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# Iterative common solutions for monotone inclusion problems, fixed point problems and equilibrium problems

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**Abstract**

Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha > 0$, and let $A$ be an $\alpha$-inverse strongly-monotone mapping of $C$ into $H$. Let $T$ be a generalized hybrid mapping of $C$ into $H$. Let $B$ and $W$ be maximal monotone operators on $H$ such that the domains of $B$ and $W$ are included in $C$. Let $0 < k < 1$, and let $g$ be a $k$-contraction of $H$ into itself. Let $V$ be a $\overline{\gamma}$-strongly monotone and $L$-Lipschitzian continuous operator with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Suppose that $F(T) \cap (A+B)^{-1}0 \cap W^{-1}0 \neq \emptyset$, where $F(T)$ and $(A+B)^{-1}0$, $W^{-1}0$ are the set of fixed points of $T$ and the sets of zero points of $A+B$ and $W$, respectively. In this paper, we prove a strong convergence theorem for finding a point $z_0$ of $F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$, where $z_0$ is a unique fixed point of $P_{F(T)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)$. This point $z_0 \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$ is also a unique solution of the variational inequality

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0.$$

Using this result, we obtain new and well-known strong convergence theorems in a Hilbert space. In particular, we solve a problem posed by Kurokawa and Takahashi (Nonlinear Anal. 73:1562-1568, 2010).
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## 1 Introduction

Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping $T : C \to H$ is called *generalized hybrid* [1] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta)$-*generalized hybrid* mapping. Kocourek, Takahashi and Yao [1] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [2] in a Hilbert space. Notice that the mapping above covers several well-known mappings. For example, an $(\alpha, \beta)$-generalized hybrid mapping $T$ is nonexpansive for $\alpha = 1$ and $\beta = 0$, *i.e.*,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is also nonspreading [3, 4] for $\alpha = 2$ and $\beta = 1$, *i.e.*,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Furthermore, it is hybrid [5] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, *i.e.*,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|x - y\|^2 + (1 - \beta)\|x - y\|^2,$$

and hence $\|x - Ty\| \leq \|x - y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. The following strong convergence theorem of Halpern's type [6] was proved by Wittmann [7]; see also [8].

**Theorem 1** *Let $C$ be a nonempty closed convex subset of $H$, and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in $C$ by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

*where $\{\alpha_n\} \subset (0, 1)$ satisfies*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty.$$

*Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

Kurokawa and Takahashi [9] also proved the following strong convergence theorem for nonspreading mappings in a Hilbert space; see also Hojo and Takahashi [10] for generalized hybrid mappings.

**Theorem 2** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a nonspreading mapping of $C$ into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in $C$ as follows*: $x_1 = x \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \\ z_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

*for all* $n = 1, 2, \ldots,$ *where* $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* $Pu$, *where* $P$ *is the metric projection of* $H$ *onto* $F(T)$.

**Remark** We do not know whether Theorem 1 for nonspreading mappings holds or not; see [9] and [10].

In this paper, we provide a strong convergence theorem for finding a point $z_0$ of $F(T) \cap (A+B)^{-1}0 \cap W^{-1}0$ such that it is a unique fixed point of

$$P_{F(T)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)$$

and a unique solution of the variational inequality

$$\big\langle (V - \gamma g)z_0, q - z_0 \big\rangle \geq 0, \quad \forall q \in F(T) \cap (A+B)^{-1}0 \cap W^{-1}0,$$

where $T, A, B, W, g$ and $V$ denote a generalized hybrid mapping of $C$ into $H$, an $\alpha$-inverse strongly-monotone mapping of $C$ into $H$ with $\alpha > 0$, maximal monotone operators on $H$ such that the domains of $B$ and $W$ are included in $C$, a $k$-contraction of $H$ into itself with $0 < k < 1$ and a $\overline{\gamma}$-strongly monotone and $L$-Lipschitzian continuous operator with $\overline{\gamma} > 0$ and $L > 0$, respectively. Using this result, we obtain new and well-known strong convergence theorems in a Hilbert space. In particular, we solve a problem posed by Kurokawa and Takahashi [9].

## 2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. When $\{x_n\}$ is a sequence in $H$, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [11] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{2.1}$$

and

$$\big\|\lambda x + (1-\lambda)y\big\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

All Hilbert spaces satisfy Opial's condition, that is,

$$\liminf_{n\to\infty}\|x_n - u\| < \liminf_{n\to\infty}\|x_n - v\| \tag{2.4}$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [12]. Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, and let $T\colon C \to H$ be a mapping. We denote by $F(T)$ the set of fixed points for $T$. A mapping $T : C \to H$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$

for all $x \in C$ and $y \in F(T)$. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [13]. For a nonempty closed convex subset $C$ of $H$, the nearest point projection of $H$ onto $C$ is denoted by $P_C$, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such $P_C$ is called the metric projection of $H$ onto $C$. We know that the metric projection $P_C$ is firmly nonexpansive; $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [14]. The following result is in [15].

**Lemma 3** *Let $H$ be a Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to H$ be a generalized hybrid mapping. Suppose that there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Then $z \in F(T)$.*

Let $B$ be a mapping of $H$ into $2^H$. The effective domain of $B$ is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping $B$ is said to be a monotone operator on $H$ if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator $B$ on $H$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on $H$. For a maximal monotone operator $B$ on $H$ and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of $B$ for $r$. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of $B$ for $r > 0$. We know from [8] that

$$A_r x \in BJ_r x, \quad \forall x \in H,\ r > 0. \tag{2.5}$$

Let $B$ be a maximal monotone operator on $H$, and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent $J_r$ is firmly nonexpansive, *i.e.*,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

We also know the following lemma from [16].

**Lemma 4** *Let $H$ be a real Hilbert space, and let $B$ be a maximal monotone operator on $H$. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds*:

$$\frac{s - t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

*for all $s, t > 0$ and $x \in H$.*

From Lemma 4, we have that

$$\|J_\lambda x - J_\mu x\| \leq \bigl(|\lambda - \mu|/\lambda\bigr)\|x - J_\lambda x\|$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [14, 17]. To prove our main result, we need the following lemmas.

