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Convergence to common solutions of various problems for nonexpansive mappings in Hilbert spaces

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Abstract

In this paper, motivated and inspired by Ceng and Yao (J. Comput. Appl. Math. 214(1):186-201, 2008), Iiduka and Takahashi (Nonlinear Anal. 61(3):341-350, 2005), Jaiboon and Kumam (Nonlinear Anal. 73(5):1180-1202, 2010), Kim (Nonlinear Anal. 73:3413-3419, 2010), Marino and Xu (J. Math. Anal. Appl. 318:43-52, 2006) and Saeidi (Nonlinear Anal. 70:4195-4208, 2009), we introduce a new iterative scheme for finding a common element of the set of solutions of a mixed equilibrium problem for an equilibrium bifunction, the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of some variational inequality problem, and the set of fixed points of a left amenable semigroup $\{T_t : t \in S\}$ of nonexpansive mappings with respect to W-mappings and a left regular sequence $\{\mu_n\}$ of means defined on an appropriate space of bounded real-valued functions of the semigroup S. Furthermore, we prove that the iterative scheme converges strongly to a common element of the above four sets. Our results extend and improve the corresponding results of many others.

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1 Introduction

Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, and let P_C be the metric projection of H onto C. Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $\theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction with $\theta(u,u) = 0$ for each $u \in C$. We consider the mixed equilibrium problem (for short, *MEP*) is to find $x^* \in C$ such that

$$MEP : \theta\left(x^*, y\right) + \varphi(y) - \varphi\left(x^*\right) \geq 0, \quad \forall y \in C.$$

In particular, if $\varphi \equiv 0$, this problem reduces to the equilibrium problem (for short, *EP*), which is to find $x^* \in C$ such that

$$EP : \theta\left(x^*, y\right) \geq 0, \quad \forall y \in C.$$

Denote the set of solutions of *MEP* by Ω. The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and the equilibrium problems as special cases.

A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. It is well known that $F(T)$ is closed convex. Recall that a mapping $f : C \to C$ is called *contractive* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|,$$

for all $x, y \in C$.

In 2000, Moudafi [1] introduced the viscosity approximation method for nonexpansive mappings (see [2] for further developments in both Hilbert and Banach spaces).

Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.1}$$

where α_n is a sequence in $(0, 1)$. It is proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.1) strongly converges to the unique solution x^* in $F(T)$ of the variational inequality

$$\langle (f - I)x^*, x - x^* \rangle \leq 0, \quad \forall x \in F(T)$$

(see [1, 2]).

Let A be a *strongly positive bounded linear operator* on H, that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2,$$

for all $x \in H$.

In 2006, Marino and Xu [3] considered the following iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.2}$$

where $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, α is a contraction coefficient of f. They proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$, for $x \in H$).

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H, and let $N_C v$ be the *normal cone* to C at $v \in C$, *i.e.*,

$$N_C v = \{\mathrm{w} \in H : \langle v - u, \mathrm{w}\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [4].

In 2005, for finding an element of $F(T) \cap VI(C, A)$, Iiduka and Takahashi [5] proposed a new iterative sequence: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T P_C(x_n - \lambda_n A x_n), \quad n \geq 1 \tag{1.3}$$

and obtained a strong convergence theorem in a Hilbert space.

Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself, and let $\{\lambda_n\}$ be a sequence of nonnegative numbers in $[0, 1]$. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I, \\ U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I, \\ U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I, \\ W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I. \end{aligned} \tag{1.4}$$

Such a mapping W_n is called the *W-mapping* generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$. The concept of W-mapping was introduced in [6, 7] and [8].

In 2008, Ceng and Yao [9] introduced the hybrid iterative scheme

$$\begin{cases} x_0 \in C \quad \text{arbitrary}, \\ \theta(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \geq 0, \quad \forall x \in C, \\ x_{n+1} = \alpha_n f(W_n x_n) + \beta_n x_n + \gamma_n W_n y_n, \end{cases} \tag{1.5}$$

where $K'(x)$ is the Fréchet derivative of K at x. They proved the sequences $\{x_n\}$ and $\{y_n\}$ generated by the hybrid iterative scheme (1.5) converge strongly to a common element of the set of solutions of *MEP* and the set of common fixed points of finitely many nonexpansive mappings.

Recall the mapping B is said to be *relaxed* (ξ,ν)*-cocoercive*, if there exist two constants $\xi,\nu > 0$ such that

$$\langle Bx - By, x - y\rangle \geq -\xi\|Bx - By\|^2 + \nu\|x - y\|^2, \quad \forall x, y \in C.$$

This class of mappings has been considered by many authors; for example, [10, 11].

In this paper, motivated and inspired by Ceng and Yao [9], Iiduka and Takahashi [5], Jaiboon and Kumam [12], Kim [13], Marino and Xu [3] and Saeidi [14], we introduce a new iterative scheme:

$$\begin{cases} x_0 \in C \quad \text{arbitrary}, \\ \theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r_n}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \\ y_n = (1 - \gamma_n)x_n + \gamma_n T_{\mu_n} W_n P_C(I - \delta_n B)z_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)T_{\mu_n} W_n y_n, \end{cases} \tag{1.6}$$

for all $x \in C$, $n \geq 0$, for finding a common element of the set of solutions of a mixed equilibrium problem for an equilibrium bifunction, the set of fixed points of an infinite family of nonexpansive mappings, the set of solutions of some variational inequality problem and the set of fixed points of a left amenable semigroup $\{T_t : t \in S\}$ of nonexpansive mappings with respect to W-mappings and a left regular sequence $\{\mu_n\}$ of means defined on an appropriate space of bounded real-valued functions of the semigroup S. Furthermore, we prove that the proposed iterative scheme (1.6) converges strongly to a common element of the above four sets. Our result extends and improves the corresponding results of many others.

2 Preliminaries

Let S be a semigroup. We denote by $B(S)$ the space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$, we define the left and right translation operators l_s and r_s on $B(S)$ by

$$(l_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for each $t \in S$ and $f \in B(S)$, respectively. Let X be a subspace of $B(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in X$.

Let X be a translation invariant subspace of $B(S)$ (*i.e.*, $l_sX \subset X$ and $r_sX \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if

$$\mu(l_sf) = \mu(f) \qquad \bigl(\text{resp. } \mu(r_sf) = \mu(f)\bigr)$$

for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [15–17]. X is said to be *left* (resp. *right*) *amenable* if X has a left (resp. right) invariant mean. X is amenable if X is left and right amenable. Moreover, $B(S)$ is amenable when S is a commutative semigroup or a solvable group. However, the free group or semigroup of two generators is not left or right amenable. In this case, we say that the semigroup S is an amenable semigroup (see [18, 19]). A semigroup S is left reversible if S has the finite intersection property for right ideals. Every left reversible semigroup S, $WAP(S)$ the space of weakly almost period functions on S has a left invariant mean. If S is both left and right reversible, then $WAP(S)$ has an invariant mean. Each group or amenable semigroup is left and right reversible (see [20, 21]).

A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp. *right*) *invariant* if

$$\lim_\alpha\bigl(\mu_\alpha(l_sf) - \mu_\alpha(f)\bigr) = 0 \qquad \Bigl(\text{resp. } \lim_\alpha\bigl(\mu_\alpha(r_sf) - \mu_\alpha(f)\bigr) = 0\Bigr)$$

for each $f \in X$ and $s \in S$, and it is said to be *left* (resp. *right*) *strongly asymptotically invariant* (or *strong regular*) if

$$\lim_\alpha\bigl\|l_s^*\mu_\alpha - \mu_\alpha\bigr\| = 0 \qquad \Bigl(\text{resp. } \lim_\alpha\bigl\|r_s^*\mu_\alpha - \mu_\alpha\bigr\| = 0\Bigr)$$

for each $s \in S$, where l_s^* and r_s^* are the adjoint operators of l_s and r_s, respectively. Such nets were first studied by Day in [18] where they were called *weak** *invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup S is left (resp. right) amenable, then the semigroup $S' = S \cup \{e\}$, where $es' = s'e = s'$ for all $s' \in S$, is also left (resp. right) amenable and conversely.

Let S be a semigroup, and let C be a closed and convex subset of H. Let $F(T)$ denote the fixed point set of T. Then $\Im = \{T_s : s \in S\}$ is called a *representation of S as nonexpansive mappings on C* if T_s is nonexpansive with $T_e = I$ and $T_{st} = T_sT_t$ for each $s, t \in S$ (*cf.* [22–30]). We denote by $F(\Im)$ the set of common fixed points of $\{T_s : s \in S\}$, *i.e.*,

$$F(\Im) = \bigcap_{s\in S} F(T_s) = \bigcap_{s\in S}\{x \in C : T_sx = x\}.$$

Lemma 2.1 ([19, 31]) *Let S be a semigroup and C be a closed convex subset of a Hilbert space H. Let $\Im = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $\{T_tu : t \in S\}$ is bounded for some $u \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_tx, y\rangle$ is an element of X for each $x \in C$ and $y \in H$, and μ be a mean on X. If we write $T_\mu x$ instead of $\int T_tx\,d\mu(t)$, then the following hold*:

(i) *T_μ is a nonexpansive mapping from C into itself,*
(ii) *$T_\mu x = x$ for each $x \in F(\Im)$,*
(iii) *$T_\mu x \in \overline{\text{co}}\{T_tx : t \in S\}$ for each $x \in C$, where $\overline{\text{co}}A$ is the closed convex hull of A.*

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ be a mapping. Then T is said to be *demiclosed at* $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C \quad \text{and} \quad Tx_n \to v \quad \text{imply} \quad Tu = v,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

Lemma 2.2 ([32]) *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.*

Let C be a nonempty subset of a normed space E, and let $x \in E$. An element $y_0 \in C$ is said to be *the best approximation to x* if

$$\|x - y_0\| = d(x, C),$$

where $d(x, C) = \inf_{y\in C} \|x - y\|$. The number $d(x, C)$ is called *the distance from x to C*. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called *the metric projection of H onto C*. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x\rangle \le 0 \tag{2.1}$$

and

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2.$$

It is easy to see that the following is true:

$$u \in VI(C, B) \iff u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{2.2}$$

In this paper, for solving the mixed equilibrium problems for an equilibrium bifunction $\theta : C \times C \to \mathbb{R}$, we assume that θ satisfies the following conditions:

(E1) $\theta(x, x) = 0$ for all $x \in C$;

(E2) θ is monotone, *i.e.*, $\theta(x,y)+\theta(y,x)\leq 0$ for all $x,y\in C$;

(E3) for each $x,y,z\in C$,

$$\lim_{t\downarrow 0}\theta\big(tz+(1-t)x,y\big)\leq\theta(x,y);$$

(E4) for each $x\in C$, the function $y\mapsto\theta(x,y)$ is convex and lower semicontinuous.

