|^2 + 2t^2 \beta^2 K^2 \|x - y\|^2 \\ &= [1 - 2t(\eta - t\beta^2 K^2)] \|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. From $0 < \eta \leq \sqrt{2}\beta K$ and $0 < t < \eta/2\beta^2 K^2$, we have

$$\|Sx - Sy\| \leq \tau_t \|x - y\|,$$

where $\tau_t = \sqrt{1 - 2t(\eta - t\beta^2 K^2)} \in (0, 1)$. Hence, S is a contraction with a contraction coefficient τ_t . □

3 Main results

We now state and prove the main results of this paper.

Theorem 3.1 *Let C be a nonempty closed convex subset and sunny nonexpansive retract of a uniformly convex and 2-uniformly smooth Banach space X . Let $F : C \rightarrow X$ be a β -Lipschitzian and η -strongly accretive operator with $0 < \eta \leq \sqrt{2}\beta K$. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for each bounded subset B of C . Suppose, in addition, that $\text{Fix}(T) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$, where $T : C \rightarrow C$ is the nonexpansive mapping defined by $Tz = \lim_{n \rightarrow \infty} T_n z$. Let Q_C be a sunny nonexpansive retraction from X onto C . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$ and satisfy the following conditions:*

- (A1) $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (A2) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = Q_C(I - \alpha_n F)x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T_n y_n, \quad n \geq 1. \end{cases} \tag{3.1}$$

Then the sequence $\{x_n\}$ strongly converges to a point $x^* \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$ which solves uniquely the variational inequality

$$\langle Fx^*, j(x^* - u) \rangle \leq 0, \quad u \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n).$$

Proof We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded. From $\lim_{n \rightarrow \infty} \alpha_n = 0$, we may assume, without loss of generality, that $0 < \alpha_n \leq \eta/2\beta^2 K^2 - \epsilon$ for all n , where ϵ is an arbitrarily small positive number. In fact, let $u \in \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$, from (3.1) and using Lemma 2.7, we have

$$\begin{aligned} \|y_n - u\| &= \|Q_C(I - \alpha_n F)x_n - Q_C u\| \\ &\leq \|(I - \alpha_n F)x_n - (I - \alpha_n F)u - \alpha_n F u\| \\ &\leq \tau_{\alpha_n} \|x_n - u\| + \alpha_n \|F u\|, \end{aligned} \tag{3.2}$$

where $\tau_{\alpha_n} = \sqrt{1 - 2\alpha_n(\eta - \alpha_n\beta^2 K^2)} \in (0, 1)$. Then from (3.1) and (3.2), we obtain

$$\begin{aligned} \|x_{n+1} - u\| &= \|(1 - \beta_n)(x_n - u) + \beta_n(T_n y_n - u)\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n\|y_n - u\| \\ &\leq (1 - \beta_n)\|x_n - u\| + \beta_n(\tau_{\alpha_n}\|x_n - u\| + \alpha_n\|F u\|) \\ &\leq [1 - \beta_n(1 - \tau_{\alpha_n})]\|x_n - u\| + \beta_n\alpha_n\|F u\| \\ &\leq \max\left\{\|x_n - u\|, \frac{\alpha_n\|F u\|}{1 - \tau_{\alpha_n}}\right\}. \end{aligned}$$

Observe that

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{1 - \tau_{\alpha_n}} = \lim_{t \rightarrow 0^+} \frac{t}{1 - \tau_t} = \frac{1}{\eta}.$$

Thus, we have $\alpha_n/(1 - \tau_{\alpha_n})$ is continuous, $\forall \alpha_n \in [0, \eta/2\beta^2K^2 - \epsilon]$. Therefore, we obtain $M_1 = \sup\{\frac{\alpha_n}{1-\tau_{\alpha_n}} : 0 < \alpha_n \leq \frac{\eta}{2\beta^2K^2} - \epsilon\} < +\infty$. By induction, we have

$$\|x_n - u\| \leq \max\{\|x_1 - u\|, M_1\|Fu\|\}.$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{T_n y_n\}$ and $\{Fx_n\}$ are bounded. Without loss of generality, we may assume that $\{x_n\}$, $\{y_n\}$, $\{T_n y_n\}$ and $\{Fx_n\} \subset B$, where B is a bounded set of C .