**Lemma 5** ([18]; see also [19]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers* *with* $\sum_{n=1}^{\infty} \beta_n < \infty$, *and let* $\{\gamma_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty} \gamma_n \leq 0$. *Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

*for all* $n = 1, 2, \ldots$. *Then* $\lim_{n\to\infty} s_n = 0$.

**Lemma 6** ([20]) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Define the sequence* $\{\tau(n)\}_{n\geq n_0}$ *of integers as follows*:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

*where* $n_0 \in \mathbb{N}$ *such that* $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. *Then the following hold*:
(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \in \mathbb{N}$.

## 3 Strong convergence theorems

Let $H$ be a real Hilbert space. A mapping $g : H \to H$ is a contraction if there exists $k \in (0,1)$ such that $\|g(x) - g(y)\| \leq k\|x - y\|$ for all $x, y \in H$. We call such a mapping $g$ a $k$-contraction. A nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\overline{\gamma} > 0$ such that $\langle x - y, Vx - Vy\rangle \geq \overline{\gamma}\|x - y\|^2$ for all $x, y \in H$. Such $V$ is also called $\overline{\gamma}$-strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian continuous if there exists $L > 0$ such that $\|Vx - Vy\| \leq L\|x - y\|$ for all $x, y \in H$. Such $V$ is also called $L$-Lipschitzian continuous. We know the following three lemmas in a Hilbert space; see Lin and Takahashi [21].

**Lemma 7** ([21]) *Let* $H$ *be a Hilbert space, and let* $V$ *be a* $\overline{\gamma}$*-strongly monotone and* $L$*-Lipschitzian continuous operator on* $H$ *with* $\overline{\gamma} > 0$ *and* $L > 0$. *Let* $t > 0$ *satisfy* $2\overline{\gamma} > tL^2$ *and* $1 > 2t\overline{\gamma}$. *Then* $0 < 1 - t(2\overline{\gamma} - tL^2) < 1$ *and* $I - tV : H \to H$ *is a contraction, where* $I$ *is the identity operator on* $H$.

**Lemma 8** ([21]) *Let* $H$ *be a Hilbert space, and let* $C$ *be a nonempty closed convex subset of* $H$. *Let* $P_C$ *be the metric projection of* $H$ *onto* $C$, *and let* $V$ *be a* $\overline{\gamma}$*-strongly monotone and* $L$*-Lipschitzian continuous operator on* $H$ *with* $\overline{\gamma} > 0$ *and* $L > 0$. *Let* $t > 0$ *satisfy* $2\overline{\gamma} > tL^2$ *and* $1 > 2t\overline{\gamma}$, *and let* $z \in C$. *Then the following are equivalent*:
(1) $z = P_C(I - tV)z$;
(2) $\langle Vz, y - z\rangle \geq 0, \forall y \in C$;
(3) $z = P_C(I - V)z$.
*Such* $z \in C$ *always exists and is unique*.

**Lemma 9** ([21]) *Let* $H$ *be a Hilbert space, and let* $g : H \to H$ *be a* $k$*-contraction with* $0 < k < 1$. *Let* $V$ *be a* $\overline{\gamma}$*-strongly monotone and* $L$*-Lipschitzian continuous operator on* $H$ *with* $\overline{\gamma} > 0$ *and* $L > 0$. *Let a real number* $\gamma$ *satisfy* $0 < \gamma < \frac{\overline{\gamma}}{k}$. *Then* $V - \gamma g : H \to H$ *is a* $(\overline{\gamma} - \gamma k)$*-strongly monotone and* $(L + \gamma k)$*-Lipschitzian continuous mapping. Furthermore, let* $C$ *be a nonempty closed convex subset of* $H$. *Then* $P_C(I - V + \gamma g)$ *has a unique fixed point* $z_0$ *in* $C$. *This point* $z_0 \in C$ *is also a unique solution of the variational inequality*

$$\big\langle (V - \gamma g)z_0, q - z_0\big\rangle \geq 0, \quad \forall q \in C.$$

Now, we prove the following strong convergence theorem of Halpern's type [6] for finding a common solution of a monotone inclusion problem for the sum of two monotone mappings, of a fixed point problem for generalized hybrid mappings and of an equilibrium problem for bifunctions in a Hilbert space.

**Theorem 10** *Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha > 0$, and let $A$ be an $\alpha$-inverse strongly-monotone mapping of $C$ into $H$. Let $B$ and $W$ be maximal monotone operators on $H$ such that the domains of $B$ and $W$ are included in $C$. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rW)^{-1}$ be resolvents of $B$ and $W$ for $\lambda > 0$ and $r > 0$, respectively. Let $S$ be a generalized hybrid mapping of $C$ into $H$. Let $0 < k < 1$, and let $g$ be a $k$-contraction of $H$ into itself. Let $V$ be a $\overline{\gamma}$-strongly monotone and $L$-Lipschitzian continuous operator with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

*Suppose $F(S) \cap (A + B)^{-1}0 \cap W^{-1}0 \neq \emptyset$. Let $x_1 = x \in H$, and let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\bigl\{\alpha_n \gamma g(x_n) + (I - \alpha_n V)SJ_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n\bigr\}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$$

$$\liminf_{n\to\infty} r_n > 0 \quad \text{and} \quad 0 < a \le \lambda_n \le b < 2\alpha.$$

*Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap (A + B)^{-1}0 \cap W^{-1}0$, where $z_0$ is a unique fixed point in $F(S) \cap (A + B)^{-1}0 \cap W^{-1}0$ of $P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)$.*

*Proof* Let $z \in F(S) \cap (A + B)^{-1}0 \cap W^{-1}0$. We have that $z = Sz$, $z = J_{\lambda_n}(I - \lambda_n A)z$ and $z = T_{r_n}z$. Putting $w_n = J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n$ and $u_n = T_{r_n}x_n$, we obtain that

$$\begin{aligned}
\|Sw_n - z\|^2 &\le \|w_n - z\|^2 \\
&= \|J_{\lambda_n}(u_n - \lambda_n Au_n) - J_{\lambda_n}(z - \lambda_n Az)\|^2 \\
&\le \|u_n - \lambda_n Au_n - (z - \lambda_n Az)\|^2 \\
&= \|u_n - z - \lambda_n(Au_n - Az)\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n\langle u_n - z, Au_n - Az\rangle + \lambda_n^2\|Au_n - Az\|^2 \\
&\le \|u_n - z\|^2 - 2\lambda_n\alpha\|Au_n - Az\|^2 + \lambda_n^2\|Au_n - Az\|^2 \\
&\le \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Az\|^2 \\
&\le \|x_n - z\|^2.
\end{aligned} \tag{3.1}$$

