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Strong convergence of the hybrid method for a finite family of nonspreading mappings and variational inequality problems

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Abstract

In this paper, we prove a strong convergence theorem by the hybrid method for finding a common element of the set of fixed points of a finite family of nonspreading mappings and the set of solutions of a finite family of variational inequality problems.

Keywords: nonspreading mapping; quasi-nonexpansive mapping; *S*-mapping

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Then a mapping $T: C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Recall that the mapping $T: C \to C$ is said to be *quasi-nonexpansive* if $\|Tx - p\| \leq \|x - p\|$, $\forall x \in C$ and $\forall p \in F(T)$, where $F(T)$ denotes the set of fixed points of T. In 2008, Kohsaka and Takahashi [1] introduced the mapping T called the nonspreading mapping in Hilbert spaces H and defined it as follows: $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2$, $\forall x, y \in C$.

Let $A: C \to H$. The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, A)$.

The variational inequality has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences; see, *e.g.*, [2–5].

A mapping A of C into H is called *inverse-strongly monotone* (see [6]) if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. Throughout this paper, we will use the following notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega(x_n) = \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

In 2008, Takahashi, Takeuchi and Kubota [7] proved the following strong convergence theorems by using the hybrid method for nonexpansive mappings in Hilbert spaces.

Theorem 1.1 *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 \in P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows*:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

In 2009, Iemoto and Takahashi [8] proved the convergence theorem of nonexpansive and nonspreading mappings as follows.

Theorem 1.2 *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself, and let T be a nonexpansive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows.*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n S x_n + (1-\beta_n)T x_n) \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then the following hold:

(i) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}$ converges weakly to $v \in F(S)$.*

(ii) *If $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}$ converges weakly to $v \in F(T)$.*

(iii) *If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}$ converges weakly to $v \in F(S) \cap F(T)$.*

Inspired and motivated by these facts and the research in this direction, we prove the strong convergence theorem by the hybrid method for finding a common element of the set of fixed points of a finite family of nonspreading mappings and the set of solutions of a finite family of variational inequality problems.

2 Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let C be a closed convex subset of a real Hilbert space H, let P_C be the metric projection of H onto C, *i.e.*, for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1 (See [9]) *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if the following inequality holds*:

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2 (See [8]) *Let C be a nonempty closed convex subset of H. Then a mapping $S : C \to C$ is nonspreading if and only if*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy\rangle$$

for all $x, y \in C$.

Example 2.3 Let $\mathcal{R}$ denote the reals with the usual norm. Let $T : \mathcal{R} \to \mathcal{R}$ be defined by

$$Tx = \begin{cases} x - 1 & \text{if } x \in (-\infty, 0], \\ -(x + 1) & \text{if } x \in (0, \infty) \end{cases}$$

for all $x \in \mathcal{R}$.

To see that T is a nonspreading mapping, if $x, y \in (0, \infty)$, then we have $Tx = -(x + 1)$ and $Ty = -(y + 1)$. From the definition of the mapping T, we have

$$\begin{aligned} |Tx - Ty|^2 &= \bigl|-(x + 1) - \bigl(-(y + 1)\bigr)\bigr|^2 \\ &= |y - x|^2 = |x - y|^2 \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle &= 2\langle x + x + 1, y + y + 1\rangle \\ &= 2\langle 2x + 1, 2y + 1\rangle \\ &= 2(2x + 1)(2y + 1) > 0 \quad (\text{since } x, y > 0). \end{aligned}$$

The above implies that

$$|Tx - Ty|^2 = |x - y|^2 < |x - y|^2 + 2\langle x - Tx, y - Ty\rangle.$$

For every $x, y \in (-\infty, 0]$, we have $Tx = x - 1$ and $Ty = y - 1$. From the definition of T, we have

$$\begin{aligned} |Tx - Ty|^2 &= \bigl|x - 1 - (y - 1)\bigr|^2 \\ &= |x - y|^2, \end{aligned}$$

and

$$2\langle x - Tx, y - Ty\rangle = 2\bigl\langle x - (x - 1), y - (y - 1)\bigr\rangle = 2.$$

From above, we have

$$|Tx - Ty|^2 = |x - y|^2 < |x - y|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Finally, for every $x \in (-\infty, 0]$ and $y \in (0, \infty)$, we have $Tx = x - 1$ and $Ty = -(y+1)$. From the definition of T, we have

$$|Tx - Ty|^2 = |x - 1 + y + 1|^2 = |x + y|^2,$$

$$\begin{aligned} |x - y|^2 &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy + y^2 - 4xy \\ &\geq x^2 + 2xy + y^2 \quad (\text{since } -4xy \geq 0) \\ &= (x + y)^2 \end{aligned}$$

and

$$\begin{aligned} 2\langle x - Tx, y - Ty\rangle &= 2\bigl\langle x - (x - 1), y + (y + 1)\bigr\rangle \\ &= 2\langle 1, 2y + 1\rangle \\ &= 2(2y + 1) > 0 \quad (\text{since } y > 0). \end{aligned}$$

From above, we have

$$\begin{aligned} |Tx - Ty|^2 &= |x + y|^2 = (x + y)^2 \\ &\leq |x - y|^2 \\ &< |x - y|^2 + 2\langle x - Tx, y - Ty\rangle. \end{aligned}$$

Hence, for all $x, y \in \mathcal{R}$, we have

$$|Tx - Ty|^2 < |x - y|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Then T is a nonspreading mapping.

Lemma 2.4 (See [1]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

Lemma 2.5 (See [9]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \quad \Leftrightarrow \quad u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.6 (See [10]) *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Lemma 2.7 (See [11]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.8 (See [12]) *Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n \in \mathbb{N},$$

then $x_n \to q$, as $n \to \infty$.

In 2009, Kangtunyakarn and Suantai [13] introduced an S-mapping generated by $T_1, \dots, T_N$ and $\lambda_1, \dots, \lambda_N$ as follows.

Definition 2.1 Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
&U_0 = I, \\
&U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
&U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
&U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\quad \vdots
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
&S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned} \tag{2.2}$$

This mapping is called an *S-mapping* generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

The next lemma is very useful for our consideration.