Step 2. We claim that $\lim_{n \rightarrow \infty} \|y_n - T_n y_n\| = 0$. Using the same method as in Step 2 of the proof in [2, Theorem 3.5], we have $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Observe that

$$\begin{aligned} \|y_n - T_n y_n\| &\leq \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - T_n y_n\| \\ &\leq \alpha_n \|Fx_n\| + \|x_n - x_{n+1}\| + (1 - \beta_n) \|x_n - T_n y_n\| \\ &\leq \alpha_n \|Fx_n\| + \|x_n - x_{n+1}\| + (1 - \beta_n) (\|x_n - y_n\| + \|y_n - T_n y_n\|) \\ &\leq 2\alpha_n \|Fx_n\| + \|x_n - x_{n+1}\| + (1 - \beta_n) \|y_n - T_n y_n\|, \end{aligned}$$

that is,

$$\|y_n - T_n y_n\| \leq \frac{1}{\beta_n} (2\alpha_n \|Fx_n\| + \|x_{n+1} - x_n\|) \rightarrow 0 \quad (n \rightarrow \infty).$$

Step 3. We claim that $\lim_{n \rightarrow \infty} \|y_n - T y_n\| = 0$. Observe that

$$\begin{aligned} \|y_n - T y_n\| &\leq \|y_n - T_n y_n\| + \|T_n y_n - T y_n\| \\ &\leq \|y_n - T_n y_n\| + \sup\{\|T_n z - T z\| : z \in B\}. \end{aligned}$$

Hence, from Step 2 and using Lemma 2.6, we have

$$\lim_{n \rightarrow \infty} \|y_n - T y_n\| = 0.$$

Step 4. We claim that $\limsup_{n \rightarrow \infty} \langle Fx^*, j(x^* - y_n) \rangle \leq 0$, where $x^* = \lim_{t \rightarrow 0^+} x_t$ and x_t is defined by $x_t = Q_C(I - tF)Tx_t$.

From $t \rightarrow 0^+$, we may assume, without loss of generality, that $t \in (0, \eta/2\beta^2K^2 - \epsilon]$. Using Lemma 2.7, it is easy to see that $Q_C(I - tF)T$ is a contraction. Thus, x_t is well defined. Next, we show that x^* is well defined. Let $u \in \text{Fix}(T)$, using Lemma 2.7, we have

$$\begin{aligned} \|x_t - u\| &= \|Q_C(I - tF)Tx_t - Q_C Tu\| \\ &\leq \|(I - tF)Tx_t - (I - tF)Tu - tFTu\| \\ &\leq \|(I - tF)Tx_t - (I - tF)Tu\| + t\|Fu\| \\ &\leq \tau_t \|x_t - u\| + t\|Fu\|, \end{aligned}$$

that is,

$$\|x_t - u\| \leq \frac{t}{1 - \tau_t} \|Fu\| \leq M_1 \|Fu\|.$$

Thus, we have $\{x_t\}$ is bounded and so is $\{FTx_t\}$. On the other hand, we have

$$\begin{aligned} \|x_t - Tx_t\| &= \|Q_C(I - tF)Tx_t - Q_C Tx_t\| \\ &\leq \|(I - tF)Tx_t - Tx_t\| = t\|FTx_t\| \rightarrow 0 \quad (t \rightarrow 0^+). \end{aligned} \tag{3.3}$$