Put $\tau = \overline{\gamma} - \frac{L^2\mu}{2}$. Using $\lim_{n\to\infty}\alpha_n = 0$, we have that for any $x, y \in H$,

$$\begin{aligned}
&\big\|(I-\alpha_n V)x - (I-\alpha_n V)y\big\|^2 \\
&\quad= \big\|x - y - \alpha_n(Vx - Vy)\big\|^2 \\
&\quad= \|x-y\|^2 - 2\alpha_n\langle x-y, Vx-Vy\rangle + \alpha_n^2\|Vx-Vy\|^2 \\
&\quad\le \|x-y\|^2 - 2\alpha_n\overline{\gamma}\|x-y\|^2 + \alpha_n^2L^2\|x-y\|^2 \\
&\quad= \big(1 - 2\alpha_n\overline{\gamma} + \alpha_n^2L^2\big)\|x-y\|^2 \\
&\quad= \big(1 - 2\alpha_n\tau - \alpha_nL^2\mu + \alpha_n^2L^2\big)\|x-y\|^2 \\
&\quad\le \big(1 - 2\alpha_n\tau - \alpha_n\big(L^2\mu - \alpha_nL^2\big) + \alpha_n^2\tau^2\big)\|x-y\|^2 \\
&\quad\le \big(1 - 2\alpha_n\tau + \alpha_n^2\tau^2\big)\|x-y\|^2 \\
&\quad= (1-\alpha_n\tau)^2\|x-y\|^2.
\end{aligned} \tag{3.2}$$

Since $1 - \alpha_n\tau > 0$, we obtain that for any $x, y \in H$,

$$\big\|(I-\alpha_n V)x - (I-\alpha_n V)y\big\| \le (1-\alpha_n\tau)\|x-y\|. \tag{3.3}$$

Putting $y_n = \alpha_n\gamma g(x_n) + (I-\alpha_n V)SJ_{\lambda_n}(I-\lambda_nA)T_{r_n}x_n$, from $z = \alpha_n Vz + z - \alpha_n Vz$, (3.1) and (3.3) we have that

$$\begin{aligned}
\|y_n - z\| &= \big\|\alpha_n\big(\gamma g(x_n) - Vz\big) + (I-\alpha_n V)Sw_n - (I-\alpha_n V)z\big\| \\
&\le \alpha_n\gamma k\|x_n - z\| + \alpha_n\big\|\gamma g(z) - Vz\big\| + (1-\alpha_n\tau)\|Sw_n - z\| \\
&\le \big\{1 - \alpha_n(\tau - \gamma k)\big\}\|x_n - z\| + \alpha_n\big\|\gamma g(z) - Vz\big\|.
\end{aligned}$$

Using this, we get

$$\begin{aligned}
\|x_{n+1} - z\| &= \big\|\beta_n(x_n - z) + (1-\beta_n)(y_n - z)\big\| \\
&\le \beta_n\|x_n - z\| + (1-\beta_n)\|y_n - z\| \\
&\le \beta_n\|x_n - z\| \\
&\qquad + (1-\beta_n)\big(\big\{1 - \alpha_n(\tau-\gamma k)\big\}\|x_n - z\| + \alpha_n\big\|\gamma g(z) - Vz\big\|\big) \\
&= \big\{1 - (1-\beta_n)\alpha_n(\tau-\gamma k)\big\}\|x_n - z\| \\
&\qquad + (1-\beta_n)\alpha_n(\tau-\gamma k)\frac{\|\gamma g(z) - Vz\|}{\tau - \gamma k}.
\end{aligned}$$

Putting $K = \max\{\|x_1 - z\|, \frac{\|\gamma g(z) - Vz\|}{\tau-\gamma k}\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. Using Lemma 9, we can take a unique $z_0 \in F(S) \cap (A+B)^{-1}0 \cap W^{-1}0$ such that

$$z_0 = P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)z_0.$$

From the definition of $\{x_n\}$, we have that

$$x_{n+1} - x_n = \beta_n x_n + (1-\beta_n)\big\{\alpha_n\gamma g(x_n) + (I-\alpha_n V)Sw_n\big\} - x_n$$

and hence

$$\begin{aligned} x_{n+1} - x_n - (1-\beta_n)\alpha_n \gamma g(x_n) &= \beta_n x_n + (1-\beta_n)(I - \alpha_n V)Sw_n - x_n \\ &= (1-\beta_n)\bigl\{(I-\alpha_n V)Sw_n - x_n\bigr\} \\ &= (1-\beta_n)(Sw_n - x_n - \alpha_n VSw_n). \end{aligned}$$

Thus, we have that

$$\begin{aligned} &\bigl\langle x_{n+1} - x_n - (1-\beta_n)\alpha_n \gamma g(x_n), x_n - z_0\bigr\rangle \\ &\quad = \bigl\langle (1-\beta_n)(Sw_n - x_n - \alpha_n VSw_n), x_n - z_0\bigr\rangle \\ &\quad = -(1-\beta_n)\langle x_n - Sw_n, x_n - z_0\rangle - (1-\beta_n)\alpha_n\langle VSw_n, x_n - z_0\rangle. \end{aligned} \tag{3.4}$$

