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Interior fixed points of unit-sphere-preserving Euclidean maps

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Abstract
Schirmer proved that there is a class of smooth self-maps of the unit sphere in Euclidean *n*-space with the property that any smooth self-map of the unit ball that extends a map of that class must have at least one fixed point in the interior of the ball. We generalize Schirmer's result by proving that a smooth self-map of Euclidean *n*-space that extends a self-map of the unit sphere of that class must have at least one fixed point in the interior of the unit ball.
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1 Introduction

For spaces X, Y and subsets $V \subseteq X$, $W \subseteq Y$, a map $f: X \to Y$ is an *extension* of a map $\phi: V \to W$ if $f(x) = \phi(x)$ for all $x \in V$. We denote by B^n the unit ball in $\mathbb{R}^n$, by S^{n-1} its boundary and by $\mathrm{int}(B^n)$ its interior.

If $f: B^1 \to B^1$ is an extension of $\phi: S^0 \to S^0 = \{-1, 1\}$ and ϕ has no fixed points, then f must have an *interior* fixed point, that is, a fixed point in $\mathrm{int}(B^1)$. However, if ϕ has a fixed point, then there need not be any interior fixed points.

If $n = 2$, the situation is more complicated. Of course the Brouwer fixed point theorem implies that a map $f: B^2 \to B^2$ must have at least one interior fixed point if it is an extension of a map $\phi: S^1 \to S^1$ that has no fixed points. But it was proved in [1] (see also [2]) that if the extension f is smooth, it may still be required to have interior fixed points for certain maps ϕ that have many fixed points. Representing the points of S^1 by complex numbers, let $\phi = \phi_d: S^1 \to S^1$, for an integer d, be the *power map* defined by $\phi_d(z) = z^d$. If $d \geq 2$ and $f: B^2 \to B^2$ is a smooth extension of ϕ_d, then f has at least one interior fixed point. It is also demonstrated in [1] that interior fixed points of extensions need not exist if $d \leq 1$ or if f is not smooth. Schirmer generalized this interior fixed point result to smooth extensions $f: B^n \to B^n$ for $n \geq 2$ to show in Example 4.7 of [3] that if f is a smooth extension of a 'sparse' map $\phi: S^{n-1} \to S^{n-1}$, a generalization of ϕ_d that is defined below, of degree d such that $(-1)^n d \geq 2$, then f must have at least one interior fixed point.

Returning to the case $n = 1$, if we extend the map $\phi: S^0 \to S^0$ without fixed points to a map $f: \mathbb{R}^1 \to \mathbb{R}^1$, there still must be a fixed point of f in $\mathrm{int}(B^1)$. The reason for the interior fixed points of the extension $f: B^1 \to B^1$ of the map of S^0 without fixed points, namely that $(-1, 1)$ and $(1, -1)$, lie in different components of $B^1 \times B^1 \setminus \Delta$, where $\Delta = \{(x, x): x \in B^1\}$, applies also to the extension $f: \mathbb{R}^1 \to \mathbb{R}^1$ since those points are also in different components of $B^1 \times \mathbb{R}^1 \setminus \Delta$.

On the other hand, the reason for the presence of fixed points in $\mathrm{int}(B^n)$ for smooth extensions of certain maps of S^{n-1} demonstrated in [3] is considerably more subtle. Therefore, it is reasonable to ask whether such fixed points would persist if, instead of smooth extensions $f: B^n \to B^n$ of $\phi: S^{n-1} \to S^{n-1}$, we consider extensions that are smooth *Euclidean maps*, that is, maps $f: \mathbb{R}^n \to \mathbb{R}^n$. Thus, we ask whether there still must be fixed points of f in $\mathrm{int}(B^n)$ if we allow f to map points of $\mathrm{int}(B^n)$ outside of B^n.

We will prove that the interior fixed points do persist, even in this more general setting. As in the case of self-maps of balls, the interior fixed points of Euclidean maps are detected by means of a theorem that relates the index of a fixed point of $\phi: S^{n-1} \to S^{n-1}$ to its index as a fixed point of an extension. We will therefore devote Section 2 to a discussion of the properties of the fixed point index that we will use. In Section 3, we prove that a smooth extension $f: \mathbb{R}^2 \to \mathbb{R}^2$ of a power map $\phi_d: S^1 \to S^1$ for $d \geq 2$ must have at least one interior fixed point. Section 4 then contains the proof that Schirmer's result generalizes to smooth extensions $f: \mathbb{R}^n \to \mathbb{R}^n$ of sparse maps $\phi: S^{n-1} \to S^{n-1}$ that satisfy the same degree restrictions.