Definition 2.1 (1) Let $F:C\to H$ and $\eta:C\times C\to H$ be two mappings. Then F is called:

(i) *η-monotone* if

$$\big\langle F(x)-F(y),\eta(x,y)\big\rangle\geq 0,\quad \forall x,y\in C,$$

(ii) *η-strongly monotone with constant α* if there exists a constant $\alpha>0$ such that

$$\big\langle F(x)-F(y),\eta(x,y)\big\rangle\geq\alpha\|x-y\|^2,\quad \forall x,y\in C,$$

(iii) *Lipschitz continuous with constant β* if there exists a constant $\beta>0$ such that

$$\big\|F(x)-F(y)\big\|\leq\beta\|x-y\|,\quad \forall x,y\in C.$$

If $\eta(x,y)=x-y$, for all $x,y\in C$, then the definitions (i) and (ii) reduce to the definition of monotonicity and strong monotonicity, respectively.

(2) A mapping $\eta:C\times C\to H$ is called *Lipschitz continuous with constant λ* if there exists a constant $\lambda>0$ such that

$$\big\|\eta(x,y)\big\|\leq\lambda\|x-y\|,\quad \forall x,y\in C.$$

(3) A differentiable function $K:C\to\mathbb{R}$ on a convex set C is called:

(i) *η-convex* [33] if

$$K(y)-K(x)\geq\big\langle K'(x),\eta(y,x)\big\rangle,\quad \forall x,y\in C,$$

where $K'(x)$ is the Fréchet derivative of K at x,

(ii) *η-strongly convex with constant σ* [34] if there exists a constant $\sigma>0$ such that

$$K(y)-K(x)-\big\langle K'(x),\eta(y,x)\big\rangle\geq\frac{\sigma}{2}\|x-y\|^2,\quad \forall x,y\in C.$$

(4) A mapping $F:C\to\mathbb{R}$ is called *sequentially continuous at x_0* [35], if $F(x_n)\to F(x_0)$ for each sequence $\{x_n\}$ satisfying $x_n\rightharpoonup x_0$. F is called *sequentially continuous on C* if it is sequentially continuous at each point of C.

Lemma 2.3 ([9]) *Let $K:C\to\mathbb{R}$ be differentiable η-strongly convex with a constant $\sigma>0$, and let $\eta:C\times C\to H$ be a mapping such that $\eta(x,y)+\eta(y,x)=0$ for all $x,y\in C$. Then $K':C\to H$ is η-strongly monotone with constant $\sigma>0$.*

Lemma 2.4 ([36]) *A Hilbert space H is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in H, the condition $x_n \rightharpoonup x$ implies that*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in H$ with $y \neq x$.

Lemma 2.5 *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x, y \in H$.

Let C be a nonempty closed convex subset of a real Hilbert space H, $\varphi : C \to \mathbb{R}$ be a real-valued function and $\theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction. Let r be a positive parameter. For a given point $x \in C$, consider the auxiliary problem for the mixed equilibrium problem (for short, $MEP(x,r)$) which consists of finding $y \in C$ such that

$$\theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\bigl\langle K'(y) - K'(x), \eta(z,y)\bigr\rangle \geq 0, \quad \forall z \in C,$$

where $\eta : C \times C \to H$ and $K'(x)$ is the Fréchet derivative of a functional $K : C \to \mathbb{R}$ at x. Let $S_r : C \to C$ be the mapping such that for each $x \in C$, $S_r(x)$ is the solution set of $MEP(x,r)$, *i.e.*,

$$\begin{aligned} S_r(x) = \Bigl\{ & y \in C : \theta(y,z) + \varphi(z) - \varphi(y) \\ & + \frac{1}{r}\bigl\langle K'(y) - K'(x), \eta(z,y)\bigr\rangle \geq 0, \forall z \in C \Bigr\} \end{aligned} \tag{2.3}$$

for all $x \in C$.

We first need the following important and interesting result.

Lemma 2.6 ([9]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (E1)-(E4). *Assume that*

(i) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*
  (a) *$\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in C$,*
  (b) *for each fixed $y \in C$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology,*

(ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is sequentially continuous from the weak topology to the strong topology,*

(iii) *for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\theta(y,z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\bigl\langle K'(y) - K'(x), \eta(z_x,y)\bigr\rangle < 0.$$

Then the following hold:

(1) *S_r is single-valued*;

(2) S_r *is a firmly nonexpansive-type mapping, i.e., for all* $x, y \in H$,

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle;$$

(3) (j) S_r *is nonexpansive if* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq \lambda\nu$;
(jj) $\langle K'(x_1) - K'(x_2), \eta(u_1, u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1, u_2)\rangle$, $\forall(x_1, x_2) \in C \times C$, *where* $u_i = S_r(x_i)$, $i = 1, 2$;

(4) $F(S_r) = \Omega$;

(5) Ω *is a closed and convex subset of* C.

Remark 2.1 In particular, from Lemma 2.6, whenever $K(x) = \frac{\|x\|^2}{2}$ and $\eta(x, y) = x - y$ for each $(x, y) \in C \times C$, then S_r is firmly nonexpansive, *i.e.*,

$$\langle x_1 - x_2, S_r(x_1) - S_r(x_2)\rangle \geq \|S_r(x_1) - S_r(x_2)\|^2, \quad \forall(x_1, x_2) \in C \times C.$$

We need the following results concerning the W-mapping W_n.

Lemma 2.7 ([37]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T_1, T_2, \ldots$ *be nonexpansive mappings of* C *into* H *such that* $\bigcap_{i=1}^{\infty} F(T_i)$ *is nonempty, and let* $\lambda_1, \lambda_2, \ldots$ *be real numbers such that* $0 < \lambda_i \leq b < 1$ *for any* $i \in \mathbb{N}$. *Then, for every* $x \in C$ *and* $k \in \mathbb{N}$, *the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists.*

Using Lemma 2.7, one can define a mapping W of C into H as

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for every $x \in C$.

Remark 2.2 ([37]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into H such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i \leq b < 1$ for any $i \in \mathbb{N}$. Then $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Remark 2.3 ([38]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into H such that $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty. If $\{x_n\}$ is an arbitrary bounded sequence in C, then we have

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

Lemma 2.8 ([39]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Hilbert space* H *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n$ *and* $\limsup_{n\to\infty} \beta_n < 1$. *Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all integers $n \geq 0$ *and*

$$\limsup_{n\to\infty} \big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.9 ([3]) *Assume A is a strongly positive linear bounded operator on a Hilbert space H with a coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.10 ([2]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n,$$

where $\{b_n\}$ is a sequence in $(0,1)$ and $\{c_n\}$ is a sequence in $\mathbb{R}$ such that
(1) $\sum_{n=1}^{\infty} b_n = \infty$,
(2) $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n c_n| < \infty$.
Then

$$\lim_{n\to\infty} a_n = 0.$$

3 Main result: strong convergence theorems

In this section, we deal with the strong convergence of hybrid viscosity approximation scheme (1.6) for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of an infinite family of nonexpansive mappings, the set of fixed points of a left amenable semigroup of nonexpansive mappings and the set of solutions of variational inequality in a Hilbert space.

Theorem 3.1 *Let S be a semigroup, $\Im = \{T_t : t \in S\}$ be a nonexpansive semigroup on H such that $F(\Im) \neq \emptyset$, X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left strong regular sequence of means on X such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$. Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_i\}$ be an infinite family of nonexpansive mappings from C into itself such that $T_i(F(\Im) \cap \Omega) \subset F(\Im)$ for each $i \in \mathbb{N}$. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex functional. Let $\theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (E1)-(E4), *and let $T_1, T_2, \ldots$ be an infinite family of nonexpansive mappings of C into H. Let $r > 0$, $\gamma > 0$ be two constants. Let f be a contraction of C into itself with a coefficient $\alpha \in (0,1)$, and let A be a strongly positive bounded linear operator with a coefficient $\bar{\gamma} > 0$ such that $0 < \alpha\gamma < \bar{\gamma} < \alpha\gamma + 1$. Let $B : C \to H$ be an L-Lipschitzian and relaxed (ξ, ν)-cocoercive mapping. Suppose that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \cap F(\Im) \cap \Omega \cap VI(C,B) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ such that $\alpha_n + \beta_n \leq 1$, and let the sequence $\{\delta_n\} \subset (0,\infty)$. Assume that*:

(C1) *$\eta : C \times C \to H$ is Lipschitz continuous with constant $\lambda > 0$ such that*
  (a) *$\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in C$,*
  (b) *for each fixed $y \in C$, $x \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology,*

(C2) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\sigma > 0$ and its derivative K' is not only sequentially continuous from the weak topology to the strong topology, but also Lipschitz continuous with constant $\nu > 0$ such that $\sigma \geq \lambda\nu$,*

(C3) *for each $x \in C$, there exist a bounded subset $D_x \subset C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0,$$

(C4) (i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,
$0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$,
(iii) $\lim_{n\to\infty}|\delta_{n+1} - \delta_n| = 0$, *$a \le \delta_n \le b$ for some a, b with* $0 \le a \le b \le \frac{2(\nu-\xi L^2)}{L^2}$,
(iv) $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$,
(C5) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Given $x_0 \in C$ is arbitrary, then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated iteratively by (1.6) *converge strongly to $x^* \in \mathcal{F}$, where $x^* = P_{\mathcal{F}}(\gamma f + (I - A))x^*$, which solves the following variational inequality*:

$$\langle(\gamma f - A)x^*, x - x^*\rangle \le 0, \quad \forall x \in \mathcal{F}.$$

Lemma 3.1 $\|(1-\beta_n)I - \alpha_n A\| \le 1 - \beta_n - \alpha_n\bar{\gamma}$.