From (2.3) and (3.1), we have that

$$\begin{aligned} 2\langle x_n - Sw_n, x_n - z_0\rangle &= \|x_n - z_0\|^2 + \|Sw_n - x_n\|^2 - \|Sw_n - z_0\|^2 \\ &\ge \|x_n - z_0\|^2 + \|Sw_n - x_n\|^2 - \|x_n - z_0\|^2 \\ &= \|Sw_n - x_n\|^2. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we also have that

$$\begin{aligned} -2\langle x_n - x_{n+1}, x_n - z_0\rangle &= 2(1-\beta_n)\alpha_n\bigl\langle \gamma g(x_n), x_n - z_0\bigr\rangle \\ &\quad - 2(1-\beta_n)\langle x_n - Sw_n, x_n - z_0\rangle - 2(1-\beta_n)\alpha_n\langle VSw_n, x_n - z_0\rangle \\ &\le 2(1-\beta_n)\alpha_n\bigl\langle \gamma g(x_n), x_n - z_0\bigr\rangle \\ &\quad - (1-\beta_n)\|Sw_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle VSw_n, x_n - z_0\rangle. \end{aligned} \tag{3.6}$$

Furthermore, using (2.3) and (3.6), we have that

$$\begin{aligned} &\|x_{n+1} - z_0\|^2 - \|x_n - x_{n+1}\|^2 - \|x_n - z_0\|^2 \\ &\quad \le 2(1-\beta_n)\alpha_n\bigl\langle \gamma g(x_n), x_n - z_0\bigr\rangle \\ &\qquad - (1-\beta_n)\|Sw_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle VSw_n, x_n - z_0\rangle. \end{aligned}$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned} &\Gamma_{n+1} - \Gamma_n - \|x_n - x_{n+1}\|^2 \\ &\quad \le 2(1-\beta_n)\alpha_n\bigl\langle \gamma g(x_n), x_n - z_0\bigr\rangle \\ &\qquad - (1-\beta_n)\|Sw_n - x_n\|^2 - 2(1-\beta_n)\alpha_n\langle VSw_n, x_n - z_0\rangle. \end{aligned} \tag{3.7}$$

Noting that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \bigl\|(1-\beta_n)\alpha_n\bigl(\gamma g(x_n) - VSw_n\bigr) + (1-\beta_n)(Sw_n - x_n)\bigr\| \\ &\le (1-\beta_n)\bigl(\|Sw_n - x_n\| + \alpha_n\|\gamma g(x_n) - VSw_n\|\bigr), \end{aligned} \tag{3.8}$$

we have that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &\le (1-\beta_n)^2\big(\|Sw_n-x_n\|+\alpha_n\|\gamma g(x_n)-VSw_n\|\big)^2\\&= (1-\beta_n)^2\|Sw_n-x_n\|^2+(1-\beta_n)^2 2\alpha_n\|Sw_n-x_n\|\big\|\gamma g(x_n)-VSw_n\big\|\\&\quad+(1-\beta_n)^2\alpha_n^2\big\|\gamma g(x_n)-VSw_n\big\|^2.\end{aligned}\tag{3.9}$$

Thus, we have from (3.7) and (3.9) that

$$\begin{aligned}\Gamma_{n+1}-\Gamma_n &\le \|x_n-x_{n+1}\|^2+2(1-\beta_n)\alpha_n\big\langle\gamma g(x_n),x_n-z_0\big\rangle\\&\quad-(1-\beta_n)\|Sw_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle\\&\le (1-\beta_n)^2\|Sw_n-x_n\|^2+(1-\beta_n)^2 2\alpha_n\|Sw_n-x_n\|\big\|\gamma g(x_n)-VSw_n\big\|\\&\quad+(1-\beta_n)^2\alpha_n^2\big\|\gamma g(x_n)-VSw_n\big\|^2+2(1-\beta_n)\alpha_n\big\langle\gamma g(x_n),x_n-z_0\big\rangle\\&\quad-(1-\beta_n)\|Sw_n-x_n\|^2-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle\end{aligned}$$

and hence

$$\begin{aligned}&\Gamma_{n+1}-\Gamma_n+\beta_n(1-\beta_n)\|Sw_n-x_n\|^2\\&\quad\le (1-\beta_n)^2 2\alpha_n\|Sw_n-x_n\|\big\|\gamma g(x_n)-VSw_n\big\|\\&\qquad+(1-\beta_n)^2\alpha_n^2\big\|\gamma g(x_n)-VSw_n\big\|^2+2(1-\beta_n)\alpha_n\big\langle\gamma g(x_n),x_n-z_0\big\rangle\\&\qquad-2(1-\beta_n)\alpha_n\langle VSw_n,x_n-z_0\rangle.\end{aligned}\tag{3.10}$$

We divide the proof into two cases.

Case 1: Suppose that $\Gamma_{n+1}\le\Gamma_n$ for all $n\in\mathbb{N}$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n)=0$. Using $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$ and $\lim_{n\to\infty}\alpha_n=0$, we have from (3.10) that

$$\lim_{n\to\infty}\|Sw_n-x_n\|=0.\tag{3.11}$$

Using (3.8), we also have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.12}$$

Since $x_{n+1}-x_n=(1-\beta_n)(y_n-x_n)$, we have from (3.12) that

$$\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{3.13}$$

We also have from (2.6) that

$$\begin{aligned}2\|u_n-z_0\|^2 &= 2\|T_{r_n}x_n-T_{r_n}z_0\|^2\\&\le 2\langle x_n-z_0,u_n-z_0\rangle\\&= \|x_n-z_0\|^2+\|u_n-z_0\|^2-\|u_n-x_n\|^2\end{aligned}$$

and hence

$$\|u_n - z_0\|^2 \le \|x_n - z_0\|^2 - \|u_n - x_n\|^2. \tag{3.14}$$

From (3.1) we have that

$$\|Sw_n - z_0\|^2 \le \|u_n - z_0\|^2 \le \|x_n - z_0\|^2 - \|u_n - x_n\|^2$$

and hence

$$\|u_n - x_n\|^2 \le \|x_n - z_0\|^2 - \|Sw_n - z_0\|^2 \le M\|Sw_n - x_n\|^2,$$

where $M = \sup\{\|x_n - z_0\| + \|Sw_n - z_0\| : n \in \mathbb{N}\}$. Thus, from (3.11) we have that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.15}$$

We show $\lim_{n\to\infty} \|Sw_n - w_n\| = 0$. Since $\|\cdot\|^2$ is a convex function, we have that

$$\|x_{n+1} - z_0\|^2 \le \beta_n\|x_n - z_0\|^2 + (1 - \beta_n)\|y_n - z_0\|^2. \tag{3.16}$$