2 The fixed point indices

Before extending the results of [1, 2] and [3] to the case of a smooth Euclidean map $f: (\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$ extending $\phi: S^{n-1} \to S^{n-1}$, we need to define the relevant fixed point indices. We will consider the restriction $f: (B^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$. Since our goal is to establish conditions for the existence of fixed points on the interior of B^n, the behavior of the function outside of B^n is not relevant. Therefore, we will make use of the indices $i(B^n, f, p)$ and $i(S^{n-1}, \phi, p)$ of an isolated fixed point $p \in S^{n-1}$. We do so by generalizing the approach used in [1] (see also [4]).

For an isolated fixed point $p \in S^{n-1}$, we can choose a small enough neighborhood U so that it contains only this fixed point and no other. We then may write f in this neighborhood of p in terms of a local coordinate system in which $U \cap B^n$ is contained in the upper half-space

$$\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$$

in such a way that p is the origin $\mathbf{0}$ in this setting and $U \cap S^{n-1}$ is contained in the subspace

$$\mathbb{R}^{n-1} = \{(x_1, x_2, \dots, x_{n-1}, 0) \in \mathbb{R}^n\}.$$

In order to calculate the index of p in each space, we consider the map $F: U \to \mathbb{R}^n$ defined by

$$F(x_1, x_2, \dots, x_n) = \begin{cases} (x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n), & \text{if } x_n \geq 0, \\ (x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_{n-1}, 0), & \text{if } x_n < 0. \end{cases}$$

Note that F sends the origin, lower half-plane and $\mathbb{R}^{n-1}$ to itself respectively. Also, $F(z) \neq z$ for $z \neq 0$. The index $i(S^{n-1}, \phi, p)$ is equal to $i(\mathbb{R}^{n-1}, F, \mathbf{0})$ in this setting as in the traditional definition of the index. Moreover, $i(B^n, f, p)$ is identified with $i(\mathbb{R}^n, F, \mathbf{0})$, which can be computed as the degree of the map $\rho \circ F: S^{n-1} \to S^{n-1}$ where $\rho: \mathbb{R}^n \setminus \mathbf{0} \to S^{n-1}$ is the retraction defined by $\rho(z) = z/|z|$.

3 Unit-circle-preserving maps of the plane

Brown, Greene and Schirmer proved

Theorem 1 [1, 2] *Let $f : B^2 \to B^2$ be a smooth map with a finite number of fixed points such that $f(\zeta) = \zeta^k$ for all $\zeta \in S^1$ for some $k \geq 2$, where B^2 is the closed two-dimensional ball with the boundary S^1. If π is a fixed point of f that lies in S^1, then either $i(B^2, f, \pi) = 0$ or $i(B^2, f, \pi) = -1$.*

The contractibility of B^2 implies the following corollary.

Corollary 2 [2] *Suppose $f : B^2 \to B^2$ is a smooth map such that $f(\zeta) = \zeta^k$ for all $\zeta \in S^1$, for some $k \geq 2$. Then there exists $z \in \mathrm{int}(B^2)$ such that $f(z) = z$.*

We will extend Theorem 1 to maps $f : (B^2, S^1) \to (\mathbb{R}^2, S^1)$ by modifying the proof of Theorem 1 in [2]. Corollary 2 will then extend to maps $f : (\mathbb{R}^2, S^1) \to (\mathbb{R}^2, S^1)$.

Theorem 3 *Let $f : (B^2, S^1) \to (\mathbb{R}^2, S^1)$ be a smooth map with a finite number of fixed points such that $f(\zeta) = \zeta^k$ for all $\zeta \in S^1$ for some $k \geq 2$. If π is a fixed point of f that lies in S^1, then either $i(B^2, f, \pi) = 0$ or $i(B^2, f, \pi) = -1$.*

