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Projection methods of iterative solutions in Hilbert spaces

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Abstract
In this paper, zero points of the sum of two monotone mappings, solutions of a monotone variational inequality, and fixed points of a nonexpansive mapping are investigated based on a hybrid projection iterative algorithm. Strong convergence of the purposed iterative algorithm is obtained in the framework of real Hilbert spaces without any compact assumptions.
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1 Introduction and preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonlinear mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Tx\}$.

Recall that S is said to be *nonexpansive* iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

If C is a bounded, closed, and convex subset of H, then $F(S)$ is not empty, closed, and convex; see [1].

Let $A : C \to H$ be a mapping. Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-inverse-strongly monotone*.

A is said to be *monotone* iff

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, we use $\mathrm{VI}(C,A)$ to denote the solution set of (1.1). It is known that $x \in C$ is a solution of (1.1) if and only if x is a fixed point of the mapping $\mathrm{Proj}_C(I - rA)$, where $r > 0$ is a constant, I stands for the identity mapping, and Proj_C stands for the metric projection from H onto C. If A is α-inverse-strongly monotone and $r \in (0, 2\alpha]$, then the mapping $\mathrm{Proj}_C(I - rA)$ is nonexpansive; see [2] for more details. It follows that $\mathrm{VI}(C,A)$ is closed and convex.

Monotone variational inequality theory has emerged as a fascinating branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, ecology, social, regional, pure, and applied sciences. In recent years, much attention has been given to developing efficient numerical methods for treating solution problems of monotone variational inequality. The gradient-projection method is a powerful tool for solving constrained convex optimization problems and has extensively been studied; see [3–5] and the references therein. It has recently been applied to solving split feasibility problems which find applications in image reconstructions and the intensity modulated radiation theory; see [6–9] and the references therein. However, the gradient-projection method requires the operator to be strongly monotone and Lipschitz continuous. These strong conditions rule out many applications. Extra gradient-projection method which was first introduce by Korpelevich [10] in the finite dimensional Euclidean space has been studied recently for relaxing the strong monotonicity of operators; see [11–13] and the references therein.

Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle > 0$. A monotone mapping $M : H \rightrightarrows H$ is *maximal* iff the graph $\mathrm{Graph}(M)$ of R is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in \mathrm{Graph}(M)$ implies $f \in Rx$.

For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r : H \to D(M)$, where $D(M)$ denotes the domain of M. It is known that J_r is firmly nonexpansive, and $M^{-1}(0) = F(J_r)$, where $F(J_r) := \{x \in D(M) : x = J_r x\}$ and $M^{-1}(0) : \{x \in H : 0 \in Mx\}$.

Recently, variational inequalities, fixed point problems, and zero point problems have been investigated by many authors based on iterative methods; see, for example, [14–32] and the references therein. In this paper, zero point problems of the sum of a maximal monotone operator and an inverse-strongly monotone mapping, solution problems of a monotone variational inequality, and fixed point problems of a nonexpansive mapping are investigated. A hybrid iterative algorithm is considered for analyzing the convergence of iterative sequences. Strong convergence theorems are established in the framework of real Hilbert spaces without any compact assumptions.

In order to prove our main results, we also need the following definitions and lemmas.

Lemma 1.1 *Let C be a nonempty, closed, and convex subset of H. Then the following inequality holds*:

$$\|x - \mathrm{Proj}_C x\|^2 + \|y - \mathrm{Proj}_C\|^2 \leq \|x - y\|^2, \quad \forall x \in H, y \in C.$$

Lemma 1.2 [1] *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $x_n - Sx_n \to 0$, then $\bar{x} \in F(S)$.*

Lemma 1.3 *Let C be a nonempty, closed, and convex subset of H, $B : C \to H$ be a mapping, and $M : H \rightrightarrows H$ be a maximal monotone operator. Then $F(J_r(I - sB)) = (B + M)^{-1}(0)$.*

Proof Notice that

$$\begin{aligned} &p \in F\big(J_r(I-sB)\big) \quad \Longleftrightarrow \quad p = J_r(I-sB)p \quad \Longleftrightarrow \quad p - sBp \in p + sMp \\ &\quad \Longleftrightarrow \quad 0 \in (B+M)^{-1}(0) \quad \Longleftrightarrow \quad p \in (B+M)^{-1}(0). \end{aligned}$$

This completes the proof. □

Lemma 1.4 [33] *Let C be a nonempty, closed, and convex subset of H, $A : C \to H$ be a Lipschitz monotone mapping, and $N_C x$ be the normal cone to C at $x \in C$; that is, $N_C x = \{y \in H : \langle x - u, y\rangle, \forall u \in C\}$. Define*

$$Wx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then W is maximal monotone and $0 \in Wx$ if and only if $x \in \mathrm{VI}(C, A)$.