Lemma 2.9 *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \dots, N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by $T_1, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.*

Proof It easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(S)$. Let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^N F(T_i)$. Since $\{T_i\}_{i=1}^N$ is a finite family of nonspreading mappings of C into itself, for every $y \in C$, we have

$$\|T_i y - x^*\|^2 \leq \frac{1}{2}\left(\|T_i y - x^*\|^2 + \|y - x^*\|^2\right). \tag{2.3}$$

This implies that

$$\|T_i y - x^*\|^2 \leq \|y - x^*\|^2, \quad \forall y \in C \text{ and } i = 1, 2, \dots, N. \tag{2.4}$$

From the definition of S and (2.4),

$$
\begin{aligned}
\|Sx_0 - x^*\|^2 &= \|\alpha_1^N T_N U_{N-1}x_0 + \alpha_2^N U_{N-1}x_0 + \alpha_3^N x_0 - x^*\|^2 \\
&= \|\alpha_1^N(T_N U_{N-1}x_0 - x^*) + \alpha_2^N(U_{N-1}x_0 - x^*) + \alpha_3^N(x_0 - x^*)\|^2 \\
&\le \alpha_1^N\|T_N U_{N-1}x_0 - x^*\|^2 + \alpha_2^N\|U_{N-1}x_0 - x^*\|^2 + \alpha_3^N\|x_0 - x^*\|^2 \\
&\le (1-\alpha_3^N)\|U_{N-1}x_0 - x^*\|^2 + \alpha_3^N\|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)\|\alpha_1^{N-1}(T_{N-1}U_{N-2}x_0 - x^*) + \alpha_2^{N-1}(U_{N-2}x_0 - x^*) \\
&\quad + \alpha_3^{N-1}(x_0 - x^*)\|^2 + \alpha_3^N\|x_0 - x^*\|^2 \\
&\le (1-\alpha_3^N)(\alpha_1^{N-1}\|T_{N-1}U_{N-2}x_0 - x^*\|^2 + \alpha_2^{N-1}\|U_{N-2}x_0 - x^*\|^2 \\
&\quad + \alpha_3^{N-1}\|x_0 - x^*\|^2) + \alpha_3^N\|x_0 - x^*\|^2 \\
&\le (1-\alpha_3^N)((1-\alpha_3^{N-1})\|U_{N-2}x_0 - x^*\|^2 + \alpha_3^{N-1}\|x_0 - x^*\|^2) \\
&\quad + \alpha_3^N\|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)(1-\alpha_3^{N-1})\|U_{N-2}x_0 - x^*\|^2 + \alpha_3^{N-1}(1-\alpha_3^N)\|x_0 - x^*\|^2 \\
&\quad + \alpha_3^N\|x_0 - x^*\|^2 \\
&= \prod_{j=N-1}^{N}(1-\alpha_3^j)\|U_{N-2}x_0 - x^*\|^2 + \left(1 - \prod_{j=N-1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&\vdots \\
&\le \prod_{j=3}^{N}(1-\alpha_3^j)\|U_2x_0 - x^*\|^2 + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&\le \prod_{j=2}^{N}(1-\alpha_3^j)\|U_1x_0 - x^*\|^2 + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=2}^{N}(1-\alpha_3^j)\|\alpha_1^1(T_1x_0 - x^*) + (1-\alpha_1^1)(x_0 - x^*)\|^2 \\
&\quad + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=2}^{N}(1-\alpha_3^j)(\alpha_1^1\|T_1x_0 - x^*\|^2 + (1-\alpha_1^1)\|x_0 - x^*\|^2 \\
&\quad - \alpha_1^1(1-\alpha_1^1)\|T_1x_0 - x_0\|) + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&\le \prod_{j=2}^{N}(1-\alpha_3^j)(\|x_0 - x^*\|^2 - \alpha_1^1(1-\alpha_1^1)\|T_1x_0 - x_0\|^2) \\
&\quad + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2.
\end{aligned}
\tag{2.5}
$$

From (2.5), we have

$$\begin{aligned}\|x_0 - x^*\|^2 &\le \prod_{j=2}^{N}(1-\alpha_3^j)\bigl(\|x_0 - x^*\|^2 - \alpha_1^1(1-\alpha_1^1)\|T_1x_0 - x_0\|^2\bigr) \\ &\quad + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2,\end{aligned}$$

which implies that

$$\|x_0 - x^*\|^2 \le \|x_0 - x^*\|^2 - \alpha_1^1(1-\alpha_1^1)\|T_1x_0 - x_0\|^2. \tag{2.6}$$

Since $\alpha_1^j \in (0,1)$ for all $j = 1, 2, \ldots, N-1$ and (2.6), we have $x_0 \in F(T_1)$. From $x_0 = T_1x_0$ and the definition of S, we have

$$U_1x_0 = \alpha_1^1 T_1x_0 + \alpha_2^1 x_0 + \alpha_3^1 x_0 = x_0.$$

From (2.5) and $x_0 \in F(U_1)$, we have

$$\begin{aligned}\|x_0 - x^*\|^2 &\le \prod_{j=3}^{N}(1-\alpha_3^j)\|U_2x_0 - x^*\|^2 + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\ &= \prod_{j=3}^{N}(1-\alpha_3^j)\|\alpha_1^2 T_2U_1x_0 + \alpha_2^2 U_1x_0 + \alpha_3^2 x_0 - x^*\|^2 \\ &\quad + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\ &= \prod_{j=3}^{N}(1-\alpha_3^j)\|\alpha_1^2(T_2x_0 - x^*) + (1-\alpha_1^2)(x_0 - x^*)\|^2 \\ &\quad + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\ &= \prod_{j=3}^{N}(1-\alpha_3^j)\bigl(\alpha_1^2\|T_2x_0 - x^*\|^2 + (1-\alpha_1^2)\|x_0 - x^*\|^2 \\ &\quad - \alpha_1^2(1-\alpha_1^2)\|T_2x_0 - x_0\|^2\bigr) \\ &\quad + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\ &\le \prod_{j=3}^{N}(1-\alpha_3^j)\bigl(\|x_0 - x^*\|^2 - \alpha_1^2(1-\alpha_1^2)\|T_2x_0 - x_0\|^2\bigr) \\ &\quad + \left(1 - \prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2,\end{aligned}$$

which implies that

$$\|x_0 - x^*\|^2 \leq \|x_0 - x^*\|^2 - \alpha_1^2(1-\alpha_1^2)\|T_2x_0 - x_0\|^2. \tag{2.7}$$

Since $\alpha_1^j \in (0,1)$ for all $j = 1,2,\dots,N-1$ and (2.7), we have $x_0 \in F(T_2)$. From the definition of S and $x_0 = T_2x_0$, we have

$$U_2x_0 = \alpha_1^2T_2U_1x_0 + \alpha_2^2U_1x_0 + \alpha_3^2x_0 = x_0.$$

By continuing in this way, we can show that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1,2,\dots,N-1$.

Finally, we shall show that $x_0 \in F(T_N)$.

Since

$$\begin{aligned} 0 = Sx_0 - x_0 &= \alpha_1^NT_NU_{N-1}x_0 + \alpha_2^NU_{N-1}x_0 + \alpha_3^Nx_0 - x_0 \\ &= \alpha_1^N(T_Nx_0 - x_0), \end{aligned}$$

and $\alpha_1^N \in (0,1]$, we obtain $T_Nx_0 = x_0$ so that $x_0 \in F(T_N)$. Then we have $x_0 \in \bigcap_{i=1}^N F(T_i)$. Hence, $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$.