Proof Let π be a fixed point of f in S^1. We can write this fixed point in the polar coordinates (r, θ) as $(1, \theta_0)$. We will introduce new coordinates on a neighborhood U of π as follows:

$$x_1 = \theta - \theta_0, \qquad x_2 = 1 - r.$$

In the new coordinate setting, the fixed point π is the origin and $U \cap S^1$ corresponds to the x_1-axis near 0, and the portion of the interior of the unit ball in U is contained in the upper half-plane. Consider the following map (as described in Section 2) in the new coordinate setting:

$$F(x_1, x_2) = \begin{cases} (x_1, x_2) - f(x_1, x_2), & \text{if } x_2 \geq 0, \\ (x_1, x_2) - f(x_1, 0), & \text{if } x_2 < 0. \end{cases}$$

Now write $F = (F_1, F_2)$ and define $g(x_1) = F_1(x_1, 0)$. Since $f(\zeta) = \zeta^k$ for $\zeta \in S^1$, we have

$$g(x_1) = F_1(x_1, 0) = x_1 - kx_1 = (1 - k)x_1.$$

Since the map f is defined to be smooth on B^2, the map F is smooth on the upper half-plane. Let F^+ denote the restriction of F to the upper half-plane. We will see that smoothness is only required in a neighborhood of the fixed point at 0. Since we are assuming that $k \geq 2$, then

$$\frac{d}{dx} F_1(x_1, 0) = g'(x_1) = 1 - k < 0$$

and the smoothness of F^+ implies that

$$\frac{\partial F_1^+(x_1, x_2)}{\partial x_1} < 0$$

for (x_1, x_2) in an ϵ-neighborhood of the origin, for $\epsilon > 0$ sufficiently small and for $x_2 \geq 0$. Let Γ be a circle of radius $\epsilon/2$ about the origin.

Let Γ^+ and Γ^- denote the half-circles above and below the x_1-axis respectively. Since F takes the lower half-plane to itself, we know that F maps Γ^- to the lower half-plane. Calculating the fixed point index of f at the origin in $\mathbb{R}^2$ is equivalent to finding the winding number of $F(\Gamma)$ around 0. Thus, we need to understand $F(\Gamma^+)$. Since Γ^+ lies in the upper half-plane, we only consider F^+. Assuming F has only a finite number of fixed points, we can choose ϵ small enough so that only one point on Γ or its interior that F maps to the origin is the origin itself. Therefore, we can homotope the restriction of F^+ to Γ^+ in $\mathbb{R}^2\backslash 0$ to the restriction of F^+ to the curve Γ^+_δ for $\delta > 0$ given by

$$\Gamma^+_\delta(t) = \left(\frac{\epsilon}{2}(2t-1), \delta\bigl(1-(1-2t)^2\bigr)\right),$$

where $0 \leq t \leq 1$.

We write the restriction of F^+ to Γ^+_δ in coordinates as

$$F^+\bigl(\Gamma^+_\delta(t)\bigr) = \bigl(F^+_1\bigl(\Gamma^+_\delta(t)\bigr), F^+_2\bigl(\Gamma^+_\delta(t)\bigr)\bigr) = \bigl(\phi_\delta(t), \psi_\delta(t)\bigr).$$

The key idea of the proof is that for δ sufficiently small, the smoothness of F^+ and the fact that

$$\frac{\partial F^+_1(x_1, x_2)}{\partial x_1} < 0 \quad \text{implies that} \quad \frac{d}{dt}\phi_\delta(t) < 0$$

for all t. This tells us that the x_1-coordinate of the curve $F^+(\Gamma^+_\delta(t))$ is a strictly monotone function of t. In particular, the curve $F^+(\Gamma^+_\delta(t))$ only crosses the x_2-axis once. This implies the desired result since the winding number of $F(\Gamma)$ can then only be either 0 or -1.

Notice that it is never specified that f maps B^2 into itself. In considering the map $F(z) = z - f(z)$, although it is assumed that F maps the exterior of the disc to the exterior of the disc, the proof allows the image of the interior of the disc under F to lie anywhere in $\mathbb{R}^2$. □

Let $\bar{f}: B^2 \to \mathbb{R}^2$ be the restriction of $f: (\mathbb{R}^2, S^1) \to (\mathbb{R}^2, S^1)$ to B^2. Since B^2 is contractible, the sum of the indices of $\rho\bar{f}: B^2 \to B^2$ equals one, and therefore $\rho\bar{f}(x) = x$ for some $x \in \mathrm{int}(B^2)$. But then $f(x) = x$ as well, so $f: (\mathbb{B}^2, S^1) \to (\mathbb{R}^2, S^1)$ has a fixed point in the interior of B^2. Therefore, we can extend Corollary 2 as the following result.