From $z_0 = \alpha_n Vz_0 + z_0 - \alpha_n Vz_0$ and (2.1), we also have that

$$\begin{aligned}
\|y_n - z_0\|^2 &= \left\|\alpha_n\left(\gamma g(x_n) - Vz_0\right) + (I - \alpha_n V)Sw_n - (I - \alpha_n V)z_0\right\|^2 \\
&\le (1 - \alpha_n\tau)^2\|Sw_n - z_0\|^2 + 2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle \\
&\le (1 - \alpha_n\tau)^2\|w_n - z_0\|^2 + 2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle \\
&\le \|w_n - z_0\|^2 + 2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle \\
&\le \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Az\|^2 \\
&\quad + 2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle.
\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17), we have that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \beta_n\|x_n - z_0\|^2 + (1 - \beta_n)\|x_n - z_0\|^2 \\
&\quad + (1 - \beta_n)\left(\lambda_n(\lambda_n - 2\alpha)\|Au_n - Az_0\|^2 + 2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle\right) \\
&= \|x_n - z_0\|^2 + (1 - \beta_n)\left(\lambda_n(\lambda_n - 2\alpha)\|Au_n - Az_0\|^2\right. \\
&\quad \left. + 2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle\right).
\end{aligned} \tag{3.18}$$

Thus, we have that

$$\begin{aligned}
&(1 - \beta_n)\lambda_n(2\alpha - \lambda_n)\|Au_n - Az\|^2 \\
&\quad \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + (1 - \beta_n)2\alpha_n\left\langle\gamma g(x_n) - Vz_0, y_n - z_0\right\rangle.
\end{aligned} \tag{3.19}$$

Then we have that

$$\lim_{n\to\infty} \|Au_n - Az_0\| = 0. \tag{3.20}$$

Since $J_{\lambda_n}$ is firmly nonexpansive, we have that

$$\begin{aligned}
2\|w_n - z_0\|^2 &= 2\big\|J_{\lambda_n}(u_n - \lambda_n Au_n) - J_{\lambda_n}(z_0 - \lambda_n Az_0)\big\|^2 \\
&\le 2\big\langle u_n - \lambda_n Au_n - (z_0 - \lambda_n Az_0), w_n - z_0\big\rangle \\
&= \big\|u_n - \lambda_n Au_n - (z_0 - \lambda_n Az_0)\big\|^2 + \|w_n - z_0\|^2 \\
&\quad - \big\|u_n - \lambda_n Au_n - (z_0 - \lambda_n Az_0) - (w_n - z_0)\big\|^2 \\
&\le \|u_n - z_0\|^2 + \|w_n - z_0\|^2 \\
&\quad - \big\|u_n - w_n - \lambda_n(Au_n - Az_0)\big\|^2 \\
&\le \|x_n - z_0\|^2 + \|w_n - z_0\|^2 - \|u_n - w_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - w_n, Au_n - Az_0\rangle - \lambda_n^2\|Au_n - Az_0\|^2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|w_n - z_0\|^2 &\le \|x_n - z_0\|^2 - \|u_n - w_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - w_n, Au_n - Az_0\rangle - \lambda_n^2\|Au_n - Az_0\|^2.
\end{aligned} \tag{3.21}$$

Using (3.17), we obtain

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|y_n - z_0\|^2 \\
&\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\big(\|w_n - z_0\|^2 + 2\alpha_n\big\langle \gamma g(x_n) - Vz_0, y_n - z_0\big\rangle\big) \\
&\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|x_n - z_0\|^2 \\
&\quad - (1-\beta_n)\|u_n - w_n\|^2 + (1-\beta_n)2\lambda_n\langle u_n - w_n, Au_n - Az_0\rangle \\
&\quad - (1-\beta_n)\lambda_n^2\|Au_n - Az_0\|^2 + (1-\beta_n)2\alpha_n\big\langle \gamma g(x_n) - Vz_0, y_n - z_0\big\rangle \\
&= \|x_n - z_0\|^2 - (1-\beta_n)\|u_n - w_n\|^2 \\
&\quad + (1-\beta_n)2\lambda_n\langle u_n - w_n, Au_n - Az_0\rangle - (1-\beta_n)\lambda_n^2\|Au_n - Az_0\|^2 \\
&\quad + (1-\beta_n)2\alpha_n\big\langle \gamma g(x_n) - Vz_0, y_n - z_0\big\rangle,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
(1-\beta_n)\|x_n - w_n\|^2 &\le \|x_n - z_0\|^2 \\
&\quad - \|x_{n+1} - z_0\|^2 + 2\lambda_n\langle u_n - w_n, Au_n - Az_0\rangle \\
&\quad - \lambda_n^2\|Au_n - Az_0\|^2 + 2\alpha_n\big\langle \gamma g(x_n) - Vz_0, y_n - z_0\big\rangle.
\end{aligned}$$

Then we have

$$\lim_{n\to\infty}\|u_n - w_n\| = 0. \tag{3.22}$$

From (3.22) and (3.15), we have that

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.23}$$

Since $\|Sw_n - w_n\| \leq \|Sw_n - x_n\| + \|x_n - w_n\|$, we have that

$$\lim_{n\to\infty} \|Sw_n - w_n\| = 0. \tag{3.24}$$

Take $\lambda_0 \in \mathbb{R}$ with $0 < a \leq \lambda_0 \leq b < 2\alpha$ arbitrarily. Put $s_n = (I - \lambda_n A)u_n$. Using $u_n = T_{r_n}x_n$ and $w_n = J_{\lambda_n}(I - \lambda_n A)u_n$, we have from Lemma 4 that

$$\begin{aligned}
\big\|J_{\lambda_0}(I-\lambda_0 A)u_n - w_n\big\| &= \big\|J_{\lambda_0}(I-\lambda_0 A)u_n - J_{\lambda_n}(I-\lambda_n A)u_n\big\| \\
&= \big\|J_{\lambda_0}(I-\lambda_0 A)u_n - J_{\lambda_0}(I-\lambda_n A)u_n \\
&\quad + J_{\lambda_0}(I-\lambda_n A)u_n - J_{\lambda_n}(I-\lambda_n A)u_n\big\| \\
&\leq \big\|(I-\lambda_0 A)u_n - (I-\lambda_n A)u_n\big\| + \|J_{\lambda_0}s_n - J_{\lambda_n}s_n\| \\
&\leq |\lambda_0 - \lambda_n|\|Au_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0}s_n - s_n\|.
\end{aligned} \tag{3.25}$$

We also have from (3.25) that

$$\big\|u_n - J_{\lambda_0}(I-\lambda_0 A)u_n\big\| \leq \|u_n - w_n\| + \big\|w_n - J_{\lambda_0}(I-\lambda_0 A)u_n\big\|. \tag{3.26}$$

We will use (3.25) and (3.26) later.