Next, we show that S is a quasi-nonexpansive mapping. Let $x \in C$ and $y \in F(S)$. From (2.5), we can imply that

$$\begin{aligned} \|Sx - y\|^2 &\leq \prod_{j=2}^N(1-\alpha_3^j)(\|x-y\|^2 - \alpha_1^1(1-\alpha_1^1)\|T_1x - x\|) \\ &\quad + \left(1 - \prod_{j=2}^N(1-\alpha_3^j)\right)\|x-y\|^2 \\ &\leq \|x-y\|^2. \end{aligned}$$

Then we have the S-mapping is quasi-nonexpansive. □

Example 2.10 Let $T_1 : [-1,1] \to [-1,1]$ be a mapping defined by

$$T_1x = \begin{cases} \frac{x+1}{2} & \text{if } x \in (0,1], \\ \frac{-x+1}{2} & \text{if } x \in [-1,0] \end{cases}$$

for all $x \in [-1,1]$.

Let $T_2 : [-1,1] \to [-1,1]$ be a mapping defined by

$$T_2x = \begin{cases} \frac{x+2}{3} & \text{if } x \in (0,1], \\ \frac{-x+2}{3} & \text{if } x \in [-1,0] \end{cases}$$

for all $x \in [-1,1]$.

To see that T_1 is a nonspreading mapping, observe that if $x, y \in (0,1]$, we have $T_1 x = \frac{x+1}{2}$ and $T_1 y = \frac{y+1}{2}$. Then we have

$$\begin{aligned}|T_1 x - T_1 y|^2 &= \left|\frac{x+1}{2} - \frac{y+1}{2}\right|^2 \\ &= \frac{1}{4}|x-y|^2\end{aligned}$$

and

$$\begin{aligned}2\langle x - T_1 x, y - T_1 y\rangle &= 2\left\langle x - \left(\frac{x+1}{2}\right), y - \left(\frac{y+1}{2}\right)\right\rangle \\ &= 2\left\langle \frac{x-1}{2}, \frac{y-1}{2}\right\rangle \\ &= \frac{1}{2}(x-1)(y-1) \\ &\geq 0 \quad \bigl(\text{since } x \leq 1, y \leq 1, \text{ then } (x-1)(y-1) \geq 0\bigr).\end{aligned}$$

From above, we have

$$\begin{aligned}|x-y|^2 + 2\langle x - T_1 x, y - T_1 y\rangle &\geq |x-y|^2 \\ &\geq \frac{1}{4}|x-y|^2 \\ &= |T_1 x - T_1 y|^2.\end{aligned}$$

For every $x, y \in [-1, 0]$, we have $T_1 x = \frac{-x+1}{2}$ and $T_1 y = \frac{-y+1}{2}$. From the definition of T_1, we have

$$\begin{aligned}|T_1 x - T_1 y|^2 &= \left|\frac{-x+1}{2} - \left(\frac{-y+1}{2}\right)\right|^2 \\ &= \left|\frac{y-x}{2}\right|^2 \\ &= \frac{1}{4}|x-y|^2\end{aligned}$$

and

$$\begin{aligned}2\langle x - T_1 x, y - T_1 y\rangle &= 2\left\langle x - \left(\frac{1-x}{2}\right), y - \left(\frac{1-y}{2}\right)\right\rangle \\ &= 2\left\langle \frac{3x-1}{2}, \frac{3y-1}{2}\right\rangle \\ &= \frac{1}{2}(3x-1)(3y-1) \\ &= \frac{1}{2}\bigl(3x(3y-1) - (3y-1)\bigr) \\ &= \frac{1}{2}(9xy - 3x - 3y + 1) \\ &> 0 \quad (\text{since } -1 \leq x, y \leq 0, \text{ then } 9xy, -3x, -3y \geq 0).\end{aligned}$$

From above, we have

$$\begin{aligned}|x-y|^2 + 2\langle x - T_1x, y - T_1y\rangle &> |x-y|^2 \\ &\geq \frac{1}{4}|x-y|^2 \\ &= |T_1x - T_1y|^2.\end{aligned}$$

Finally, for every $x \in (0,1]$ and $y \in [-1,0]$, we have $T_1x = \frac{x+1}{2}$ and $T_1y = \frac{-y+1}{2}$. From the definition of T_1, we have

$$\begin{aligned}|T_1x - T_1y|^2 &= \left|\frac{x+1}{2} - \frac{-y+1}{2}\right|^2 \\ &= \frac{1}{4}|x+y|^2\end{aligned}$$

and

$$\begin{aligned}2\langle x - T_1x, y - T_1y\rangle &= 2\left\langle x - \left(\frac{x+1}{2}\right), y - \left(\frac{-y+1}{2}\right)\right\rangle \\ &= 2\left\langle \frac{x-1}{2}, \frac{3y-1}{2}\right\rangle \\ &= \frac{1}{2}(x-1)(3y-1) \\ &= \frac{1}{2}\bigl(x(3y-1) - (3y-1)\bigr) \\ &= \frac{1}{2}(3xy - x - 3y + 1) \\ &= \frac{1}{2}\bigl(3y(x-1) + (1-x)\bigr) \\ &\geq 0 \quad \bigl(\text{since } 0 < x \leq 1 \text{ and } -1 \leq y \leq 0, \text{ then } 3y(x-1), (1-x) \geq 0\bigr).\end{aligned}$$

From above, we have

$$\begin{aligned}|x-y|^2 + 2\langle x - T_1x, y - T_1y\rangle &\geq |x-y|^2 \\ &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy + y^2 - 4xy \\ &\geq x^2 + 2xy + y^2 \quad (\text{since } -4xy \geq 0) \\ &= (x+y)^2 \\ &\geq \frac{1}{4}(x+y)^2 \\ &= |T_1x - T_1y|^2.\end{aligned}$$

Then for all $x, y \in [-1,1]$, we have

$$|T_1x - T_1y|^2 \leq |x-y|^2 + \langle x - T_1x, y - T_1y\rangle.$$

Hence, we have T_1 is a nonspreading mapping.

Next, we show that T_2 is a nonspreading mapping. Let $x, y \in (0, 1]$, then we have $T_2x = \frac{x+2}{3}$ and $T_2y = \frac{y+2}{3}$. From the definition of T_2, we have

$$\begin{aligned}
|T_2x - T_2y|^2 &= \left|\frac{x+2}{3} - \frac{y+2}{3}\right|^2 \\
&= \frac{1}{9}|x-y|^2
\end{aligned}$$

and

$$\begin{aligned}
2\langle x - T_2x, y - T_2y\rangle &= 2\left\langle x - \left(\frac{x+2}{3}\right), y - \left(\frac{y+2}{3}\right)\right\rangle \\
&= 2\left\langle \frac{2x-2}{3}, \frac{2y-2}{3}\right\rangle \\
&= \frac{8}{9}(x-1)(y-1) \\
&\geq 0 \quad \bigl(\text{since } 0 < x, y \leq 1, \text{ then } (x-1)(y-1) \geq 0\bigr).
\end{aligned}$$

From above, we have

$$\begin{aligned}
|x-y|^2 + 2\langle x - T_2x, y - T_2y\rangle &\geq |x-y|^2 \\
&\geq \frac{1}{9}|x-y|^2 \\
&= |T_2x - T_2y|^2.
\end{aligned}$$