Corollary 4 *Let $f: (\mathbb{R}^2, S^1) \to (\mathbb{R}^2, S^1)$ be a smooth map such that $f(\zeta) = \zeta^k$ for all $\zeta \in S^1$ for some $k \geq 2$. Then there exist $z \in \mathrm{int}(B^2)$ such that $f(z) = z$.*

4 Interior fixed points of a map $f: (\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$

Definition 1 ([3], p.34) A smooth map $\phi: S^{n-1} \to S^{n-1}$ with finitely many fixed points is *transversely fixed* if $d\phi_p - I: T_p(S^{n-1}) \to T_p(S^{n-1})$ is a nonsingular linear map for each fixed point p. For $F = \{p_1, \ldots, p_r\}$ a fixed point class of ϕ, let

$$i(F) = \sum_{j=1}^{r} i\bigl(S^{n-1}, \phi, p_j\bigr).$$

The *transverse Nielsen number* $N_{\pitchfork}(\phi)$ is defined by

$$N_{\pitchfork}(\phi) = \sum_{F \in \mathfrak{F}} \left| i(F) \right|,$$

where $\mathfrak{F}$ is the set of fixed point classes of ϕ.

A smooth map $\phi \colon S^{n-1} \to S^{n-1}$ is *sparse* if it is transversely fixed and it has $N_{\pitchfork}(\phi)$ fixed points.

In [3], p.45 Schirmer obtained the following result.

Theorem 5 *Let $\phi \colon S^{n-1} \to S^{n-1}$ be a sparse map of degree d and suppose $f : (B^n, S^{n-1}) \to (B^n, S^{n-1})$ is a smooth map extending ϕ. If $(-1)^n d \geq 2$, then f must have a fixed point in* $\operatorname{int}(B^n)$.

We will extend this result as a consequence of the following

Theorem 6 *Given a smooth map $\phi : S^{n-1} \to S^{n-1}$ and a smooth map $f : (B^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$ extending ϕ, suppose that $p \in S^{n-1}$ is an isolated fixed point of f and that $d\phi_p - I : T_p(S^{n-1}) \to T_p(S^{n-1})$ is a nonsingular linear transformation. Then either $i(B^n, f, p) = 0$ or $i(B^n, f, p) = i(S^{n-1}, \phi, p)$.*

Proof The following proof is a modified version of Theorem 5.1 in [1]. We again write f in a small ball that contains $p \in S^{n-1}$ as described in Section 2 and the map F is also as defined there. Moreover, for $\varepsilon > 0$ small enough, let

$$D_\varepsilon(x_1, \dots, x_n) = \begin{cases} (\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_n) - f(\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_n), & \text{if } x_n \geq 0, \\ (\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_n) - f(\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_{n-1}, 0), & \text{if } x_n < 0. \end{cases}$$

This then means that the index $i(B^n, f, p) = i(B^n, D_\varepsilon, \mathbf{0})$ is the degree of $\rho \circ D_\varepsilon : S^{n-1} \to S^{n-1}$, where $\rho(x) = x/|x|$ for $x \in \mathbb{R}^n \backslash \mathbf{0}$.

Note that

$$\left| F(\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_n) - dF_p(\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_n) \right| \leq o(\varepsilon)$$

because F is a C^1 function. We also have

$$\left| dF_p(\varepsilon x_1, \varepsilon x_2, \dots, \varepsilon x_n) \right| \geq C\varepsilon$$

for some $C > 0$ independent of ε and $(x_1, x_2, \dots, x_n)$ due to the fact that $dF_p = -(df_p - I)$ is nonsingular by hypothesis.

Since dF_p is a nonsingular linear map, this last degree is easily seen to be 0, or ± 1. But the images of D_ε of the upper and lower hemisphere are each contained entirely in either the lower or upper half-space. This means that D_ε is of degree 0 or D_ε is homotopic in $\mathbb{R}^n \backslash \mathbf{0}$ to the suspension of $D_\varepsilon|_{S^{n-2}}$. □

We can use Theorem 6 to extend Theorem 5 to the case $f : (\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$. The following is a modified version of part of [3]. Note that despite the fact that the case $n = 2$

is solved in the previous section, the new material presented below extends the solution to all the cases.

Suppose we have $\phi : S^{n-1} \to S^{n-1}$ and a smooth map $f : (\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$ extending ϕ. A fixed point class F of f is called a *common fixed point class* of f and ϕ if there exists an essential fixed point class $\overline{F}$ of ϕ which is contained in F.