Proof Since $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \le (1 - \beta_n)\|A\|^{-1}$. Since A is a linear bounded self-adjoint operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\langle((1-\beta_n)I - \alpha_n A)x, x\rangle &= 1 - \beta_n - \alpha_n\langle Ax, x\rangle \\ &\ge 1 - \beta_n - \alpha_n\|A\| \\ &\ge 0,\end{aligned}$$

which shows that $(1-\beta_n)I - \alpha_n A$ is positive. By Lemma 2.9, we have

$$\|(1-\beta_n)I - \alpha_n A\| \le 1 - \beta_n - \alpha_n\bar{\gamma}.$$

□

Lemma 3.2 *Let B be an L-Lipschitzian and relaxed (ξ, ν)-cocoercive mapping and $\delta_n \le \frac{2(\nu-\xi L^2)}{L^2}$, then*

$$\|(I - \delta_n B)x - (I - \delta_n B)y\| \le \|x - y\|,$$

for all $x, y \in C$.

Proof Since B is an L-Lipschitzian and relaxed (ξ, ν)-cocoercive mapping and $\delta_n \le \frac{2(\nu-\xi L^2)}{L^2}$, we have

$$\begin{aligned}&\|(I - \delta_n B)x - (I - \delta_n B)y\|^2 \\ &\quad = \|x - y\|^2 - 2\delta_n\langle Bx - By, x - y\rangle + \delta_n^2\|Bx - By\|^2 \\ &\quad \le \|x - y\|^2 - 2\delta_n(-\xi\|Bx - By\|^2 + \nu\|x - y\|^2) + \delta_n^2 L^2\|x - y\|^2 \\ &\quad \le \|x - y\|^2 + 2\delta_n\xi L^2\|x - y\|^2 - 2\delta_n\nu\|x - y\|^2 + \delta_n^2 L^2\|x - y\|^2 \\ &\quad = (1 + 2\delta_n\xi L^2 - 2\delta_n\nu + \delta_n^2 L^2)\|x - y\|^2 \le \|x - y\|^2,\end{aligned}$$

for all $x, y \in C$. Thus,

$$\|(I - \delta_n B)x - (I - \delta_n B)y\| \le \|x - y\|,$$

for all $x, y \in C$. □

Lemma 3.3 $\|z_n - p\| \le \|x_n - p\|$, $\forall p \in \mathcal{F}$.

Proof From (2.3), we note that $z_n = S_{r_n} x_n$. From Lemma 2.6, we get

$$\|z_n - p\| \le \|S_{r_n} x_n - S_{r_n} p\| \le \|x_n - p\|$$

for all $p \in \mathcal{F}$. □

Lemma 3.4 $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{W_n y_n\}$, $\{W_n x_n\}$ *and* $\{f(W_n x_n)\}$ *are all bounded.*

Proof Let $p \in \mathcal{F}$. Since $p \in VI(C, B)$, from (2.2), we get $p = P_C(I - \delta_n B)p$. From Lemma 2.1, Lemma 3.3 and W_n, P_C being nonexpansive, we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T_{\mu_n} W_n P_C(I - \delta_n B)z_n - p\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n \|T_{\mu_n} W_n P_C(I - \delta_n B)z_n - p\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n \|(I - \delta_n B)z_n - (I - \delta_n B)p\| \\
&\le (1-\gamma_n)\|x_n - p\| + \gamma_n \|z_n - p\| \\
&\le \|x_n - p\|.
\end{aligned}$$

From (1.6) and Lemma 3.1, we obtain

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \|\alpha_n \gamma f(W_n x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big) T_{\mu_n} W_n y_n - p\| \\
&\quad = \|\alpha_n\big(\gamma f(W_n x_n) - Ap\big) + \beta_n(x_n - p) \\
&\qquad + \big((1-\beta_n)I - \alpha_n A\big)(T_{\mu_n} W_n y_n - p)\| \\
&\quad \le \alpha_n \|\gamma f(W_n x_n) - Ap\| + \beta_n \|x_n - p\| + \|(1-\beta_n)I - \alpha_n A\| \|y_n - p\| \\
&\quad \le (1 - \beta_n - \alpha_n \bar{\gamma})\|y_n - p\| + \beta_n \|x_n - p\| \\
&\qquad + \alpha_n \gamma \|f(W_n x_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| \\
&\quad \le (1 - \alpha_n \bar{\gamma})\|x_n - p\| + \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| \\
&\quad = \big(1 - \alpha_n(\bar{\gamma} - \gamma\alpha)\big)\|x_n - p\| + \alpha_n(\bar{\gamma} - \gamma\alpha)\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha},
\end{aligned} \tag{3.1}$$

for all $n \ge 0$. It follows by mathematical induction that

$$\|x_{n+1} - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad n \ge 0.$$

Therefore, $\{x_n\}$ is bounded. We also deduce that $\{y_n\}$, $\{z_n\}$, $\{W_n y_n\}$, $\{W_n x_n\}$ and $\{f(W_n x_n)\}$ are all bounded. □

Lemma 3.5 *Let the mapping W_n be generated iteratively by* (1.5). *If $\{\omega_n\}$ is a bounded sequence in H, then*

(1) $\lim_{n\to\infty}\|W_{n+1}\omega_n - W_n\omega_n\| = 0$.
(2) $\lim_{n\to\infty}\|T_{\mu_{n+1}}\omega_n - T_{\mu_n}\omega_n\| = 0$.

Proof (1) We shall use M to denote the possible different constants appearing in the following argument. From (1.5), since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}\|W_{n+1}\omega_n - W_n\omega_n\| &= \big\|\lambda_1 T_1 U_{n+1,2}\omega_n + (1-\lambda_1)\omega_n - \lambda_1 T_1 U_{n,2}\omega_n - (1-\lambda_1)\omega_n\big\| \\ &\le \lambda_1\|U_{n+1,2}\omega_n - U_{n,2}\omega_n\| \\ &= \lambda_1\big\|\lambda_2 T_2 U_{n+1,3}\omega_n + (1-\lambda_2)\omega_n - \lambda_2 T_2 U_{n,3}\omega_n - (1-\lambda_2)\omega_n\big\| \\ &\le \lambda_1\lambda_2\|U_{n+1,3}\omega_n - U_{n,3}\omega_n\| \\ &\le \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}\omega_n - U_{n,n+1}\omega_n\| \\ &\le M\prod_{i=1}^{n}\lambda_i,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|W_{n+1}\omega_n - W_n\omega_n\| = 0.$$

(2) Let $q \in F(\Im)$. Then $\|T_s\omega_n - q\| \le \|\omega_n - q\|$. Also, we have

$$\|T_s\omega_n\| \le \|\omega_n - q\| + \|q\|$$

for all $s \in S$ and $n \ge 0$. Since $\{\omega_n\}$ is bounded and $\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0$, we get

$$\begin{aligned}\|T_{\mu_{n+1}}\omega_n - T_{\mu_n}\omega_n\| &= \sup\big\{\big|\langle T_{\mu_{n+1}}\omega_n - T_{\mu_n}\omega_n, z\rangle\big| : \|z\| = 1\big\} \\ &= \sup\big\{\big|\mu_{n+1}(s)\langle T_s\omega_n, z\rangle - \mu_n(s)\langle T_s\omega_n, z\rangle\big| : \|z\| = 1\big\} \\ &\le \|\mu_{n+1} - \mu_n\|\cdot\sup_{s\in S}\|T_s\omega_n\| \\ &\le \|\mu_{n+1} - \mu_n\|\big(\|\omega_n - q\| + \|q\|\big) \\ &\longrightarrow 0.\end{aligned}$$

□

Lemma 3.6 $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|z_{n+1} - z_n\| = 0$.