Let us show that $\limsup_{n\to\infty}\langle (V-\gamma g)z_0, x_n - z_0\rangle \geq 0$. Put

$$A = \limsup_{n\to\infty}\big\langle (V-\gamma g)z_0, x_n - z_0\big\rangle.$$

Without loss of generality, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $A = \lim_{i\to\infty}\langle (V-\gamma g)z_0, x_{n_i} - z_0\rangle$ and $\{x_{n_i}\}$ converges weakly to some point $w \in H$. From $\|x_n - w_n\| \to 0$ and $\|x_n - u_n\| \to 0$, we also have that $\{w_{n_i}\}$ and $\{u_{n_i}\}$ converge weakly to $w \in C$. On the other hand, from $\{\lambda_{n_i}\} \subset [a,b]$ there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in [a,b]$. Without loss of generality, we assume that $w_{n_i} \to w$, $u_{n_i} \to w$ and $\lambda_{n_i} \to \lambda_0$. From (3.24) we know $\lim_{n\to\infty}\|Sw_n - w_n\| = 0$. Thus, we have from Lemma 3 that $w = Sw$. Since $W$ is a monotone operator and $\frac{x_{n_i} - u_{n_i}}{r_{n_i}} \in Wu_{n_i}$, we have that for any $(u,v) \in W$,

$$\left\langle u - u_{n_i}, v - \frac{x_{n_i} - u_{n_i}}{r_{n_i}}\right\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_i} \rightharpoonup w$ and $x_{n_i} - u_{n_i} \to 0$, we have

$$\langle u - w, v\rangle \geq 0.$$

Since $W$ is a maximal monotone operator, we have $0 \in Ww$ and hence $w \in W^{-1}0$. Since $\lambda_{n_i} \to \lambda_0$, we have from (3.25) that

$$\big\|J_{\lambda_0}(I-\lambda_0 A)u_{n_i} - w_{n_i}\big\| \to 0.$$

Furthermore, we have from (3.26) that

$$\big\|u_{n_i} - J_{\lambda_0}(I-\lambda_0 A)u_{n_i}\big\| \to 0.$$

Since $J_{\lambda_0}(I - \lambda_0 A)$ is nonexpansive, we have that $w = J_{\lambda_0}(I - \lambda_0 A)w$. This means that $0 \in Aw + Bw$. Thus, we have

$$w \in F(T) \cap (A + B)^{-1}0 \cap W^{-1}0.$$

Then we have

$$A = \lim_{i\to\infty} \langle (V - \gamma g)z_0, x_{n_i} - z_0 \rangle = \langle (V - \gamma g)z_0, w - z_0 \rangle \geq 0. \tag{3.27}$$

Since $y_n - z_0 = \alpha_n(\gamma g(x_n) - Vz_0) + (I - \alpha_n V)Sw_n - (I - \alpha_n V)z_0$, we have

$$\|y_n - z_0\|^2 \leq (1 - \alpha_n \tau)^2 \|Sw_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Vz_0, y_n - z_0 \rangle.$$

Thus, we have

$$\|y_n - z_0\|^2 \leq (1 - \alpha_n \tau)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Vz_0, y_n - z_0 \rangle.$$

Consequently, we have that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \beta_n \|x_n - z_0\|^2 + (1 - \beta_n)\|y_n - z_0\|^2 \\
&\leq \beta_n \|x_n - z_0\|^2 \\
&\quad + (1 - \beta_n)\big((1 - \alpha_n \tau)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle \gamma g(x_n) - Vz_0, y_n - z_0 \rangle\big) \\
&= \big(\beta_n + (1 - \beta_n)(1 - \alpha_n \tau)^2\big) \|x_n - z_0\|^2 \\
&\quad + 2(1 - \beta_n)\alpha_n \langle \gamma g(x_n) - Vz_0, y_n - z_0 \rangle \\
&\leq \big(1 - (1 - \beta_n)\big(2\alpha_n \tau - (\alpha_n \tau)^2\big)\big) \|x_n - z_0\|^2 \\
&\quad + 2(1 - \beta_n)\alpha_n \gamma k \|x_n - z_0\|^2 + 2(1 - \beta_n)\alpha_n \langle \gamma g(z_0) - Vz_0, y_n - z_0 \rangle \\
&= \big(1 - 2(1 - \beta_n)\alpha_n(\tau - \gamma k)\big) \|x_n - z_0\|^2 \\
&\quad + (1 - \beta_n)(\alpha_n \tau)^2 \|x_n - z_0\|^2 + 2(1 - \beta_n)\alpha_n \langle \gamma g(z_0) - Vz_0, y_n - z_0 \rangle \\
&= \big(1 - 2(1 - \beta_n)\alpha_n(\tau - \gamma k)\big) \|x_n - z_0\|^2 \\
&\quad + 2(1 - \beta_n)\alpha_n(\tau - \gamma k)\left(\frac{\alpha_n \tau^2 \|x_n - z_0\|^2}{2(\tau - \gamma k)} + \frac{\langle \gamma g(z_0) - Vz_0, y_n - z_0 \rangle}{\tau - \gamma k}\right).
\end{aligned}$$