For every $x, y \in [-1, 0]$, we have $T_2x = \frac{2-x}{3}$ and $T_2y = \frac{2-y}{3}$. From the definition of T_2, we have

$$\begin{aligned}
|T_2x - T_2y|^2 &= \left|\frac{2-x}{3} - \frac{2-y}{3}\right|^2 \\
&= \left|\frac{y-x}{3}\right|^2 \\
&= \frac{1}{9}|x-y|^2
\end{aligned}$$

and

$$\begin{aligned}
2\langle x - T_2x, y - T_2y\rangle &= 2\left\langle x - \left(\frac{2-x}{3}\right), y - \left(\frac{2-y}{3}\right)\right\rangle \\
&= 2\left\langle \frac{4x-2}{3}, \frac{4y-2}{3}\right\rangle \\
&= \frac{8}{9}(2x-1)(2y-1) \\
&= \frac{8}{9}\bigl(2x(2y-1) - (2y-1)\bigr) \\
&= \frac{8}{9}(4xy - 2x - 2y + 1) \\
&> 0 \quad (\text{since } -1 \leq x, y \leq 0, \text{ then } 4xy, -2x, -2y \geq 0).
\end{aligned}$$

From above, we have

$$\begin{aligned}
|x-y|^2+2\langle x-T_2x,y-T_2y\rangle &> |x-y|^2 \\
&\geq \frac{1}{9}|x-y|^2 \\
&= |T_2x-T_2y|^2.
\end{aligned}$$

Finally, for every $x \in (0,1]$ and $y \in [-1,0]$, we have $T_2x = \frac{x+2}{3}$ and $T_2y = \frac{2-y}{3}$. From the definition of T_2, we have

$$\begin{aligned}
|T_2x-T_2y|^2 &= \left|\frac{x+2}{3}-\frac{2-y}{3}\right|^2 \\
&= \frac{1}{9}|x+y|^2
\end{aligned}$$

and

$$\begin{aligned}
2\langle x-T_2x,y-T_2y\rangle &= 2\left\langle x-\left(\frac{x+2}{3}\right),y-\left(\frac{2-y}{3}\right)\right\rangle \\
&= 2\left\langle\frac{2x-2}{3},\frac{4y-2}{3}\right\rangle \\
&= \frac{8}{9}(x-1)(2y-1) \\
&= \frac{8}{9}\bigl(x(2y-1)-(2y-1)\bigr) \\
&= \frac{8}{9}(2xy-x-2y+1) \\
&= \frac{8}{9}\bigl(2y(x-1)+(1-x)\bigr) \\
&\geq 0 \quad \bigl(\text{since } 0<x\leq 1 \text{ and } -1\leq y\leq 0, \text{ then } 2y(x-1),(1-x)\geq 0\bigr).
\end{aligned}$$

From above, we have

$$\begin{aligned}
|x-y|^2+2\langle x-T_2x,y-T_2y\rangle &\geq |x-y|^2 \\
&= x^2-2xy+y^2 \\
&= x^2+2xy+y^2-4xy \\
&\geq (x+y)^2 \quad (\text{since } -4xy\geq 0) \\
&\geq \frac{1}{9}|x+y|^2 \\
&= |T_2x-T_2y|^2.
\end{aligned}$$

Then for every $x,y \in [-1,1]$, we have

$$|T_2x-T_2y|^2 \leq |x-y|^2+2\langle x-T_2x,y-T_2y\rangle.$$

Hence, we have T_2 is a nonspreading mapping. Observe that $1 \in F(T_1) \cap F(T_2)$. Let the mapping $S : [-1,1] \to [-1,1]$ be the S-mapping generated by T_1, T_2 and α_1, α_2, where $\alpha_1 = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$ and $(\frac{4}{15}, \frac{5}{15}, \frac{6}{15})$. From Lemma 2.9, we have $1 \in F(S)$.

3 Main result

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H. For every $i = 1,2,\ldots,N$, let $A_i : C \to H$ be an α_i-inverse strongly monotone mapping, and let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings with $\mathfrak{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N VI(C,A_i) \neq \emptyset$. For every $i = 1,2,\ldots,N$, define the mapping $G_i : C \to C$ by $G_i x = P_C(I - \lambda A_i)x$ $\forall x \in C$ and $\lambda \in [c,d] \subset (0,2\alpha_i)$. Let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,3,\ldots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$ $\alpha_2^j \in (0,1)$ for all $j = 1,2,\ldots,N$, and let S be the S-mapping generated by $T_1, T_2, \ldots, T_N$ and $\rho_1, \rho_2, \ldots, \rho_N$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C_1 = C$ and*

$$\begin{cases} z_n = \sum_{i=1}^N \delta_n^i G_i x_n, \\ y_n = \alpha_n x_n + \beta_n S x_n + \gamma_n z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n = 1$ and suppose the following conditions hold:

(i) $\lim_{n\to\infty} \delta_n^i = \delta^i \in (0,1), \quad \forall i = 1,2,\ldots,N$ *and* $\sum_{i=1}^N \delta_n^i = 1$,

(ii) $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [a,b] \subset (0,1)$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathfrak{F}} x_1$.

Proof First, we show that $(I - \lambda A_i)$ is a nonexpansive mapping for every $i = 1,2,\ldots,N$. Let $x, y \in C$. Since A is an α_i-inverse strongly monotone and $\lambda < 2\alpha_i$, we have

$$\begin{aligned} \big\|(I - \lambda A_i)x - (I - \lambda A_i)y\big\|^2 &= \big\|x - y - \lambda(A_i x - A_i y)\big\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, A_i x - A_i y\rangle + \lambda^2\|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2 - 2\alpha_i\lambda\|A_i x - A_i y\|^2 + \lambda^2\|A_i x - A_i y\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha_i)\|A_i x - A_i y\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus $(I - \lambda A_i)$ is a nonexpansive mapping for every $i = 1,2,\ldots,N$. Since P_C is a nonexpansive mapping, we have G_i is a nonexpansive mapping for every $i = 1,2,\ldots,N$. From Lemma 2.5, we have

$$F(G_i) = F\big(P_C(I - \lambda A_i)\big) = VI(C,A_i), \quad \forall i = 1,2,\ldots,N. \tag{3.2}$$

From (3.2), $VI(C,A_i)$ is closed and convex. Let $z \in \mathfrak{F}$. From (3.2), we have $z \in F(P_C(I - \lambda A_i))$ for every $i = 1,2,\ldots,N$. By nonexpansiveness of G_i, we have

$$\|z_n - z\| = \left\| \sum_{i=1}^{N} \delta_n^i (G_i x_n - z) \right\| \le \sum_{i=1}^{N} \delta_n^i \|x_n - z\| = \|x_n - z\|. \tag{3.3}$$

Next, we show that C_n is closed and convex for every $n \in \mathbb{N}$. It is obvious that C_n is closed. In fact, we know that for $z \in C_n$,

$$\|y_n - z\| \le \|x_n - z\| \quad \text{is equivalent to} \quad \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \le 0.$$