Proof Define a sequence $\{u_n\}$ by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)u_n$$

for all $n \ge 0$. Observe that from the definition of u_n, we get

$$\begin{aligned}u_{n+1} - u_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(W_{n+1}x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)T_{\mu_{n+1}}W_{n+1}y_{n+1}}{1-\beta_{n+1}}\end{aligned}$$

$$\begin{aligned}
&-\frac{\alpha_n\gamma f(W_nx_n)+((1-\beta_n)I-\alpha_nA)T_{\mu_n}W_ny_n}{1-\beta_n}\\
&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(W_{n+1}x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(W_nx_n)+T_{\mu_{n+1}}W_{n+1}y_{n+1}\\
&\quad-T_{\mu_n}W_ny_n+\frac{\alpha_n}{1-\beta_n}AT_{\mu_n}W_ny_n-\frac{\alpha_{n+1}}{1-\beta_n}AT_{\mu_{n+1}}W_{n+1}y_{n+1}\\
&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\gamma f(W_{n+1}x_{n+1})-AT_{\mu_{n+1}}W_{n+1}y_{n+1}\big)\\
&\quad+\frac{\alpha_n}{1-\beta_n}\big(AT_{\mu_n}W_ny_n-\gamma f(W_nx_n)\big)+T_{\mu_{n+1}}W_{n+1}y_{n+1}\\
&\quad-T_{\mu_{n+1}}W_{n+1}y_n+T_{\mu_{n+1}}W_{n+1}y_n-T_{\mu_n}W_ny_n.
\end{aligned}\tag{3.2}$$

From (2.3), we note that $z_n=S_{r_n}x_n$ and $z_{n+1}=S_{r_{n+1}}x_{n+1}$, we have

$$\theta(z_n,x)+\varphi(x)-\varphi(z_n)+\frac{1}{r_n}\big\langle K'(z_n)-K'(x_n),\eta(x,z_n)\big\rangle\geq 0,\tag{3.3}$$

$$\theta(z_{n+1},x)+\varphi(x)-\varphi(z_{n+1})+\frac{1}{r_{n+1}}\big\langle K'(z_{n+1})-K'(x_{n+1}),\eta(x,z_{n+1})\big\rangle\geq 0\tag{3.4}$$

for all $x\in C$. Putting $x=z_{n+1}$ in (3.3) and $x=z_n$ in (3.4), we have

$$\theta(z_n,z_{n+1})+\varphi(z_n+1)-\varphi(z_n)+\frac{1}{r_n}\big\langle K'(z_n)-K'(x_n),\eta(z_{n+1},z_n)\big\rangle\geq 0,\tag{3.5}$$

$$\theta(z_{n+1},z_n)+\varphi(z_n)-\varphi(z_{n+1})+\frac{1}{r_{n+1}}\big\langle K'(z_{n+1})-K'(x_{n+1}),\eta(z_n,z_{n+1})\big\rangle\geq 0.\tag{3.6}$$

After multiplying (3.5) and (3.6) by r_n and adding them together, we obtain

$$\left\langle\eta(z_{n+1},z_n),K'(z_n)-K'(x_n)-\frac{r_n}{r_{n+1}}\big(K'(z_{n+1})-K'(x_{n+1})\big)\right\rangle\geq 0.$$

Hence,

$$\begin{aligned}
&\bigg\langle\eta(z_{n+1},z_n),K'(z_n)-K'(z_{n+1})+K'(x_{n+1})-K'(x_n)\\
&\quad+\left(1-\frac{r_n}{r_{n+1}}\right)\big(K'(z_{n+1})-K'(x_{n+1})\big)\bigg\rangle\geq 0.
\end{aligned}$$

Then, by Lemma 2.3, we have

$$\begin{aligned}
&\left\langle\eta(z_{n+1},z_n),K'(x_{n+1})-K'(x_n)+\left(1-\frac{r_n}{r_{n+1}}\right)\big(K'(z_{n+1})-K'(x_{n+1})\big)\right\rangle\\
&\quad\geq\big\langle\eta(z_n,z_{n+1}),K'(z_n)-K'(z_{n+1})\big\rangle\geq\sigma\|z_n-z_{n+1}\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
&\sigma\|z_{n+1}-z_n\|^2\\
&\quad\leq\big\|\eta(z_{n+1},z_n)\big\|\left(\big\|K'(x_{n+1})-K'(x_n)\big\|+\left(1-\frac{r_n}{r_{n+1}}\right)\big\|K'(z_{n+1})-K'(x_{n+1})\big\|\right)\\
&\quad\leq\lambda\|z_{n+1}-z_n\|\left(\nu\|x_{n+1}-x_n\|+\left(1-\frac{r_n}{r_{n+1}}\right)\nu\|z_{n+1}-x_{n+1}\|\right).
\end{aligned}$$

Without loss of generality, we assume that there exists a real number k such that $r_n > k > 0$ for all $n \geq 0$, we have

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \frac{\lambda\nu}{\sigma}\|x_{n+1} - x_n\| + \frac{\nu}{\sigma}\frac{1}{k}|r_{n+1} - r_n|\|z_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{\nu}{\sigma k}|r_{n+1} - r_n|M', \end{aligned} \tag{3.7}$$

where $M' = \sup\{\|z_n - x_n\| : n \geq 0\}$.

Setting $v_n = P_C(I - \delta_n B)z_n$ for all $n \geq 0$, from Lemma 3.2, we have

$$\begin{aligned}\|v_{n+1} - v_n\| &\leq \|P_C(I - \delta_{n+1}B)z_{n+1} - P_C(I - \delta_n B)z_n\| \\ &\leq \|(I - \delta_{n+1}B)z_{n+1} - (I - \delta_n B)z_n\| \\ &\leq \|(I - \delta_{n+1}B)z_{n+1} - (I - \delta_{n+1}B)z_n\| + \|(\delta_{n+1} - \delta_n)Bz_n\| \\ &\leq \|z_{n+1} - z_n\| + |\delta_{n+1} - \delta_n|\|Bz_n\|.\end{aligned}$$

From (3.7), we get

$$\begin{aligned}&\|T_{\mu_{n+1}}W_{n+1}v_{n+1} - T_{\mu_n}W_nv_n\| \\ &\quad\leq \|T_{\mu_{n+1}}W_{n+1}v_{n+1} - T_{\mu_{n+1}}W_{n+1}v_n\| + \|T_{\mu_{n+1}}W_{n+1}v_n - T_{\mu_{n+1}}W_nv_n\| \\ &\qquad + \|T_{\mu_{n+1}}W_nv_n - T_{\mu_n}W_nv_n\| \\ &\quad\leq \|v_{n+1} - v_n\| + \|W_{n+1}v_n - W_nv_n\| + \|T_{\mu_{n+1}}W_nv_n - T_{\mu_n}W_nv_n\| \\ &\quad\leq \|z_{n+1} - z_n\| + |\delta_{n+1} - \delta_n|\|Bz_n\| \\ &\qquad + \|W_{n+1}v_n - W_nv_n\| + \|T_{\mu_{n+1}}W_nv_n - T_{\mu_n}W_nv_n\| \\ &\quad\leq \|x_{n+1} - x_n\| + \frac{\nu}{\sigma k}|r_{n+1} - r_n|M' + |\delta_{n+1} - \delta_n|\|Bz_n\| \\ &\qquad + \|W_{n+1}v_n - W_nv_n\| + \|T_{\mu_{n+1}}W_nv_n - T_{\mu_n}W_nv_n\|.\end{aligned} \tag{3.8}$$

Also, we have

$$\begin{aligned}&\|y_{n+1} - y_n\| \\ &\quad= \|(1 - \gamma_{n+1})x_{n+1} + \gamma_{n+1}T_{\mu_{n+1}}W_{n+1}v_{n+1} - (1 - \gamma_n)x_n - \gamma_nT_{\mu_n}W_nv_n\| \\ &\quad= \|(1 - \gamma_{n+1})(x_{n+1} - x_n) + (\gamma_n - \gamma_{n+1})x_n \\ &\qquad + (\gamma_{n+1} - \gamma_n)T_{\mu_n}W_nv_n + \gamma_n(T_{\mu_{n+1}}W_{n+1}v_{n+1} - T_{\mu_n}W_nv_n)\| \\ &\quad\leq (1 - \gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_n - \gamma_{n+1}|\|x_n\| \\ &\qquad + |\gamma_{n+1} - \gamma_n|\|T_{\mu_n}W_nv_n\| + \gamma_{n+1}\|T_{\mu_{n+1}}W_{n+1}v_{n+1} - T_{\mu_n}W_nv_n\|.\end{aligned} \tag{3.9}$$

From (3.2), we obtain

$$\begin{aligned}&\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\| \\ &\quad\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\left(\|\gamma f(W_{n+1}x_{n+1})\| + \|AT_{\mu_{n+1}}W_{n+1}y_{n+1}\|\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_n}{1-\beta_n}\left(\|AT_{\mu_n}W_ny_n\|+\|\gamma f(W_nx_n)\|\right)\\
&+\|T_{\mu_{n+1}}W_{n+1}y_{n+1}-T_{\mu_{n+1}}W_{n+1}y_n\|\\
&+\|T_{\mu_{n+1}}W_{n+1}y_n-T_{\mu_n}W_ny_n\|-\|x_{n+1}-x_n\|\\
\le{}&\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(W_{n+1}x_{n+1})\|+\|AT_{\mu_{n+1}}W_{n+1}y_{n+1}\|\right)\\
&+\frac{\alpha_n}{1-\beta_n}\left(\|AT_{\mu_n}W_ny_n\|+\|\gamma f(W_nx_n)\|\right)\\
&+\|y_{n+1}-y_n\|+\|T_{\mu_{n+1}}W_{n+1}y_n-T_{\mu_n}W_ny_n\|-\|x_{n+1}-x_n\|.
\end{aligned}
\tag{3.10}
$$

Combining (3.8), (3.9) and (3.10), we obtain

$$
\begin{aligned}
&\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|\\
&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(W_{n+1}x_{n+1})\|+\|AT_{\mu_{n+1}}W_{n+1}y_{n+1}\|\right)\\
&\qquad+\frac{\alpha_n}{1-\beta_n}\left(\|AT_{\mu_n}W_ny_n\|+\|\gamma f(W_nx_n)\|\right)\\
&\quad\le(1-\gamma_{n+1})\|x_{n+1}-x_n\|+|\gamma_n-\gamma_{n+1}|\|x_n\|\\
&\qquad+|\gamma_{n+1}-\gamma_n|\|T_{\mu_n}W_nv_n\|+\gamma_{n+1}\Bigg(\|x_{n+1}-x_n\|\\
&\qquad+\frac{\nu}{\sigma k}|r_{n+1}-r_n|M'+|\delta_{n+1}-\delta_n|\|Bz_n\|+\|W_{n+1}v_n-W_nv_n\|\\
&\qquad+\|T_{\mu_{n+1}}W_nv_n-T_{\mu_n}W_nv_n\|\Bigg)+\|T_{\mu_{n+1}}W_{n+1}y_n-T_{\mu_n}W_{n+1}y_n\|\\
&\qquad+\|T_{\mu_n}W_{n+1}y_n-T_{\mu_n}W_ny_n\|-\|x_{n+1}-x_n\|.
\end{aligned}
\tag{3.11}
$$

Thus, it follows from (3.11), Lemma 3.4, Lemma 3.5 and condition (C4) that

$$\limsup_{n\to\infty}\left(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|\right)\le 0.$$

By Lemma 2.8, we get

$$\lim_{n\to\infty}\|u_n-x_n\|=0.$$

Consequently, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|u_n-x_n\|=0. \tag{3.12}$$

From (3.7), we get

$$\lim_{n\to\infty}\|z_{n+1}-z_n\|=0. \qquad \square$$

Lemma 3.7 $\lim_{n\to\infty}\|x_n-T_tx_n\|=0$ *for all* $t\in S$.