By (3.27) and Lemma 5, we obtain that $x_n \to z_0$, where

$$z_0 = P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)z_0.$$

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 6 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Thus, we have from (3.10) that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\beta_{\tau(n)}(1-\beta_{\tau(n)})\|Sw_{\tau(n)} - x_{\tau(n)}\|^2 \\
&\quad \le (1-\beta_{\tau(n)})^2 2\alpha_{\tau(n)}\|Sw_{\tau(n)} - x_{\tau(n)}\| \big\|\gamma g(x_{\tau(n)}) - VSw_{\tau(n)}\big\| \\
&\qquad + (1-\beta_{\tau(n)})^2\alpha_{\tau(n)}^2 \big\|\gamma g(x_{\tau(n)}) - VSw_{\tau(n)}\big\|^2 \\
&\qquad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\big\langle \gamma g(x_{\tau(n)}), x_{\tau(n)} - z_0\big\rangle \\
&\qquad - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\langle VSw_{\tau(n)}, x_{\tau(n)} - z_0\rangle. 
\end{aligned} \tag{3.28}$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have from (3.28) and Lemma 6 that

$$\lim_{n\to\infty}\big\|Sw_\tau(n) - x_\tau(n)\big\| = 0. \tag{3.29}$$

As in the proof of Case 1, we also have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{3.30}$$

and

$$\lim_{n\to\infty}\|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.31}$$

Furthermore, we have that $\lim_{n\to\infty}\|u_{\tau(n)} - x_{\tau(n)}\| = 0$, $\lim_{n\to\infty}\|Au_{\tau(n)} - Az_0\| = 0$, $\lim_{n\to\infty}\|u_{\tau(n)} - w_{\tau(n)}\| = 0$ and $\lim_{n\to\infty}\|x_{\tau(n)} - w_{\tau(n)}\| = 0$. From these we have that $\lim_{n\to\infty}\|Sw_{\tau(n)} - w_{\tau(n)}\| = 0$. As in the proof of Case 1, we can show that

$$\limsup_{n\to\infty}\big\langle (V-\gamma g)z_0, x_{\tau(n)} - z_0\big\rangle \ge 0.$$

We also have that

$$\|y_{\tau(n)} - z_0\|^2 \le (1-\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - z_0\|^2 + 2\alpha_{\tau(n)}\big\langle \gamma g(x_{\tau(n)}) - Vz_0, y_{\tau(n)} - z_0\big\rangle$$

and hence

$$\begin{aligned}
\|x_{\tau(n)+1} - z_0\|^2 &\le \big(1 - 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}(\tau - \gamma k)\big)\|x_{\tau(n)} - z_0\|^2 \\
&\quad + (1-\beta_{\tau(n)})(\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - z_0\|^2 \\
&\quad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\big\langle \gamma g(z_0) - Vz_0, y_{\tau(n)} - z_0\big\rangle.
\end{aligned}$$

From $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have that

$$\begin{aligned}
&2(1-\beta_{\tau(n)})\alpha_{\tau(n)}(\tau - \gamma k)\|x_{\tau(n)} - z_0\|^2 \\
&\quad \le (1-\beta_{\tau(n)})(\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - z_0\|^2 \\
&\qquad + 2(1-\beta_{\tau(n)})\alpha_{\tau(n)}\big\langle \gamma g(z_0) - Vz_0, y_{\tau(n)} - z_0\big\rangle.
\end{aligned}$$

Since $(1 - \beta_{\tau(n)})\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned} &2(\tau - \gamma k)\|x_{\tau(n)} - z_0\|^2 \\ &\quad \le \alpha_{\tau(n)}\tau^2\|x_{\tau(n)} - z_0\|^2 + 2\langle \gamma g(z_0) - Vz_0, y_{\tau(n)} - z_0\rangle. \end{aligned}$$

Thus, we have that

$$\limsup_{n\to\infty} 2(\tau - \gamma k)\|x_{\tau(n)} - z_0\|^2 \le 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$ as $n \to \infty$. Since $x_{\tau(n)} - x_{\tau(n)+1} \to 0$, we have $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 6 again, we obtain that

$$\|x_n - z_0\| \le \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

## 4 Applications

In this section, using Theorem 10, we can obtain well-known and new strong convergence theorems in a Hilbert space. Let $H$ be a Hilbert space, and let $f$ be a proper lower semicontinuous convex function of $H$ into $(-\infty, \infty]$. Then the subdifferential $\partial f$ of $f$ is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \le f(y), \forall y \in H\}$$

for all $x \in H$. From Rockafellar [22], we know that $\partial f$ is a maximal monotone operator. Let $C$ be a nonempty closed convex subset of $H$, and let $i_C$ be the indicator function of $C$, *i.e.*,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, $i_C$ is a proper lower semicontinuous convex function on $H$. So, we can define the resolvent $J_\lambda$ of $\partial i_C$ for $\lambda > 0$, *i.e.*,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We know that $J_\lambda x = P_C x$ for all $x \in H$ and $\lambda > 0$; see [11].

**Theorem 11** *Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $S$ be a generalized hybrid mapping of $C$ into $C$. Suppose $F(S) \ne \emptyset$. Let $u, x_1 \in C$, and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (1 - \alpha_n)Sx_n\}$$

*for all $n \in \mathbb{N}$, where $\{\beta_n\} \subset (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty$$

*and*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

*Then the sequence* $\{x_n\}$ *converges strongly to* $z_0 \in F(S)$, *where* $z_0 = P_{F(S)}u$.

*Proof* Put $A = 0$, $B = W = \partial i_C$ and $\lambda_n = r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 10. Then we have $J_{\lambda_n} = T_{r_n} = P_C$ for all $n \in \mathbb{N}$. Furthermore, put $g(x) = u$ and $V(x) = x$ for all $x \in H$. Then we can take $\overline{\gamma} = L = 1$. Thus, we can take $\mu = 1$. On the other hand, since $\|g(x) - g(y)\| = 0 \le \frac{1}{3}\|x - y\|$ for all $x, y \in H$, we can take $k = \frac{1}{3}$. So, we can take $\gamma = 1$. Then for $u, x_1 \in C$, we get that

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n u + (I - \alpha_n)Sx_n\}$$

for all $n \in \mathbb{N}$. So, we have $\{x_n\} \subset C$. We also have

$$z_0 = P_{F(S)\cap C}(I - V + \gamma g)z_0 = P_{F(S)}(z_0 - z_0 + 1 \cdot u) = P_{F(S)}u.$$

Thus, we obtain the desired result by Theorem 10. □

Theorem 11 solves the problem posed by Kurokawa and Takahashi [9]. The following result is a strong convergence theorem of Halpern's type [6] for finding a common solution of a monotone inclusion problem for the sum of two monotone mappings, of a fixed point problem for nonexpansive mappings and of an equilibrium problem for bifunctions in a Hilbert space.