So, for every $z_1, z_2 \in C_n$ and $t \in (0,1)$, it follows that

$$\begin{aligned}
&\|y_n - x_n\|^2 + 2\bigl\langle y_n - x_n, x_n - \bigl(tz_1 + (1-t)z_2\bigr)\bigr\rangle \\
&\quad = t\bigl(2\langle y_n - x_n, x_n - z_1\rangle + \|y_n - x_n\|^2\bigr) \\
&\qquad + (1-t)\bigl(2\langle y_n - x_n, x_n - z_2\rangle + \|y_n - x_n\|^2\bigr) \\
&\quad \le 0,
\end{aligned}$$

then, we have C_n is convex. Since $VI(C,A_i)$ is closed and convex for every $i = 1,2,\ldots,N$, we have $\bigcap_{i=1}^N VI(C,A_i)$ is closed and convex. From Lemma 2.4, we have $\bigcap_{i=1}^N F(T_i)$ is closed and convex. Hence, we have $\mathfrak{F}$ is closed and convex. This implies that $P_{\mathfrak{F}}$ is well defined. Next, we show that $\mathfrak{F} \subset C_n$ for every $n \in \mathbb{N}$. Let $z \in \mathfrak{F}$, then we have

$$\begin{aligned}
\|y_n - z\| &= \bigl\|\alpha_n(x_n - z) + \beta_n(Sx_n - z) + \gamma_n(z_n - z)\bigr\| \\
&\le \alpha_n\|x_n - z\| + \beta_n\|Sx_n - z\| + \gamma_n\|z_n - z\| \\
&\le \|x_n - z\|.
\end{aligned}$$

It follows that $z \in C_n$. Hence, we have $\mathfrak{F} \subset C_n$ for every $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined. Since $x_n = P_{C_n}x_1$, for every $w \in C_n$, we have

$$\|x_n - x_1\| \le \|w - x_1\|, \quad \forall n \in \mathbb{N}. \tag{3.4}$$

In particular, we have

$$\|x_n - x_1\| \le \|P_{\mathfrak{F}}x_1 - x_1\|. \tag{3.5}$$

By (3.4) we have $\{x_n\}$ is bounded, so are $\{G_i x_n\}$, $\{T_i x_n\}$ for every $i = 1,2,\ldots,N$, $\{z_n\}$, $\{y_n\}$ and $\{Sx_n\}$. Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$ and $x_n = P_{C_n}x_1$, we have

$$\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_n - x_1\|^2 + \|x_n - x_1\|\|x_1 - x_{n+1}\|,
\end{aligned}$$

which implies that

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|.$$

Hence, we have $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&\le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2, \qquad (3.6)
\end{aligned}$$

it implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \qquad (3.7)$$

Since $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|.$$

By (3.7) we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \qquad (3.8)$$

Since

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

by (3.7) and (3.8), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \qquad (3.9)$$

Next, we will show that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \qquad (3.10)$$

For every $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
&\big\|P_C(I - \lambda A_i)x_n - z\big\|^2 \\
&\quad = \big\|P_C(I - \lambda A_i)x_n - P_C(I - \lambda A_i)z\big\|^2 \\
&\quad \le \big\|(I - \lambda A_i)x_n - (I - \lambda A_i)z\big\|^2 \\
&\quad = \big\|x_n - z - \lambda(A_i x_n - A_i z)\big\|^2 \\
&\quad = \|x_n - z\|^2 + \lambda^2 \|A_i x_n - A_i z\|^2 - 2\lambda\langle x_n - z, A_i x_n - A_i z\rangle
\end{aligned}$$

$$\begin{aligned}&\leq \|x_n - z\|^2 + \lambda^2\|A_i x_n - A_i z\|^2 - 2\lambda\alpha_i\|A_i x_n - A_i z\|^2\\&= \|x_n - z\|^2 - \lambda(2\alpha_i - \lambda)\|A_i x_n - A_i z\|^2.\end{aligned} \tag{3.11}$$

From the definition of y_n and (3.11), we have

$$\begin{aligned}\|y_n - z\|^2 &\leq \alpha_n\|x_n - z\|^2 + \beta_n\|Sx_n - z\|^2 + \gamma_n\|z_n - z\|^2\\&\leq \alpha_n\|x_n - z\|^2 + \beta_n\|Sx_n - z\|^2 + \gamma_n\sum_{i=1}^N \delta_n^i\big\|P_C(I - \lambda A_i)x_n - z\big\|^2\\&\leq \alpha_n\|x_n - z\|^2 + \beta_n\|Sx_n - z\|^2\\&\quad + \gamma_n\sum_{i=1}^N \delta_n^i\big(\|x_n - z\|^2 - \lambda(2\alpha_i - \lambda)\|A_i x_n - A_i z\|^2\big)\\&= \alpha_n\|x_n - z\|^2 + \beta_n\|Sx_n - z\|^2 + \gamma_n\|x_n - z\|^2\\&\quad - \gamma_n\sum_{i=1}^N \delta_n^i\lambda(2\alpha_i - \lambda)\|A_i x_n - A_i z\|^2\\&\leq \|x_n - z\|^2 - \gamma_n\sum_{i=1}^N \delta_n^i\lambda(2\alpha_i - \lambda)\|A_i x_n - A_i z\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\sum_{i=1}^N \delta_n^i\lambda(2\alpha_i - \lambda)\|A_i x_n - A_i z\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2\\&\leq \big(\|x_n - z\| + \|y_n - z\|\big)\|y_n - x_n\|.\end{aligned}$$

From conditions (i), (ii) and (3.9), it implies that

$$\lim_{n\to\infty}\|A_i x_n - A_i z\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{3.12}$$

Since

$$\begin{aligned}\big\|P_C(I - \lambda A_i)x_n - z\big\|^2 &\leq \big\langle (I - \lambda A_i)x_n - (I - \lambda A_i)z, P_C(I - \lambda A_i)x_n - z\big\rangle\\&= \frac{1}{2}\Big(\big\|(I - \lambda A_i)x_n - (I - \lambda A_i)z\big\|^2 + \big\|P_C(I - \lambda A_i)x_n - z\big\|^2\\&\quad - \big\|(I - \lambda A_i)x_n - (I - \lambda A_i)z - P_C(I - \lambda A_i)x_n + z\big\|^2\Big)\\&\leq \frac{1}{2}\Big(\|x_n - z\|^2 + \big\|P_C(I - \lambda A_i)x_n - z\big\|^2\\&\quad - \big\|x_n - P_C(I - \lambda A_i)x_n - \lambda(A_i x_n - A_i z)\big\|^2\Big)\\&= \frac{1}{2}\Big(\|x_n - z\|^2 + \big\|P_C(I - \lambda A_i)x_n - z\big\|^2\\&\quad - \big\|x_n - P_C(I - \lambda A_i)x_n\big\|^2 - \big\|\lambda(A_i x_n - A_i z)\big\|^2\\&\quad + 2\lambda\big\langle x_n - P_C(I - \lambda A_i)x_n, A_i x_n - A_i z\big\rangle\Big),\end{aligned}$$

it implies that

$$\begin{aligned}\big\|P_C(I-\lambda A_i)x_n - z\big\|^2 \le{}& \|x_n - z\|^2 - \big\|x_n - P_C(I-\lambda A_i)x_n\big\|^2 \\ &+ 2\lambda\big\|x_n - P_C(I-\lambda A_i)x_n\big\|\|A_ix_n - A_iz\|.\end{aligned} \tag{3.13}$$