Proof Let $p\in\mathcal{F}$ and put

$$M_0=\max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\right\}.$$

Set $D = \{y \in H : \|y - p\| \leq M_0\}$. We remark that D is a bounded closed convex set, $\{x_n\}, \{y_n\}, \{z_n\} \subset D$ and it is invariant under $\Im$ and W_n for all $n \in \mathbb{N}$. We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \|T_{\mu_n}y - T_t T_{\mu_n}y\| = 0, \tag{3.13}$$

for all $t \in S$. Let $\varepsilon > 0$, by [40] (Theorem 1.2), there exists $\delta > 0$ such that

$$\overline{\text{co}} F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D) \tag{3.14}$$

for all $t \in S$. By [40] (Corollary 1.1), there exists a natural number N such that

$$\left\| \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s}y - T_t\left(\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\right)\right\| \leq \delta, \tag{3.15}$$

for all $t, s \in S$ and $y \in D$. Since $\{\mu_n\}$ is left strong regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l^*_{t^i}\mu_n\| \leq \frac{\delta}{M_0 + \|p\|}$ for $n \geq n_0$ and $i = 1, 2, \ldots, N$. Then we have

$$\begin{aligned}
&\sup_{y\in D}\left\| T_{\mu_n}y - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y \, d\mu_n(s)\right\| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \langle T_{\mu_n}y, z\rangle - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\, d\mu_n(s), z\right\rangle\right| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \mu_n(s)\langle T_s y, z\rangle - \mu_n(s)\left\langle \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y, z\right\rangle\right| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \frac{1}{N+1}\sum_{i=0}^{N}\mu_n(s)\langle T_s y, z\rangle - \frac{1}{N+1}\sum_{i=0}^{N}\mu_n(s)\langle T_{t^i s}y, z\rangle\right| \\
&\quad \leq \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}\left|\mu_n(s)\langle T_s t, z\rangle - l^*_{t^i}\mu_n(s)\langle T_s y, z\rangle\right| \\
&\quad \leq \max_{i=0,1,2,\ldots,N}\frac{1}{N+1}\sum_{i=0}^{N}\left\|\mu_n - l^*_{t^i}\mu_n\right\| \|T_s y\| \\
&\quad \leq \max_{i=0,1,2,\ldots,N}\left\|\mu_n - l^*_{t^i}\mu_n\right\|\left(M_0 + \|p\|\right) \leq \delta,
\end{aligned} \tag{3.16}$$

for all $n \geq n_0$. By Lemma 2.1, we have

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\, d\mu_n(s) \in \overline{\text{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s y) : s \in S\right\}. \tag{3.17}$$

It follows from (3.14)-(3.17) that

$$\begin{aligned}
T_{\mu_n}y &= \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\, d\mu_n(s) + \left(T_{\mu_n}y - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\, d\mu_n(s)\right) \\
&\quad \in \overline{\text{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y : s \in S\right\} + B_\delta
\end{aligned}$$

$$\subset \overline{\text{co}} F_\delta(T_t; D) + B_\delta$$
$$\subset T_\varepsilon(T_t; D),$$

for all $y \in D$ and $n \geq n_0$. Therefore,

$$\limsup_{n\to\infty} \sup_{y\in D} \|T_t T_{\mu_n} y - T_{\mu_n} y\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (3.13). Now, let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$, which satisfies (3.14). Take $L_0 = (\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|$. From $\lim_{n\to\infty} \alpha_n = 0$, (3.12) and (3.13), there exists $k \in \mathbb{N}$ such that $\alpha_n < \frac{\delta(1-\beta_n)}{2L_0}$, $T_{\mu_n} y \in F_\delta(T_t; D)$ for all $y \in D$ and $\|x_n - x_{n+1}\| < \frac{\delta(1-\beta_n)}{2\beta_n}$ for all $n \geq k$. We note that

$$\begin{aligned}
&\big\|\alpha_n\big(\gamma f(W_n x_n) - AT_{\mu_n} W_n y_n\big)\big\| \\
&\quad \leq \alpha_n\big(\gamma \big\|f(W_n x_n) - f(p)\big\| + \big\|\gamma f(p) - Ap\big\| + \|AT_{\mu_n} W_n y_n - Ap\|\big) \\
&\quad \leq \alpha_n\big(\gamma\alpha \|W_n x_n - p\| + \big\|\gamma f(p) - Ap\big\| + \|A\| \|T_{\mu_n} W_n y_n - p\|\big) \\
&\quad \leq \alpha_n\big(\gamma\alpha \|x_n - p\| + \big\|\gamma f(p) - Ap\big\| + \|A\| \|y_n - p\|\big) \\
&\quad \leq \alpha_n\big(\big(\gamma\alpha + \|A\|\big)\|x_n - p\| + \big\|\gamma f(p) - Ap\big\|\big) \\
&\quad \leq \alpha_n\big(\big(\gamma\alpha + \|A\|\big)M_0 + \big\|\gamma f(p) - Ap\big\|\big) \\
&\quad \leq \frac{\delta(1-\beta_n)}{2},
\end{aligned}$$

for all $n \geq k$. Since

$$\begin{aligned}
&x_{n+1} - \beta_n x_{n+1} \\
&\quad = (1-\beta_n) T_{\mu_n} W_n y_n + \beta_n x_n + \alpha_n\big(\gamma f(W_n x_n) - AT_{\mu_n} W_n y_n\big) - \beta_n x_{n+1},
\end{aligned}$$

we get

$$\begin{aligned}
x_{n+1} &= T_{\mu_n} W_n y_n + \frac{\beta_n}{1-\beta_n}(x_n - x_{n+1}) \\
&\quad + \frac{\alpha_n}{1-\beta_n}\big(\gamma f(W_n x_n) - AT_{\mu_n} W_n y_n\big) \\
&\in F_\delta(T_t; D) + B_{\frac{\delta}{2}} + B_{\frac{\delta}{2}} \\
&\subset F_\delta(T_t; D) + B_\delta \\
&\subset F_\varepsilon(T_t; D),
\end{aligned}$$

for all $n \geq k$. This shows that

$$\limsup_{n\to\infty} \|x_n - T_t x_n\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$. □

Lemma 3.8 $\lim_{n\to\infty} \|x_n - T_{\mu_n} W_n y_n\| = 0$.

Proof Since $x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_{\mu_n} W_n y_n$, we have

$$\begin{aligned}\|x_n - T_{\mu_n} W_n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{\mu_n} W_n y_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n \big\|\gamma f(W_n x_n) - AT_{\mu_n} W_n y_n\big\| + \beta_n \|x_n - T_{\mu_n} W_n y_n\|,\end{aligned}$$

that is,

$$\|x_n - T_{\mu_n} W_n y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\big\|\gamma f(W_n x_n) - AT_{\mu_n} W_n y_n\big\|.$$

It follows from condition (C4) and Lemma 3.6 that

$$\lim_{n\to\infty} \|x_n - T_{\mu_n} W_n y_n\| = 0. \qquad \square$$

Lemma 3.9 $\lim_{n\to\infty} \|x_n - z_n\| = 0$.