Assume that $\{t_n\} \in (0, \eta/2\beta^2K^2 - \epsilon]$ such that $t_n \rightarrow 0^+$ as $n \rightarrow \infty$. Put $\tilde{x}_n := x_{t_n}$. It follows from (3.3) that $\|\tilde{x}_n - T\tilde{x}_n\| \rightarrow 0$ ($n \rightarrow \infty$). Since $\{x_t\}$ is bounded, without loss of generality, we may assume that $\tilde{x}_n \rightharpoonup x^*$. We can use Lemma 2.4 to get $x^* \in \text{Fix}(T)$. Therefore, using Lemma 2.2 and Lemma 2.7, we have

$$\begin{aligned} \|x_t - x^*\|^2 &= \langle x_t - x^*, j(x_t - x^*) \rangle \\ &= \langle x_t - (I - tF)Tx_t, j(x_t - x^*) \rangle + \langle (I - tF)Tx_t - x^*, j(x_t - x^*) \rangle \\ &\leq \langle (I - tF)Tx_t - x^*, j(x_t - x^*) \rangle \\ &\leq \langle (I - tF)Tx_t - (I - tF)Tx^*, j(x_t - x^*) \rangle + t\langle Fx^*, j(x^* - x_t) \rangle \\ &\leq \tau_t \|x_t - x^*\|^2 + t\langle Fx^*, j(x^* - x_t) \rangle, \end{aligned}$$

that is,

$$\|x_t - x^*\|^2 \leq \frac{t}{1 - \tau_t} \langle Fx^*, j(x^* - x_t) \rangle \leq M_1 \langle Fx^*, j(x^* - x_t) \rangle.$$

In particular,

$$\|\tilde{x}_n - x^*\|^2 \leq M_1 \langle Fx^*, j(x^* - \tilde{x}_n) \rangle.$$

Consequently, the weak convergence of $\{\tilde{x}_n\}$ to x^* actually implies that $x_n \rightarrow x^*$. Therefore, $x^* = \lim_{t \rightarrow 0^+} x_t$ is well defined.

Since y_n is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which converges weakly to ω . From Step 3 and using Lemma 2.4, we have $\omega \in \text{Fix}(T)$. Observe that

$$\limsup_{n \rightarrow \infty} \langle Fx^*, j(x^* - y_n) \rangle = \lim_{k \rightarrow \infty} \langle Fx^*, j(x^* - y_{n_k}) \rangle = \langle Fx^*, j(x^* - \omega) \rangle. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned} x_t &= Q_C(I - tF)Tx_t - (I - tF)Tx_t - (I - tF)x_t + (I - tF)Tx_t + x_t - tFx_t \\ \Rightarrow Fx_t &= \frac{1}{t} [Q_C(I - tF)Tx_t - (I - tF)Tx_t - (I - tF)x_t + (I - tF)Tx_t]. \end{aligned}$$

Therefore, for $\omega \in \text{Fix}(T)$, we can use Lemma 2.2 to get

$$\begin{aligned} \langle Fx_t, j(x_t - \omega) \rangle &= \frac{1}{t} \langle Q_C(I - tF)Tx_t - (I - tF)Tx_t, j(x_t - \omega) \rangle \\ &\quad - \frac{1}{t} \langle (I - tF)x_t - (I - tF)Tx_t, j(x_t - \omega) \rangle \\ &\leq -\frac{1}{t} \langle x_t - Tx_t, j(x_t - \omega) \rangle + \langle Fx_t - FTx_t, j(x_t - \omega) \rangle \end{aligned}$$

$$\begin{aligned} &\leq -\frac{1}{t} \langle (I - T)x_t - (I - T)\omega, j(x_t - \omega) \rangle + \beta \|x_t - Tx_t\| \|x_t - \omega\| \\ &\leq \beta M_2 \|x_t - Tx_t\|, \end{aligned} \tag{3.5}$$

where $M_2 = \sup\{\|x_t - \omega\|, t \in (0, \eta/2\beta^2 K^2 - \epsilon]\}$. Now replacing t in (3.5) with t_n and letting $n \rightarrow \infty$, we have

$$\langle Fx^*, j(x^* - \omega) \rangle \leq 0. \tag{3.6}$$

From (3.4) and (3.6), we obtain $\limsup_{n \rightarrow \infty} \langle Fx^*, j(x^* - y_n) \rangle \leq 0$.