From the definition of y_n and (3.13), we have

$$\begin{aligned}\|y_n - z\|^2 &\le \alpha_n\|x_n - z\|^2 + \beta_n\|Sx_n - z\|^2 + \gamma_n\|z_n - z\|^2 \\ &\le (1-\gamma_n)\|x_n - z\|^2 + \gamma_n\sum_{i=1}^N \delta_n^i\big\|P_C(I-\lambda A_i)x_n - z\big\|^2 \\ &\le (1-\gamma_n)\|x_n - z\|^2 + \gamma_n\sum_{i=1}^N \delta_n^i\big(\|x_n - z\|^2 - \big\|x_n - P_C(I-\lambda A_i)x_n\big\|^2 \\ &\quad + 2\lambda\big\|x_n - P_C(I-\lambda A_i)x_n\big\|\|A_ix_n - A_iz\|\big) \\ &= \|x_n - z\|^2 - \gamma_n\sum_{i=1}^N \delta_n^i\big\|x_n - P_C(I-\lambda A_i)x_n\big\|^2 \\ &\quad + 2\gamma_n\sum_{i=1}^N \delta_n^i\lambda\big\|x_n - P_C(I-\lambda A_i)x_n\big\|\|A_ix_n - A_iz\|,\end{aligned}$$

which implies that

$$\begin{aligned}\gamma_n\sum_{i=1}^N \delta_n^i\big\|x_n - P_C(I-\lambda A_i)x_n\big\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\ &\quad + 2\gamma_n\sum_{i=1}^N \delta_n^i\lambda\big\|x_n - P_C(I-\lambda A_i)x_n\big\|\|A_ix_n - A_iz\| \\ &\le \big(\|x_n - z\| + \|y_n - z\|\big)\|y_n - x_n\| \\ &\quad + 2\gamma_n\sum_{i=1}^N \delta_n^i\lambda\big\|x_n - P_C(I-\lambda A_i)x_n\big\|\|A_ix_n - A_iz\|.\end{aligned}$$

From conditions (i), (ii), (3.9) and (3.12), we have

$$\lim_{n\to\infty}\big\|P_C(I-\lambda A_i)x_n - x_n\big\| = 0, \quad \forall i = 1,2,\ldots,N. \tag{3.14}$$

Since

$$\|z_n - x_n\| \le \sum_{i=1}^N \delta_n^i\big\|P_C(I-\lambda A_i)x_n - x_n\big\|,$$

from (3.14), we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.15}$$

Since

$$y_n - x_n = \beta_n(Sx_n - x_n) + \gamma_n(z_n - x_n)$$

from (3.9) and (3.15), we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Next, we will show that

$$\lim_{n\to\infty} \|T_i U_{i-1}x_n - U_{i-1}x_n\| = 0, \quad \forall i = 1, 2, \ldots, N. \tag{3.16}$$

From the definition of y_n, we have

$$\begin{aligned}
\|y_n - z\|^2 &\le \alpha_n\|x_n - z\|^2 + \beta_n\|Sx_n - z\|^2 + \gamma_n\|z_n - z\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big\|\alpha_1^N(T_N U_{N-1}x_n - z) \\
&\quad + \alpha_2^N(U_{N-1}x_n - z) + \alpha_3^N(x_n - z)\big\| \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big(\alpha_1^N\|T_N U_{N-1}x_n - z\|^2 + \alpha_2^N\|U_{N-1}x_n - z\|^2 \\
&\quad + \alpha_3^N\|x_n - z\|^2 - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\big) \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big(\big(1-\alpha_3^N\big)\|U_{N-1}x_n - z\|^2 \\
&\quad + \alpha_3^N\|x_n - z\|^2 - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\big) \\
&= (1-\beta_n)\|x_n - z\|^2 + \beta_n\big(\big(1-\alpha_3^N\big)\big\|\alpha_1^{N-1}(T_{N-1}U_{N-2}x_n - z) \\
&\quad + \alpha_2^{N-1}(U_{N-2}x_n - z) + \alpha_3^{N-1}(x_n - z)\big\|^2 \\
&\quad + \alpha_3^N\|x_n - z\|^2 - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\big) \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big(\big(1-\alpha_3^N\big)\big(\alpha_1^{N-1}\|T_{N-1}U_{N-2}x_n - z\|^2 \\
&\quad + \alpha_2^{N-1}\|U_{N-2}x_n - z\|^2 + \alpha_3^{N-1}\|x_n - z\|^2 \\
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2\big) \\
&\quad + \alpha_3^N\|x_n - z\|^2 - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\big) \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\big(\big(1-\alpha_3^N\big)\big(\big(1-\alpha_3^{N-1}\big)\|U_{N-2}x_n - z\|^2 \\
&\quad + \alpha_3^{N-1}\|x_n - z\|^2 - \alpha_1^{N-1}\alpha_2^{N-1}\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2\big) \\
&\quad + \alpha_3^N\|x_n - z\|^2 - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\big) \\
&= (1-\beta_n)\|x_n - z\|^2 + \beta_n\big(\big(1-\alpha_3^N\big)\big(1-\alpha_3^{N-1}\big)\|U_{N-2}x_n - z\|^2 \\
&\quad + \big(1-\alpha_3^N\big)\alpha_3^{N-1}\|x_n - z\|^2 - \alpha_1^{N-1}\alpha_2^{N-1}\big(1-\alpha_3^N\big)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad + \alpha_3^N\|x_n - z\|^2 - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\big) \\
&= (1-\beta_n)\|x_n - z\|^2 + \beta_n\left(\prod_{j=N-1}^{N}\big(1-\alpha_3^j\big)\|U_{N-2}x_n - z\|^2\right. \\
&\quad + \left(1 - \prod_{j=N-1}^{N}\big(1-\alpha_3^j\big)\right)\|x_n - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n - U_{N-1}x_n\|^2\Biggr) \\
&= (1-\beta_n)\|x_n - z\|^2 + \beta_n\Biggl(\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\bigl\|\alpha_1^{N-2}(T_{N-2}U_{N-3}x_n - z) \\
&\quad + \alpha_2^{N-2}(U_{N-3}x_n - z) + \alpha_3^{N-2}(x_n - z)\bigr\|^2 \\
&\quad + \Biggl(1-\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_n - z\|^2 \\
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n - U_{N-1}x_n\|^2\Biggr) \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\Biggl(\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\bigl(\alpha_1^{N-2}\|T_{N-2}U_{N-3}x_n - z\|^2 \\
&\quad + \alpha_2^{N-2}\|U_{N-3}x_n - z\|^2 + \alpha_3^{N-2}\|x_n - z\|^2 \\
&\quad - \alpha_1^{N-2}\alpha_2^{N-2}\|T_{N-2}U_{N-3}x_n - U_{N-3}x_n\|^2\bigr) \\
&\quad + \Biggl(1-\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_n - z\|^2 \\
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n - U_{N-1}x_n\|^2\Biggr) \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\Biggl(\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\bigl(\bigl(1-\alpha_3^{N-2}\bigr)\|U_{N-3}x_n - z\|^2 \\
&\quad + \alpha_3^{N-2}\|x_n - z\|^2 - \alpha_1^{N-2}\alpha_2^{N-2}\|T_{N-2}U_{N-3}x_n - U_{N-3}x_n\|^2\bigr) \\
&\quad + \Biggl(1-\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_n - z\|^2 \\
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \alpha_1^N\alpha_2^N\|T_NU_{N-1}x_n - U_{N-1}x_n\|^2\Biggr) \\
&= (1-\beta_n)\|x_n - z\|^2 + \beta_n\Biggl(\prod_{j=N-2}^{N}\bigl(1-\alpha_3^j\bigr)\|U_{N-3}x_n - z\|^2 \\
&\quad + \Biggl(1-\prod_{j=N-2}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_n - z\|^2 \\
&\quad - \alpha_1^{N-2}\alpha_2^{N-2}\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\|T_{N-2}U_{N-3}x_n - U_{N-3}x_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\Bigr) \\
&\le \\
&\vdots \\
&\le (1-\beta_n)\|x_n - z\|^2 + \beta_n\Biggl(\prod_{j=1}^{N}\bigl(1-\alpha_3^j\bigr)\|U_0x_n - z\|^2 \\
&\quad + \Biggl(1-\prod_{j=1}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_n - z\|^2 \\
&\quad - \alpha_1^1\alpha_2^1\prod_{j=2}^{N}\bigl(1-\alpha_3^j\bigr)\|T_1U_0x_n - U_0x_n\|^2 \\
&\quad \vdots \\
&\quad - \alpha_1^{N-2}\alpha_2^{N-2}\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\|T_{N-2}U_{N-3}x_n - U_{N-3}x_n\|^2 \\
&\quad - \alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2\Biggr) \\
&= \|x_n - z\|^2 \\
&\quad - \beta_n\alpha_1^1\alpha_2^1\prod_{j=2}^{N}\bigl(1-\alpha_3^j\bigr)\|T_1x_n - x_n\|^2 \\
&\quad \vdots \\
&\quad - \beta_n\alpha_1^{N-2}\alpha_2^{N-2}\prod_{j=N-1}^{N}\bigl(1-\alpha_3^j\bigr)\|T_{N-2}U_{N-3}x_n - U_{N-3}x_n\|^2 \\
&\quad - \beta_n\alpha_1^{N-1}\alpha_2^{N-1}\bigl(1-\alpha_3^N\bigr)\|T_{N-1}U_{N-2}x_n - U_{N-2}x_n\|^2 \\
&\quad - \beta_n\alpha_1^N\alpha_2^N\|T_N U_{N-1}x_n - U_{N-1}x_n\|^2.
\end{aligned}
\tag{3.17}
$$