We will again consider the restriction $f : (B^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$. In the notation of [3], p.39, let

$$u(F) = \max\Big\{0, \sum\big(i\big(S^{n-1}, \phi, \overline{F}\big)\big|\overline{F} \subset F \text{ and } i\big(S^{n-1}, \phi, \overline{F}\big) > 0\big)\Big\},$$
$$l(F) = \min\Big\{0, \sum\big(i\big(S^{n-1}, \phi, \overline{F}\big)\big|\overline{F} \subset F \text{ and } i\big(S^{n-1}, \phi, \overline{F}\big) < 0\big)\Big\}.$$

Then F is a *transversally common fixed point class* of f and ϕ if

$$l(F) \leq i(F) \leq u(F).$$

Definition 2 The *boundary transversal Nielsen number* of $f : (B^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$ is

$$N\big(f; B^n, S^{n-1}_{\pitchfork}\big) = N_{\pitchfork}(\phi) + N(f) - N(f, \phi_{\pitchfork}),$$

where $N(f, \phi_{\pitchfork})$ is the number of essential and transversally common fixed point classes of f and ϕ.

Suppose that $\phi : S^{n-1} \to S^{n-1}$ has degree d. Then $f : B^n \to \mathbb{R}^n$ has one fixed point class F with $i(F) = 1$, and so $l(F) \leq 0 < i(F)$. If $n = 2$, then ϕ has $|1 - d|$ essential fixed point classes, each of the same index, and $\sum i(S^1, \phi, \overline{F}) = L(\phi) = 1 - d$. Hence, $i(F) \leq u(F)$ if and only if $d \leq 0$, and

$$N(f, \phi_{\pitchfork}) = \begin{cases} 1 & \text{if } d \leq 0, \\ 0 & \text{if } d > 0. \end{cases}$$

If $n \geq 3$, then ϕ has one fixed point class $\overline{F}$ with $i(S^{n-1}, \phi, \overline{F}) = 1 + (-1)^{n-1}d$ and so

$$N(f, \phi_{\pitchfork}) = \begin{cases} 1 & \text{if } (-1)^{n-1}d \geq 0, \\ 0 & \text{if } (-1)^{n-1}d < 0. \end{cases}$$

Note that this formula is still true for the case $n = 2$.

If $\phi : S^{n-1} \to S^{n-1}$ is a sparse map of degree d then $N_{\pitchfork}(\phi) = |1 - (-1)^n d|$ for all d and all $n \geq 2$.

Since $N(f) = 1$, for the case that $(-1)^n d \leq 0$, the boundary transversal Nielsen number is

$$\begin{aligned} N\big(f; B^n, S^{n-1}_{\pitchfork}\big) &= N_{\pitchfork}(\phi) + N(f) - N(f, \phi_{\pitchfork}) \\ &= \big|1 - (-1)^n d\big| + 1 - 1 \\ &= \big|1 - (-1)^n d\big|. \end{aligned}$$

As for the case that $(-1)^n d > 0$, the boundary transversal Nielsen number is

$$\begin{aligned} N\big(f;B^n,S^{n-1}_{\pitchfork}\big) &= N_{\pitchfork}(\phi) + N(f) - N(f,\phi_{\pitchfork}) \\ &= \big|1-(-1)^n d\big| + 1 - 0 \\ &= \big|1-(-1)^n d\big| + 1. \end{aligned}$$

Hence, we have just proven the following

Proposition 7 *If* $f:(B^n,S^{n-1}) \to (\mathbb{R}^n,S^{n-1})$ *is a smooth extension of a sparse map* $\phi: S^{n-1} \to S^{n-1}$ *of degree* d*, with* $n \geq 2$*, then the boundary transversal Nielsen number is*

$$N\big(f;B^n,S^{n-1}_{\pitchfork}\big) = \begin{cases} |1-(-1)^n d| & \text{if } (-1)^n d \leq 0, \\ |1-(-1)^n d| + 1 & \text{if } (-1)^n d > 0. \end{cases}$$

As defined in [1], p.2, *the extension Nielsen number* $N(f|\phi)$ is a lower bound for the number of fixed points on S^{n-1} of continuous extensions of a continuous map ϕ. It is equal to the number of essential (classical) fixed point classes F of f with $F \cap S^{n-1} = \emptyset$. A fixed point class F is *representable* on S^{n-1} if there exists a subset $F' \subset F \cap S^{n-1}$ with $i(B^n,f,F) = i(S^{n-1},\phi,F')$. The *smooth extension number* $N^1(f|\phi)$ is the number of essential (classical) fixed point classes F of f which are not representable on S^{n-1}. It is a lower bound for the number of fixed points in S^{n-1} of a smooth extension of a smooth and transversally fixed map ϕ.