Step 5. We claim that $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$. From (3.1) and using Lemma 2.2, we have

$$\langle Q_C(I - \alpha_n F)x_n - (I - \alpha_n F)x_n, j(y_n - x^*) \rangle \leq 0.$$

Observe that

$$\begin{aligned} \|y_n - x^*\|^2 &= \langle Q_C(I - \alpha_n F)x_n - x^*, j(y_n - x^*) \rangle \\ &= \langle Q_C(I - \alpha_n F)x_n - (I - \alpha_n F)x_n, j(y_n - x^*) \rangle + \langle (I - \alpha_n F)x_n - x^*, j(y_n - x^*) \rangle \\ &\leq \langle (I - \alpha_n F)x_n - x^*, j(y_n - x^*) \rangle \\ &\leq \langle (I - \alpha_n F)x_n - (I - \alpha_n F)x^*, j(y_n - x^*) \rangle + \alpha_n \langle Fx^*, j(x^* - y_n) \rangle \\ &\leq \|(I - \alpha_n F)x_n - (I - \alpha_n F)x^*\| \|y_n - x^*\| + \alpha_n \langle Fx^*, j(x^* - y_n) \rangle \\ &\leq \tau_{\alpha_n} \|x_n - x^*\| \|y_n - x^*\| + \alpha_n \langle Fx^*, j(x^* - y_n) \rangle \\ &\leq \frac{\tau_{\alpha_n}^2}{2} \|x_n - x^*\|^2 + \frac{1}{2} \|y_n - x^*\|^2 + \alpha_n \langle Fx^*, j(x^* - y_n) \rangle, \end{aligned}$$

that is,

$$\|y_n - x^*\|^2 \leq \tau_{\alpha_n} \|x_n - x^*\|^2 + 2\alpha_n \langle Fx^*, j(x^* - y_n) \rangle. \tag{3.7}$$

By (3.1) and (3.7), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|T_n y_n - x^*\|^2 \\ &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|y_n - x^*\|^2 \\ &\leq (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n [\tau_{\alpha_n} \|x_n - x^*\|^2 + 2\alpha_n \langle Fx^*, j(x^* - y_n) \rangle] \\ &\leq [1 - \beta_n(1 - \tau_{\alpha_n})] \|x_n - x^*\|^2 + 2M_1 \beta_n (1 - \tau_{\alpha_n}) \langle Fx^*, j(x^* - y_n) \rangle \\ &= (1 - \lambda_n) \|x_n - x^*\|^2 + \lambda_n \delta_n, \end{aligned}$$

where $\lambda_n = \beta_n(1 - \tau_{\alpha_n})$, $\delta_n = 2M_1 \langle Fx^*, j(x^* - y_n) \rangle$. It is easy to see that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\limsup_{n \rightarrow \infty} \delta_n \leq 0$. Hence, by Lemma 2.5, the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$.

Step 6. We claim that x^* is a unique solution of the variational inequality $\langle Fx^*, j(x^* - u) \rangle \leq 0$, $u \in \bigcap_{n=1}^{\infty} F(T_n)$.

From Step 4, we have shown that x^* is a solution of the variational inequality $\langle Fx^*, j(x^* - u) \rangle \leq 0$, $u \in \bigcap_{n=1}^{\infty} F(T_n)$. Without loss of generality, we assume that $\tilde{x} \in \text{Fix}(T)$ is also a solution of the variational inequality. Therefore, we have

$$\langle Fx^*, j(x^* - \tilde{x}) \rangle \leq 0, \quad (3.8)$$

and

$$\langle F\tilde{x}, j(\tilde{x} - x^*) \rangle \leq 0. \quad (3.9)$$

Adding up (3.8) and (3.9), we get

$$\langle Fx^* - F\tilde{x}, j(x^* - \tilde{x}) \rangle \leq 0.$$

The strong accretivity of F implies that $x^* = \tilde{x}$ and the uniqueness is proved. \square

Remark 3.2 Obviously, Theorem 3.1 extends the results of Yao *et al.* [1] and Wang and Hu [2] to the more general uniformly convex and 2-uniformly smooth Banach space.

Competing interests

The author declares that she has no competing interests.

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