**Theorem 12** *Let* $H$ *be a real Hilbert space, and let* $C$ *be a nonempty closed convex subset of* $H$. *Let* $\alpha > 0$, *and let* $A$ *be an* $\alpha$*-inverse strongly-monotone mapping of* $C$ *into* $H$. *Let* $B$ *and* $W$ *be maximal monotone operators on* $H$ *such that the domains of* $B$ *and* $W$ *are included in* $C$. *Let* $J_\lambda = (I + \lambda B)^{-1}$ *and* $T_r = (I + rW)^{-1}$ *be resolvents of* $B$ *and* $W$ *for* $\lambda > 0$ *and* $r > 0$, *respectively. Let* $S$ *be a nonexpansive mapping of* $C$ *into* $H$. *Let* $0 < k < 1$, *and let* $g$ *be a* $k$*-contraction of* $H$ *into itself. Let* $V$ *be a* $\overline{\gamma}$*-strongly monotone and* $L$*-Lipschitzian continuous operator with* $\overline{\gamma} > 0$ *and* $L > 0$. *Take* $\mu, \gamma \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

*Suppose* $F(S) \cap (A + B)^{-1}0 \cap W^{-1}0 \neq \emptyset$. *Let* $x_1 = x \in H$, *and let* $\{x_n\} \subset H$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n V)SJ_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n\}$$

*for all* $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,\infty)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$$

$$\liminf_{n\to\infty} r_n > 0 \quad \textit{and} \quad 0 < a \le \lambda_n \le b < 2\alpha.$$

*Then the sequence* $\{x_n\}$ *converges strongly to* $z_0 \in F(S) \cap (A+B)^{-1}0 \cap W^{-1}0$, *where* $z_0 = P_{F(S)\cap(A+B)^{-1}0\cap W^{-1}0}(I - V + \gamma g)z_0$.

*Proof* We know that a nonexpansive mapping $T$ of $C$ into $H$ is a $(1,0)$-generalized hybrid mapping. So, we obtain the desired result by Theorem 10. □

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (with respect to $C$) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, *i.e.*,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) $f$ is monotone, *i.e.*, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f\bigl(tz + (1-t)x, y\bigr) \leq f(x,y);$$

(A4) for all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

The following lemmas were given in Combettes and Hirstoaga [23] and Takahashi, Takahashi and Toyoda [16]; see also [24, 25].

**Lemma 13** ([23]) *Let* $H$ *be a real Hilbert space, and let* $C$ *be a nonempty closed convex subset of* $H$. *Assume that* $f : C \times C \to \mathbb{R}$ *satisfies* (A1)-(A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows*:

$$T_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

*for all* $x \in H$. *Then the following hold*:

(1) $T_r$ *is single-valued*;

(2) $T_r$ *is a firmly nonexpansive mapping, i.e., for all* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ *is closed and convex.*

We call such $T_r$ the resolvent of $f$ for $r > 0$.

**Lemma 14** ([16]) *Let* $H$ *be a Hilbert space, and let* $C$ *be a nonempty closed convex subset of* $H$. *Let* $f : C \times C \to \mathbb{R}$ *satisfy* (A1)-(A4). *Let* $A_f$ *be a set-valued mapping of* $H$ *into itself* *defined by*

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

*Then* $EP(f) = A_f^{-1}0$ *and* $A_f$ *is a maximal monotone operator with* $D(A_f) \subset C$. *Furthermore, for any* $x \in H$ *and* $r > 0$, *the resolvent* $T_r$ *of* $f$ *coincides with the resolvent of* $A_f$, *i.e.*,

$$T_r x = (I + rA_f)^{-1}x.$$

Using Lemmas 13, 14 and Theorem 10, we also obtain the following result for generalized hybrid mappings of $C$ into $H$ with equilibrium problem in a Hilbert space; see also [26–28].

**Theorem 15** *Let* $H$ *be a real Hilbert space, and let* $C$ *be a nonempty closed convex subset of* $H$. *Let* $S$ *be a generalised hybrid mapping of* $C$ *into* $H$. *Let* $f$ *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $0 < k < 1$, *and let* $g$ *be a* $k$*-contraction of* $H$ *into itself. Let* $V$ *be a* $\overline{\gamma}$*-strongly monotone and* $L$*-Lipschitzian continuous operator of* $H$ *into itself with* $\overline{\gamma} > 0$ *and* $L > 0$. *Take* $\mu, \gamma \in \mathbb{R}$ *as follows*:

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \qquad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

*Suppose that* $F(S) \cap EP(f) \neq \emptyset$. *Let* $x_1 = x \in H$, *and let* $\{x_n\} \subset H$ *be a sequence generated by*

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \beta_n x_n + (1 - \beta_n)\{\alpha_n \gamma g(x_n) + (I - \alpha_n V)Su_n\}$$

*for all* $n \in \mathbb{N}$, *where* $\{\beta_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \liminf_{n\to\infty} r_n > 0,$$
$$\text{and} \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

*Then the sequence* $\{x_n\}$ *converges strongly to* $z_0 \in F(S) \cap EP(f)$, *where* $z_0 = P_{F(S)\cap EP(f)}(I - V + \gamma g)z_0$.

*Proof* Put $A = 0$ and $B = \partial i_C$ in Theorem 10. Furthermore, for the bifunction $f : C \times C \to \mathbb{R}$, define $A_f$ as in Lemma 14. Put $W = A_f$ in Theorem 10, and let $T_{r_n}$ be the resolvent of $A_f$ for $r_n > 0$. Then we obtain that the domain of $A_f$ is included in $C$ and $T_{r_n}x_n = u_n$ for all $n \in \mathbb{N}$. Thus, we obtain the desired result by Theorem 10. □

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All three authors take equal roles in deriving results and writing of this paper.

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