From (3.17) and condition (ii), we have

$$
\begin{aligned}
\beta_n\alpha_1^1\alpha_2^1\prod_{j=2}^{N}\bigl(1-\alpha_3^j\bigr)\|T_1x_n - x_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&\le \bigl(\|x_n - z\| + \|y_n - z\|\bigr)\|y_n - x_n\|.
\end{aligned}
$$

Form (3.9), we have

$$
\lim_{n\to\infty}\|T_1x_n - x_n\| = 0. \tag{3.18}
$$

By using the same method as (3.18), we can conclude that

$$\lim_{n\to\infty} \|T_i U_{i-1} x_n - U_{i-1} x_n\| = 0, \quad \forall i = 1, 2, \dots, N.$$

Let $\omega(x_n)$ be the set of all weakly ω-limit of $\{x_n\}$. We shall show that $\omega(x_n) \subset \mathfrak{F}$. Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. Let $q \in \omega(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to q.

Put $Q : C \to C$ defined by

$$Qx = \sum_{i=1}^{N} \delta^i G_i x, \quad \forall x \in C. \tag{3.19}$$

Since $G_i = P_C(I - \lambda A_i)$ is a nonexpansive mapping, for every $i = 1, 2, \dots, N$, from Lemma 2.6 and 2.5, we have

$$F(Q) = \bigcap_{i=1}^{N} F(G_i) = \bigcap_{i=1}^{N} VI(C, A_i). \tag{3.20}$$

Since

$$\begin{aligned} \|x_n - Qx_n\| &\le \|x_n - z_n\| + \|z_n - Qx_n\| \\ &= \|x_n - z_n\| + \left\| \sum_{i=1}^{N} \delta_n^i G_i x_n - \sum_{i=1}^{N} \delta^i G_i x_n \right\| \\ &= \|x_n - z_n\| + \left\| \sum_{i=1}^{N} (\delta_n^i - \delta^i) G_i x_n \right\| \\ &\le \|x_n - z_n\| + \sum_{i=1}^{N} |\delta_n^i - \delta^i| \|G_i x_n\|, \end{aligned}$$

from the condition (i) and (3.15), we have

$$\lim_{n\to\infty} \|x_n - Qx_n\| = 0. \tag{3.21}$$

From (3.21), we have

$$\lim_{i\to\infty} \|x_{n_i} - Qx_{n_i}\| = 0.$$

From (3.19), it is easy to see that Q is a nonexpansive mapping. By Lemma 2.7 and $x_{n_i} \rightharpoonup q$ as $i \to \infty$, we have $q \in F(Q) = \bigcap_{i=1}^{N} F(G_i)$ From (3.2), we have

$$q \in \bigcap_{i=1}^{N} VI(C, A_i). \tag{3.22}$$

Next, we will show that $q \in F(S)$. Assume that $q \neq Sq$. From the Opial property, (3.10) and