Proposition 8 *If* $\phi: S^{n-1} \to S^{n-1}$ *is sparse, then*

$$\begin{aligned} &N^1(f|\phi) = N\big(f;B^n,S^{n-1}_{\pitchfork}\big) - N_{\pitchfork}(\phi), \\ &N(f|\phi) = N\big(f;B^n,S^{n-1}\big) - N(\phi). \end{aligned}$$

Proof Our proof is modeled on the proofs of Proposition 4.3 and Corollary 4.4 in [3]. For any essential fixed point class F of f, since ϕ is sparse, $F \cap S^{n-1}$ contains $u(F)$ fixed points p such that $i(S^{n-1},\phi,p) = 1$ and $l(F)$ fixed points p such that $i(S^{n-1},\phi,p) = -1$. This means that F is representable on S^{n-1} if and only if $l(F) \leq i(F) \leq u(F)$. By the definitions of all the Nielsen numbers involved, we have the result stated above for $N^1(f|\phi)$.

The result for $N(f|\phi)$ can be obtained in a similar manner by using Corollary 2.6 from [1] along with the fact that all fixed point classes of a sparse map are essential. □

By the definitions of $N(f,\phi)$ in [5] and the definition of $N^1(f|\phi)$ defined early, we obtain

Proposition 9 *If* $\phi: S^{n-1} \to S^{n-1}$ *is sparse, then the number of essential fixed point classes of* f *which are common but not transversally common is*

$$N^1(f|\phi) - N(f|\phi) = N(f,\phi) - N(f,\phi_{\pitchfork}).$$

We are now ready to prove the following Theorem.

Theorem 10 *Let $n \geq 2$, and let $\phi : S^{n-1} \to S^{n-1}$ be a sparse map of degree d and suppose $f : (B^n, S^{n-1}) \to (\mathbb{R}^n, S^{n-1})$ is a smooth map extending ϕ. If $(-1)^n d \geq 2$, then f must have a fixed point in* $\operatorname{int}(B^n)$.

Proof Since ϕ has degree d and it is sparse, by definition

$$N(f|\phi) = \begin{cases} 1 & \text{if } d \neq (-1)^n, \\ 0 & \text{if } d = (-1)^n \end{cases}$$

and

$$\begin{aligned} N^1(f|\phi) - N(f|\phi) &= N\bigl(f; B^n, S^{n-1}_{\pitchfork}\bigr) - N_{\pitchfork}(\phi) - N(f|\phi) \quad \text{(from Proposition 8)} \\ &= N\bigl(f; B^n, S^{n-1}_{\pitchfork}\bigr) - \bigl|1 - (-1)^n d\bigr| - N(f|\phi). \end{aligned}$$

Applying Proposition 7, we have

$$N^1(f|\phi) - N(f|\phi) = \begin{cases} 0 & \text{if } (-1)^n d \leq 1, \\ 1 & \text{if } (-1)^n d \geq 2. \end{cases}$$

Thus, every smooth extension over B^n of a sparse map of S^{n-1} of degree d with $(-1)^n d \geq 2$ has a fixed point on the interior of B^n. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References

1. Brown, RF, Greene, RE, Schirmer, H: Fixed points of map extensions. In: Topological Fixed Point Theory and Applications, Tianjin, 1988. Lecture Notes in Math., vol. 1411, pp. 24-45. Springer, Berlin (1989)
2. Brown, RF, Greene, RE: An interior fixed point property of the disc. Am. Math. Mon. **101**(1), 39-47 (1994)
3. Schirmer, H: Nielsen theory of transversal fixed point sets. Fundam. Math. **141**(1), 31-59 (1992). With an appendix by Robert E. Greene
4. Dold, A: Fixed point index and fixed point theorem for Euclidean neighborhood retract. Topology **4**, 1-9 (1965)
5. Schirmer, H: A relative Nielsen number. Pac. J. Math. **122**, 459-473 (1986)

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