Proof For $p \in \mathcal{F}$, since S_r is firmly nonexpansive, we have

$$\begin{aligned}\|z_n - p\|^2 &= \|S_{r_n} x_n - S_{r_n} p\|^2 \\ &\le \langle S_{r_n} x_n - S_{r_n} p, x_n - p\rangle = \langle z_n - p, x_n - p\rangle \\ &= \frac{1}{2}\big(\|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2\big)\end{aligned}$$

and hence

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|z_n - x_n\|^2. \tag{3.18}$$

Note that the following equality holds:

$$\big\|tx + (1-t)y\big\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$$

for all $t \in [0,1]$ and $x, y \in H$. So, from (1.6) and (3.18), we get

$$\begin{aligned}\|y_n - p\|^2 &= \big\|(1-\gamma_n)(x_n - p) + \gamma_n\big(T_{\mu_n} W_n P_C(I - \delta_n B)z_n - p\big)\big\|^2 \\ &\le \gamma_n \big\|T_{\mu_n} W_n P_C(I - \delta_n B)z_n - p\big\|^2 + (1-\gamma_n)\|x_n - p\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\big\|T_{\mu_n} W_n P_C(I - \delta_n B)z_n - x_n\big\|^2 \\ &= \gamma_n\|z_n - p\|^2 + (1-\gamma_n)\|x_n - p\|^2 - (1-\gamma_n)\|y_n - x_n\|^2 \\ &\le \gamma_n\big(\|x_n - p\|^2 - \|z_n - x_n\|^2\big) + (1-\gamma_n)\|x_n - p\|^2 \\ &= \|x_n - p\|^2 - \gamma_n\|z_n - x_n\|^2.\end{aligned} \tag{3.19}$$

Therefore, from Lemma 2.5, Lemma 3.1 and (3.19), we have

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad = \big\|\alpha_n\big(\gamma f(W_n x_n) - Ap\big) + \beta_n(x_n - T_{\mu_n} W_n y_n) + (I - \alpha_n A)(T_{\mu_n} W_n y_n - p)\big\|^2\end{aligned}$$

$$
\begin{aligned}
&\le \left\| \beta_n(x_n - T_{\mu_n} W_n y_n) + (I - \alpha_n A)(T_{\mu_n} W_n y_n - p) \right\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(W_n x_n) - Ap, x_{n+1} - p \rangle \\
&\le \left(\|I - \alpha_n A\| \|y_n - p\| + \beta_n \|x_n - T_{\mu_n} W_n y_n\| \right)^2 \\
&\quad + 2\alpha_n \left\| \gamma f(W_n x_n) - Ap \right\| \|x_{n+1} - p\| \\
&\le \left((1 - \alpha_n \bar{\gamma}) \|y_n - p\| + \beta_n \|x_n - T_{\mu_n} W_n y_n\| \right)^2 \\
&\quad + 2\alpha_n \left\| \gamma f(W_n x_n) - Ap \right\| \|x_{n+1} - p\| \\
&\le (1 - \alpha_n \bar{\gamma})^2 \|y_n - p\|^2 + \beta_n^2 \|x_n - T_{\mu_n} W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n \bar{\gamma}) \beta_n \|y_n - p\| \|x_n - T_{\mu_n} W_n y_n\| \\
&\quad + 2\alpha_n \left\| \gamma f(W_n x_n) - Ap \right\| \|x_{n+1} - p\| \\
&\le (1 - \alpha_n \bar{\gamma})^2 \left(\|x_n - p\|^2 - \gamma \|z_n - x_n\|^2 \right) + \beta_n^2 \|x_n - T_{\mu_n} W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n \bar{\gamma}) \beta_n \|y_n - p\| \|x_n - T_{\mu_n} W_n y_n\| \\
&\quad + 2\alpha_n \left\| \gamma f(W_n x_n) - Ap \right\| \|x_{n+1} - p\|.
\end{aligned}
$$

Then we derive

$$
\begin{aligned}
&(1 - \alpha_n \bar{\gamma})^2 \gamma_n \|z_n - x_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2 \bar{\gamma}^2 \|x_n - p\|^2 + \beta_n^2 \|x_n - T_{\mu_n} W_n y_n\|^2 \\
&\qquad + 2(1 - \alpha_n \bar{\gamma}) \beta_n \|y_n - p\| \|x_n - T_{\mu_n} W_n y_n\| \\
&\qquad + 2\alpha_n \left\| \gamma f(W_n x_n) - Ap \right\| \|x_{n+1} - p\| \\
&\quad \le \left(\|x_n - p\| + \|x_{n+1} - p\| \right) \|x_{n+1} - x_n\| + \alpha_n^2 \bar{\gamma}^2 \|x_n - p\|^2 \\
&\qquad + \beta_n^2 \|x_n - T_{\mu_n} W_n y_n\|^2 + 2(1 - \alpha_n \bar{\gamma}) \beta_n \|y_n - p\| \|x_n - T_{\mu_n} W_n y_n\| \\
&\qquad + 2\alpha_n \left\| \gamma f(W_n x_n) - Ap \right\| \|x_{n+1} - p\|.
\end{aligned} \tag{3.20}
$$

So, from (C4), Lemma 3.6, Lemma 3.8 and (3.20), we obtain

$$
\lim_{n \to \infty} \|x_n - z_n\| = 0. \qquad \square
$$

Lemma 3.10 $\lim_{n\to\infty} \|y_n - v_n\| = \lim_{n\to\infty} \|z_n - v_n\| = 0$, *where* $v_n = P_C(I - \delta_n B) z_n$ *for all* $n \ge 0$.

Proof Let $p \in \mathcal{F}$. Setting $v_n = P_C(I - \delta_n B) z_n$ for all $n \ge 0$, since $p \in VI(C, B)$, we have $p = P_C(I - \delta_n B)p$. From the L-Lipschitzian and relaxed (ξ, ν)-cocoercive mapping on B and Lemma 3.3, we have

$$
\begin{aligned}
&\|v_n - p\|^2 \\
&\quad = \left\| P_C(I - \delta_n B) z_n - P_C(I - \delta_n B)p \right\|^2 \\
&\quad \le \left\| z_n - p - \delta_n (B z_n - Bp) \right\|^2 \\
&\quad = \|z_n - p\|^2 - 2\delta_n \langle B z_n - Bp, z_n - p \rangle + \delta_n^2 \|B z_n - Bp\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \|x_n - p\|^2 - 2\delta_n\big(-\xi\|Bz_n - Bp\|^2 + \nu\|z_n - p\|^2\big) + \delta_n^2\|Bz_n - Bp\|^2 \\
&\le \|x_n - p\| + \left(2\delta_n\xi + \delta_n^2 - \frac{2\delta_n\nu}{L^2}\right)\|Bz_n - Bp\|^2. \qquad (3.21)
\end{aligned}
$$

From (1.6) and (3.21), we get

$$
\begin{aligned}
\|y_n - p\|^2 &= \big\|(1-\gamma_n)x_n + \gamma_n T_{\mu_n} W_n v_n - p\big\|^2 \\
&= \big\|(1-\gamma_n)(x_n - p) + \gamma_n(T_{\mu_n} W_n v_n - p)\big\|^2 \\
&\le (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|v_n - p\|^2 \\
&\le \|x_n - p\|^2 + \gamma_n\left(2\delta_n\xi + \delta_n^2 - \frac{2\delta_n\nu}{L^2}\right)\|Bz_n - Bp\|^2. \qquad (3.22)
\end{aligned}
$$

From (1.6), Lemma 3.1 and (3.22), we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\|\alpha_n\big(\gamma f(W_n x_n) - Ap\big) + \beta_n(x_n - p) \\
&\qquad + \big((1-\beta_n)I - \alpha_n A\big)(T_{\mu_n} W_n y_n - p)\big\|^2 \\
&\quad \le \big(\alpha_n\big\|\gamma f(W_n x_n) - Ap\big\| + \beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\bar{\gamma})\|y_n - p\|\big)^2 \\
&\quad = \alpha_n^2\big\|\gamma f(W_n x_n) - Ap\big\|^2 + \beta_n^2\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})^2\|y_n - p\|^2 \\
&\qquad + 2\alpha_n\big(\beta_n\big\|\gamma f(W_n x_n) - Ap\big\|\|x_n - p\| \\
&\qquad + (1-\beta_n-\alpha_n\bar{\gamma})\big\|\gamma f(W_n x_n) - Ap\big\|\|y_n - p\|^2\big) \\
&\qquad + 2\beta_n(1-\beta_n-\alpha_n\bar{\gamma})\|x_n - p\|\|y_n - p\| \\
&\quad \le \alpha_n M + \beta_n^2\|x_n - p\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})^2\|y_n - p\| \\
&\qquad + \beta_n(1-\beta_n-\alpha_n\bar{\gamma})\big(\|x_n - p\|^2 + \|y_n - p\|^2\big) \\
&\quad = \alpha_n M + \beta_n(1-\alpha_n\bar{\gamma})\|x_n - p\|^2 + (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|y_n - p\|^2 \\
&\quad \le \alpha_n M + \beta_n(1-\alpha_n\bar{\gamma})\|x_n - p\|^2 + (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma}) \\
&\qquad \times \left(\|x_n - p\|^2 + \gamma_n\left(2\delta_n\xi + \delta_n^2 - \frac{2\delta_n\nu}{L^2}\right)\|Bz_n - Bp\|^2\right) \\
&\quad \le \alpha_n M + (1-\alpha_n\bar{\gamma})^2\|x_n - p\|^2 \\
&\qquad + (1-\alpha_n\bar{\gamma})^2\gamma_n\left(2\delta_n\xi + \delta_n^2 - \frac{2\delta_n\nu}{L^2}\right)\|Bz_n - Bp\|^2 \\
&\quad = \alpha_n M + \|x_n - p\|^2 \\
&\qquad + (1-\alpha_n\bar{\gamma})^2\gamma_n\left(2\delta_n\xi + \delta_n^2 - \frac{2\delta_n\nu}{L^2}\right)\|Bz_n - Bp\|^2. \qquad (3.23)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 &\le -(1-\alpha_n\bar{\gamma})^2\gamma_n\left(2\delta_n\xi + \delta_n^2 - \frac{2\delta_n\nu}{L^2}\right)\|Bz_n - Bp\|^2 \\
&\le \alpha_n M + \|x_n - p\|^2 - \|x_{n+1} - p\|^2
\end{aligned}
$$

$$\begin{aligned}&\le \alpha_n M + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\|\\ &\longrightarrow 0,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|Bz_n - Bp\| = 0. \tag{3.24}$$