(3.16), we have

$$
\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i}-q\|^2 &< \liminf_{i\to\infty}\|x_{n_i}-Sq\|^2 \\
&= \liminf_{i\to\infty}\big\|x_{n_i}-Sx_{n_i}+(Sx_{n_i}-Sq)\big\|^2 \\
&= \liminf_{i\to\infty}\big(\|x_{n_i}-Sx_{n_i}\|^2+\|Sx_{n_i}-Sq\|^2+2\langle x_{n_i}-Sx_{n_i},Sx_{n_i}-Sq\rangle\big) \\
&= \liminf_{i\to\infty}\|Sx_{n_i}-Sq\|^2 \\
&= \liminf_{i\to\infty}\big\|\alpha_1^N T_N U_{N-1}x_{n_i}+\alpha_2^N U_{N-1}x_{n_i}+\alpha_3^N x_{n_i} \\
&\quad -\alpha_1^N T_N U_{N-1}q-\alpha_2^N U_{N-1}q-\alpha_3^N q\big\|^2 \\
&= \liminf_{i\to\infty}\big\|\alpha_1^N(T_N U_{N-1}x_{n_i}-T_N U_{N-1}q) \\
&\quad +\alpha_2^N(U_{N-1}x_{n_i}-U_{N-1}q)+\alpha_3^N(x_{n_i}-q)\big\|^2 \\
&\le \liminf_{i\to\infty}\big(\alpha_1^N\|T_N U_{N-1}x_{n_i}-T_N U_{N-1}q\|^2 \\
&\quad +\alpha_2^N\|U_{N-1}x_{n_i}-U_{N-1}q\|^2+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&\le \liminf_{i\to\infty}\big(\alpha_1^N\big(\|U_{N-1}x_{n_i}-U_{N-1}q\|^2 \\
&\quad +2\langle U_{N-1}x_{n_i}-T_N U_{N-1}x_{n_i},U_{N-1}q-T_N U_{N-1}q\rangle\big) \\
&\quad +\alpha_2^N\|U_{N-1}x_{n_i}-U_{N-1}q\|^2+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&= \liminf_{i\to\infty}\big(\big(1-\alpha_3^N\big)\|U_{N-1}x_{n_i}-U_{N-1}q\|^2+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&= \liminf_{i\to\infty}\big(\big(1-\alpha_3^N\big)\big\|\alpha_1^{N-1}(T_{N-1}U_{N-2}x_{n_i}-T_{N-1}U_{N-2}q) \\
&\quad +\alpha_2^{N-1}(U_{N-2}x_{n_i}-U_{N-2}q)+\alpha_3^{N-1}(x_{n_i}-q)\big\|^2+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&\le \liminf_{i\to\infty}\big(\big(1-\alpha_3^N\big)\big(\alpha_1^{N-1}\|T_{N-1}U_{N-2}x_{n_i}-T_{N-1}U_{N-2}q\|^2 \\
&\quad +\alpha_2^{N-1}\|U_{N-2}x_{n_i}-U_{N-2}q\|^2+\alpha_3^{N-1}\|x_{n_i}-q\|^2\big)+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&\le \liminf_{i\to\infty}\big(\big(1-\alpha_3^N\big)\big(\alpha_1^{N-1}\big(\|U_{N-2}x_{n_i}-U_{N-2}q\|^2 \\
&\quad +2\langle U_{N-2}x_{n_i}-T_{N-1}U_{N-2}x_{n_i},U_{N-2}q-T_{N-1}U_{N-2}q\rangle\big) \\
&\quad +\alpha_2^{N-1}\|U_{N-2}x_{n_i}-U_{N-2}q\|^2+\alpha_3^{N-1}\|x_{n_i}-q\|^2\big)+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&= \liminf_{i\to\infty}\big(\big(1-\alpha_3^N\big)\big(\big(1-\alpha_3^{N-1}\big)\|U_{N-2}x_{n_i}-U_{N-2}q\|^2 \\
&\quad +\alpha_3^{N-1}\|x_{n_i}-q\|^2\big)+\alpha_3^N\|x_{n_i}-q\|^2\big) \\
&= \liminf_{i\to\infty}\left(\prod_{j=N-1}^{N}\big(1-\alpha_3^j\big)\|U_{N-2}x_{n_i}-U_{N-2}q\|^2\right. \\
&\quad \left.+\left(1-\prod_{j=N-1}^{N}\big(1-\alpha_3^j\big)\right)\|x_{n_i}-q\|^2\right) \\
&\le \\
&\ \vdots
\end{aligned}
$$

$$\begin{aligned}
&\leq \liminf_{i\to\infty}\Biggl(\prod_{j=1}^{N}\bigl(1-\alpha_3^j\bigr)\|U_0x_{n_i}-U_0q\|^2 \\
&\quad + \Biggl(1-\prod_{j=1}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_{n_i}-q\|^2\Biggr) \\
&= \liminf_{i\to\infty}\Biggl(\prod_{j=1}^{N}\bigl(1-\alpha_3^j\bigr)\|x_{n_i}-q\|^2 \\
&\quad + \Biggl(1-\prod_{j=1}^{N}\bigl(1-\alpha_3^j\bigr)\Biggr)\|x_{n_i}-q\|^2\Biggr) \\
&= \liminf_{i\to\infty}\|x_{n_i}-q\|^2.
\end{aligned}$$

This is a contradiction. Then, we have $q \in F(S)$. From Lemma 2.9, we have

$$q \in \bigcap_{i=1}^{N} F(T_i). \tag{3.23}$$

From (3.22) and (3.23), we have $q \in \mathfrak{F}$. Hence, $\omega(x_n) \subset \mathfrak{F}$. Therefore, by (3.5) and Lemma 2.8, we have $\{x_n\}$ converges strongly to $P_{\mathfrak{F}}x_1$. This completes the proof. □

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H. For every $i = 1,2,\dots,N$, let $A_i : C \to H$ be an α_i-inverse strongly monotone mapping, and let $T : C \to C$ be a nonspreading mapping with $\mathfrak{F} = F(T) \cap \bigcap_{i=1}^{N} VI(C,A_i) \neq \emptyset$. For every $i = 1,2,\dots,N$, define the mapping $G_i : C \to C$ by $G_ix = P_C(I-\lambda A_i)x$ $\forall x \in C$ and $\lambda \in [c,d] \subset (0,2\alpha_i)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C_1 = C$ and*

$$\begin{cases}
z_n = \sum_{i=1}^{N} \delta_n^i G_i x_n, \\
y_n = \alpha_n x_n + \beta_n T x_n + \gamma_n z_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1,
\end{cases} \tag{3.24}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n = 1$ and suppose the following conditions hold:

(i) $\lim_{n\to\infty} \delta_n^i = \delta^i \in (0,1), \quad \forall i = 1,2,\dots,N$ *and* $\sum_{i=1}^{N} \delta_n^i = 1$;

(ii) $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subseteq [a,b] \subset (0,1)$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathfrak{F}}x_1$.

Corollary 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping, and let $\{T_i\}_{i=1}^{N}$ be a finite family of nonspreading mappings with $\mathfrak{F} = \bigcap_{i=1}^{N} F(T_i) \cap VI(C,A) \neq \emptyset$. Let $\rho_j = (\alpha_1^j,\alpha_2^j,\alpha_3^j) \in I \times I \times I$, $j = 1,2,3,\dots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j,\alpha_3^j \in (0,1)$ for all $j = 1,2,\dots,N-1$ and*

$\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in (0,1)$ for all $j = 1,2,\dots,N$, and let S be the S-mapping generated by $T_1, T_2, \dots, T_N$ and $\rho_1, \rho_2, \dots, \rho_N$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C_1 = C$ and

$$\begin{cases} y_n = \alpha_n x_n + \beta_n S x_n + \gamma_n P_C(I - \lambda A)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases} \tag{3.25}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [a,b] \subset (0,1)$, $\alpha_n + \beta_n + \gamma_n = 1$ and $\lambda \subseteq [c,d] \subset (0,2\alpha)$. Then the sequence $\{x_n\}$ converges strongly to $P_{\mathfrak{F}} x_1$.

Competing interests
The authors declare that they have no competing interests.

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