On the other hand, since P_C is firmly nonexpansive, by Lemma 3.2, we have

$$\begin{aligned}\|v_n - p\|^2 &= \big\|P_C(I - \delta_n B)z_n - P_C(I - \delta_n B)p\big\|^2\\ &\le \big\langle (I - \delta_n B)z_n - (I - \delta_n B)p, v_n - p\big\rangle\\ &= \frac{1}{2}\Big\{\big\|(I - \delta_n B)z_n - (I - \delta_n B)p\big\|^2 + \|v_n - p\|^2 - \big\|z_n - v_n - \gamma_n(Bz_n - Bp)\big\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|z_n - p\|^2 + \|v_n - p\|^2 - \|z_n - v_n\|^2\\ &\quad + 2\gamma_n\|z_n - v_n\|\|Bz_n - Bp\| - \gamma_n^2\|Bz_n - Bp\|^2\Big\},\end{aligned}$$

which yields that

$$\begin{aligned}\|v_n - p\|^2 &\le \|z_n - p\|^2 - \|z_n - v_n\|^2 + 2\gamma_n\|z_n - v_n\|\|Bz_n - Bp\|\\ &\le \|x_n - p\|^2 - \|z_n - v_n\|^2 + 2\gamma_n\|z_n - v_n\|\|Bz_n - Bp\|.\end{aligned} \tag{3.25}$$

Combining (3.22) and (3.25), we obtain

$$\begin{aligned}\|y_n - p\|^2 &\le (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n\|v_n - p\|^2\\ &\le (1 - \gamma_n)\|x_n - p\|^2 + \gamma_n\big(\|x_n - p\|^2 - \|z_n - v_n\|^2\\ &\quad + 2\gamma_n\|z_n - v_n\|\|Bz_n - Bp\|\big)\\ &= \|x_n - p\|^2 - \gamma_n\|z_n - v_n\|^2 + 2\gamma_n\|z_n - v_n\|\|Bz_n - Bp\|.\end{aligned} \tag{3.26}$$

Therefore, from (3.23) and (3.26), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n M + \beta_n(1 - \alpha_n\bar{\gamma})\|x_n - p\|^2 + (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - p\|^2\\ &\le \alpha_n M + \beta_n(1 - \alpha_n\bar{\gamma})\|x_n - p\|^2 + (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\\ &\quad \times \big(\|x_n - p\|^2 - \gamma\|z_n - v_n\|^2 + 2\gamma_n^2\|z_n - v_n\|\|Bz_n - Bp\|\big)\\ &\le \alpha_n M + (1 - \alpha_n\bar{\gamma})^2\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\gamma_n\|z_n - v_n\|^2\\ &\quad + 2(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\gamma_n^2\|z_n - v_n\|\|Bz_n - Bp\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}&(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\gamma_n\|z_n - v_n\|^2\\ &\quad \le \alpha_n M + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_{n+1} - x_n\|\\ &\qquad + 2(1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\gamma_n^2\|z_n - v_n\|\|Bz_n - Bp\|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|z_n - v_n\| = 0. \tag{3.27}$$

Observe that

$$\begin{aligned}
\|y_n - v_n\| &\le \|y_n - x_n\| + \|x_n - z_n\| + \|z_n - v_n\| \\
&\le \gamma_n \|T_{\mu_n} W_n v_n - x_n\| + \|x_n - z_n\| + \|z_n - v_n\| \\
&\le \gamma_n \big(\|T_{\mu_n} W_n v_n - T_{\mu_n} W_n y_n\| + \|T_{\mu_n} W_n y_n - x_n\|\big) \\
&\quad + \|x_n - z_n\| + \|z_n - v_n\| \\
&\le \gamma_n \big(\|v_n - y_n\| + \|T_{\mu_n} W_n y_n - x_n\|\big) + \|x_n - z_n\| + \|z_n - v_n\|,
\end{aligned}$$

and hence

$$(1-\gamma_n)\|y_n - v_n\| \le \gamma_n \|T_{\mu_n} W_n y_n - x_n\| + \|x_n - z_n\| + \|z_n - v_n\|.$$

Thus, from Lemma 3.8, Lemma 3.9, (3.27) and (C4), we derive

$$\lim_{n\to\infty} \|y_n - v_n\| = 0. \qquad \square$$

Lemma 3.11 *$P_{\mathcal{F}}(\gamma f + (I - A))$ is a contraction of H into itself.*

Proof From Lemma 3.1, we have

$$\begin{aligned}
&\big\|P_{\mathcal{F}}\big(\gamma f + (I-A)\big)x - P_{\mathcal{F}}\big(\gamma f + (I-A)\big)y\big\| \\
&\quad \le \big\|\gamma f(x) + (I-A)x - \gamma f(y) - (I-A)y\big\| \\
&\quad \le \gamma \big\|f(x) - f(y)\big\| + \big\|(I-A)(x-y)\big\| \\
&\quad \le \gamma\alpha \|x-y\| + (1-\bar{\gamma})\|x-y\| \\
&\quad = \big(1 - (\bar{\gamma} - \gamma\alpha)\big)\|x-y\|,
\end{aligned}$$

for all $x, y \in H$. From the condition $\bar{\gamma}$, $0 < \alpha\gamma < \bar{\gamma} < \alpha\gamma + 1$, we obtain $1 - (\bar{\gamma} - \gamma\alpha) \in (0,1)$. Therefore, $P_{\mathcal{F}}(\gamma f + (I - A))$ is a contraction. □

Now, we prove Theorem 3.1.

Proof of Theorem 3.1 From Lemma 3.11 and the Banach contraction principle, $P_{\mathcal{F}}(\gamma f + (I - A))$ has a unique fixed point, say $x^* \in H$. That is, $x^* = P_{\mathcal{F}}(\gamma f + (I - A))x^*$. Then, using (2.1), x^* is the unique solution of the variational inequality

$$\big\langle (\gamma f - A)x^*, x - x^* \big\rangle \le 0 \tag{3.28}$$

for all $x \in \mathcal{F}$. Now, we show that

$$\limsup_{n\to\infty} \big\langle (\gamma f - A)x^*, x_n - x^* \big\rangle \le 0. \tag{3.29}$$

To show this, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle(\gamma f - A)x^*, x_{n_i} - x^*\rangle. \tag{3.30}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup z$. We need to show that $z \in \mathcal{F} = F(\Im) \cap \Omega \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap VI(C,B)$.

(I) Since $x_{n_i} \rightharpoonup z$, by Lemma 2.2 and Lemma 3.7, we get

$$T_t z = z$$

for all $t \in S$. Therefore, $z \in F(\Im)$.

(II) Now, we show that $z \in \Omega$. Since $z_n = S_{r_n} x_n$, we derive

$$\theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r_n}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0,$$

for all $x \in C$. From the monotonicity of θ, we have

$$\theta(x, z_n) \leq -\theta(z_n, x) \leq \varphi(x) - \varphi(z_n) + \frac{1}{r_n}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle,$$

and hence

$$\theta(x, z_{n_i}) \leq \varphi(x) - \varphi(z_{n_i}) + \left\langle \frac{K'(z_{n_i}) - K'(x_{n_i})}{r_{n_i}}, \eta(x, z_{n_i})\right\rangle.$$

Since $\frac{K'(z_{n_i}) - K'(x_{n_i})}{r_{n_i}} \to 0$ and $z_{n_i} \rightharpoonup z$, from the lower semicontinuity of φ and (E4), we have

$$\theta(x, z) + \varphi(z) - \varphi(x) \leq 0,$$

for all $x \in C$. For t with $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)z$. Since $x \in C$ and $z \in C$, we have $x_t \in C$ and

$$\theta(x_t, z) + \varphi(z) - \varphi(x_t) \leq 0.$$

From (E1), (E4) and the convexity of φ, we get

$$\begin{aligned} 0 &= \theta(x_t, x_t) + \varphi(x_t) - \varphi(x_t) \\ &\leq t\theta(x_t, x) + (1-t)\theta(x_t, z) + t\varphi(x) + (1-t)\varphi(z) - \varphi(x_t) \\ &\leq t\big(\theta(x_t, x) + \varphi(x) - \varphi(x_t)\big). \end{aligned}$$

Hence,

$$\theta(x_t, x) + \varphi(x) - \varphi(x_t) \geq 0,$$

for all $x \in C$. From (E3) and the lower semicontinuity of φ, we have

$$\theta(z,x) + \varphi(x) - \varphi(z) \geq 0,$$

for all $x \in C$. Therefore, $z \in \Omega$.

(III) We show that $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $z \notin F(W)$, then $z \neq Wz$. Since $z \in F(\Im) \cap \Omega$, by our assumption, we have $T_i z \in F(\Im)$, $\forall i \in \mathbb{N}$ and then $W_n z \in F(\Im)$. From Lemma 2.1, we get

$$T_{\mu_n} W_n z = W_n z, \tag{3.31}$$

for all $n \in \mathbb{N}$. Since

$$\|y_n - x_n\| \leq \|y_n - v_n\| + \|v_n - z_n\| + \|z_n - x_n\|,$$

from Lemma 3.9 and Lemma 3.10, we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.32}$$

By Lemma 2.4, Lemma 3.8, (3.31) and (3.32), we obtain

$$\begin{aligned}
&\liminf_{i\to\infty} \|x_{n_i} - z\| \\
&\quad < \liminf_{i\to\infty} \|x_{n_i} - Wz\| \\
&\quad \leq \liminf_{i\to\infty} \bigl(\|x_{n_i} - T_{\mu_{n_i}} W_{n_i} y_{n_i}\| + \|T_{\mu_{n_i}} W_{n_i} y_{n_i} - T_{\mu_{n_i}} W_{n_i} x_{n_i}\| \\
&\qquad + \|T_{\mu_{n_i}} W_{n_i} x_{n_i} - T_{\mu_{n_i}} W_{n_i} z\| + \|T_{\mu_{n_i}} W_{n_i} z - Wz\|\bigr) \\
&\quad \leq \liminf_{i\to\infty} \bigl(\|y_{n_i} - x_{n_i}\| + \|x_{n_i} - z\|\bigr) \\
&\quad \leq \liminf_{i\to\infty} \|x_{n_i} - z\|.
\end{aligned}$$

This is a contraction. Therefore, $z \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

(IV) We show that $z \in VI(C,B)$. Let $U : H \to 2^H$ be a set-valued mapping defined by

$$Ux = \begin{cases} Bx + N_C x, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases}$$

where $N_C x$ is the normal cone to C at $x \in C$. By assumption of B, we have

$$\begin{aligned}
\langle Bx - By, x - y\rangle &\geq -\xi \|Bx - By\|^2 + \nu \|x - y\|^2 \\
&\geq \bigl(\nu - \xi L^2\bigr)\|x - y\|^2 \\
&\geq 0,
\end{aligned} \tag{3.33}$$

which implies that B is monotone. Thus, U is a maximal monotone. Let $(u,v) \in G(U)$. Since $v - Bu \in N_C u$ and $v_n = P_C(I - \delta_n B)z_n \in C$, we have

$$\langle u - v_n, v - Bu\rangle \geq 0. \tag{3.34}$$

On the other hand, from (2.1), we have

$$\langle u - v_n, v_n - (I - \delta_n B)z_n\rangle \geq 0$$

and hence

$$\left\langle u - v_n, \frac{v_n - z_n}{\delta_n} + Bz_n \right\rangle \geq 0.$$

It follows by (3.33) and (3.34) that

$$\begin{aligned}
\langle u - v_{n_i}, v\rangle &\geq \langle u - v_{n_i}, Bu\rangle \\
&\geq \langle u - v_{n_i}, Bu\rangle - \left\langle u - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\delta_{n_i}} + Bz_{n_i} \right\rangle \\
&= \left\langle u - v_{n_i}, Bu - Bz_{n_i} - \frac{v_{n_i} - z_{n_i}}{\delta_{n_i}} \right\rangle \\
&= \langle u - v_{n_i}, Bu - Bv_{n_i}\rangle + \langle u - v_{n_i}, Bv_{n_i} - Bz_{n_i}\rangle \\
&\quad - \left\langle u - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\delta_{n_i}} \right\rangle \\
&\geq \langle u - v_{n_i}, Bv_{n_i} - Bz_{n_i}\rangle - \left\langle u - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\delta_{n_i}} \right\rangle.
\end{aligned}$$

From Lemma 3.9 and Lemma 3.10, we obtain $\langle u - z, v\rangle \geq 0$ as $i \to \infty$. Since U is maximal monotone, we have $z \in U^{-1}(0)$. Therefore, $z \in VI(C,B)$. By (I)-(IV), $z \in \mathcal{F} = F(\Im) \cap \Omega \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap VI(C,B)$. Since $x^* = P_{\mathcal{F}}(\gamma f + (I - A))x^*$, from (3.28), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle &= \limsup_{i\to\infty}\langle(\gamma f - A)x^*, x_{n_i} - x^*\rangle \\
&= \limsup_{i\to\infty}\langle(\gamma f - A)x^*, z_{n_i} - x^*\rangle \\
&= \langle(\gamma f - A)x^*, z - x^*\rangle \\
&\leq 0.
\end{aligned}$$

(V) Finally, we prove that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to x^*. From (1.6), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Ax^*) + \beta_n(x_n - x^*) \\
&\quad + ((1 - \beta_n)I - \alpha_n A)(T_{\mu_n} W_n y_n - x^*)\|^2 \\
&\leq \|\beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n A)(T_{\mu_n} W_n y_n - x^*)\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(W_n x_n) - Ax^*, x_{n+1} - x^*\rangle \\
&\leq (\beta_n\|x_n - x^*\| + \|((1 - \beta_n)I - \alpha_n A)(T_{\mu_n} W_n y_n - x^*)\|)^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\alpha_n\gamma\langle f(W_nx_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\le \left(\beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - x^*\|\right)^2 \\
&\quad + 2\alpha_n\gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \frac{1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\quad = \left(1 - \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right)\|x_n - x^*\|^2 + \frac{(\alpha_n\bar{\gamma})^2}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\
&\qquad + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\
&\quad = \left(1 - \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right)\|x_n - x^*\|^2 \\
&\qquad + \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\left(\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma\alpha)}\|x_n - x^*\|^2 + \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle\right).
\end{aligned}
$$

It follows that

$$
\|x_{n+1} - x^*\|^2 \le (1 - b_n)\|x_n - x^*\| + b_nc_n, \tag{3.35}
$$

where

$$
\begin{aligned}
b_n &= \frac{2(\bar{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}, \\
c_n &= \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma\alpha)}\|x_n - x^*\|^2 + \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle.
\end{aligned}
$$

From (C4)-(i), we have $\sum_{n=0}^{\infty} b_n = \infty$, and by (3.29), we get $\limsup_{n\to\infty} c_n \le 0$. Consequently, applying Lemma 2.10 to (3.35), we get $\|x_n - x^*\| \to 0$. Therefore,

$$
\lim_{n\to\infty} x_n = x^*.
$$

From Lemma 3.9 and (3.32), we obtain

$$
\lim_{n\to\infty} y_n = x^* \quad \text{and} \quad \lim_{n\to\infty} z_n = x^*.
$$

This completes the proof of Theorem 3.1. □

Corollary 3.1 *Let* $H, C, S, \Im, X, \{\mu_n\}, \varphi, \theta, T_i, f, A, \{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *be as in Theorem* 3.1. *Suppose that* $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \cap F(\Im) \cap \Omega \neq \emptyset$. *Assume that*

(C1) $\eta : C \times C \to H$ *is Lipschitz continuous with constant* $\lambda > 0$ *such that*
  (a) $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in C$,
  (b) *for each fixed* $y \in C$, $x \mapsto \eta(y,x)$ *is sequentially continuous from the weak topology to the weak topology*;

(C2) $K : C \to \mathbb{R}$ *is* η*-strongly convex with constant* $\sigma > 0$ *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology, but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq \lambda\nu$;

(C3) *for each* $x \in C$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \in C \setminus D_x$,

$$\theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0;$$

(C4) (i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
  (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
  $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,
  (iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(C5) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Given $x_0 \in C$ *is arbitrary, let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be generated by*

$$\begin{cases} \theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r_n}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \\ y_n = (1-\gamma_n)x_n + \gamma_n T_{\mu_n} W_n P_C z_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) T_{\mu_n} W_n y_n, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $x^* \in \mathcal{F}$, *where* $x^* = P_{\mathcal{F}}(\gamma f + (I - A))x^*$, *which solves the following variational inequality*:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

Proof Setting $B = 0$ in Theorem 3.1, we obtain the required result. □

Corollary 3.2 *Let* H, C, S, $\Im$, X, $\{\mu_n\}$, T_i, f, A, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be as in Theorem* 3.1. *Suppose that* $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \cap F(\Im) \neq \emptyset$. *Assume that*

(C1) (i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
  (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
  $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,
  (iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Given $x_0 \in C$ *is arbitrary, let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be generated by*

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T_{\mu_n} W_n P_C x_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) T_{\mu_n} W_n y_n, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $x^* \in \mathcal{F}$, *where* $x^* = P_{\mathcal{F}}(\gamma f + (I - A))x^*$, *which solves the following variational inequality*:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}.$$

Proof Set $B = 0$, $\theta(x,y) = 0$ for all $x, y \in C$, $\varphi = 0$ and $r_n = 1$ for all $n \geq 1$. Take $K(x) = \frac{\|x\|^2}{2}$ and $\eta(x,y) = x - y$ for all $x, y \in C$. From (1.6), we have

$$\begin{cases} y_n = (1 - \gamma_n)x_n + \gamma_n T_{\mu_n} W_n P_C x_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) T_{\mu_n} W_n y_n, \quad n \geq 0. \end{cases}$$

Then the conclusion immediately follows from Theorem 3.1. □

Corollary 3.3 *Let* H, C, S, $\Im$, X, $\{\mu_n\}$, φ, θ, f, A, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be as in Theorem* 3.1. *Suppose that* $\Omega \neq \emptyset$. *Assume that*

(C1) $\eta : C \times C \to H$ *is Lipschitz continuous with constant* $\lambda > 0$ *such that*
  (a) $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in C$,
  (b) *for each fixed* $y \in C$, $x \mapsto \eta(y,x)$ *is sequentially continuous from the weak topology to the weak topology*;

(C2) $K : C \to \mathbb{R}$ *is* η-*strongly convex with constant* $\sigma > 0$, *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology, but also Lipschitz continuous with constant* $\nu > 0$ *such that* $\sigma \geq \lambda\nu$;

(C3) *for each* $x \in C$, *there exist a bounded subset* $D_x \subset C$ *and* $z_x \in C$ *such that for any* $y \in C \setminus D_x$,

$$\theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y) \rangle < 0;$$

(C4) (i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
  (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
  $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,
  (iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$;

(C5) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Given $x_0 \in C$ *is arbitrary, let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be generated by*

$$\begin{cases} \theta(z_n, x) + \varphi(x) - \varphi(z_n) + \frac{1}{r_n}\langle K'(z_n) - K'(x_n), \eta(x, z_n)\rangle \geq 0, \\ y_n = (1 - \gamma_n)x_n + \gamma_n P_C z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)y_n, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $x^* \in \Omega$, *where* $x^* = P_{\Omega}(\gamma f + (I - A))x^*$, *which solves the following variational inequality*:

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega.$$

Proof Set $B = 0$ and $T_i x = x$ for all $i = 1, 2, \ldots$ in (1.6). Then $W_n x = x$ for all $x \in C$. The conclusion immediately follows from Theorem 3.1. □

Competing interests
The author declares that they have no competing interests.

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