2 Main results

Now, we are in a position to give our main results.

Theorem 2.1 *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set, $A : C \to H$ be an α-Lipschitz continuous and monotone mapping, and $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $M : H \rightrightarrows H$ be a maximal monotone operator such that $D(M) \subset C$. Assume that $\mathcal{F} := F(S) \cap (B + M)^{-1}(0) \cap \mathrm{VI}(C, A)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \mathrm{Proj}_C(J_{s_n}(x_n - s_n Bx_n) - r_n A J_{s_n}(x_n - s_n Bx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n) S \,\mathrm{Proj}_C(J_{s_n}(x_n - s_n Bx_n) - r_n A z_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases} \tag{2.1}$$

where $J_{s_n} = (I + s_n M)^{-1}$, $\{r_n\}$ is a sequence in $(0, \frac{1}{\alpha})$, $\{s_n\}$ is a sequence in $(0, 2\beta)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$. Assume that the following restrictions are satisfied:

(a) $0 < a \le r_n \le b < \frac{1}{\alpha}$;
(b) $0 < c \le s_n \le d < 2\beta$;
(c) $0 \le \alpha_n \le e < 1$,

where a, b, c, d, and e are real constants. Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathcal{F}} x_1$.

Proof First, we show that C_n is closed and convex for each $n \ge 1$. From the assumption, we see that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some

$m \geq 1$. We show that C_{m+1} is closed and convex for the same m. Let $v_1, v_2 \in C_{m+1}$ and $v = tv_1 + (1-t)v_2$, where $t \in (0,1)$. Notice that

$$\|y_m - v\| \leq \|x_m - v\|$$

is equivalent to

$$\|y_m\|^2 - \|x_m\|^2 - 2\langle v, y_m - x_m \rangle \geq 0.$$

It is clear that $v \in C_{m+1}$. This shows that C_n is closed and convex for each $n \geq 1$.

Next, we show that $\mathcal{F} \subset C_n$ for each $n \geq 1$. Put $u_n = \mathrm{Proj}_C(v_n - r_n A z_n)$, where $v_n = J_{s_n}(x_n - s_n B x_n)$. From the assumption, we see that $\mathcal{F} \subset C = C_1$. Suppose that $\mathcal{F} \subset C_m$ for some $m \geq 1$. For any $p \in \mathcal{F} \subset C_m$, we see from Lemma 1.1 that

$$\begin{aligned}
\|u_m - p\|^2 &\leq \|v_m - r_m A z_m - p\|^2 - \|v_m - r_m A z_m - u_m\|^2 \\
&= \|v_m - p\|^2 - \|v_m - u_m\|^2 + 2r_m \langle A z_m, p - u_m \rangle \\
&= \|v_m - p\|^2 - \|v_m - u_m\|^2 + 2r_m \bigl(\langle A z_m - Ap, p - z_m \rangle + \langle Ap, p - z_m \rangle \\
&\quad + \langle A z_m, z_m - u_m \rangle \bigr) \\
&\leq \|v_m - p\|^2 - \|v_m - z_m + z_m - u_m\|^2 + 2r_m \langle A z_m, z_m - u_m \rangle \\
&= \|v_m - p\|^2 - \|v_m - z_m\|^2 - \|z_m - u_m\|^2 \\
&\quad + 2\langle v_m - z_m - r_m A z_m, u_m - z_m \rangle.
\end{aligned} \tag{2.2}$$

Notice that A is Lipschitz continuous. In view of $z_m = \mathrm{Proj}_C(v_m - r_m A v_m)$, we find that

$$\begin{aligned}
&\langle v_m - z_m - r_m A z_m, u_m - z_m \rangle \\
&\quad = \langle v_m - z_m - r_m A v_m, u_m - z_m \rangle + \langle r_m A v_m - r_m A z_m, u_m - z_m \rangle \\
&\quad \leq r_m \alpha \|v_m - z_m\| \|u_m - z_m\|.
\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain that

$$\begin{aligned}
\|u_m - p\|^2 &\leq \|v_m - p\|^2 - \|v_m - z_m\|^2 - \|z_m - u_m\|^2 + 2r_m \alpha \|v_m - z_m\| \|u_m - z_m\| \\
&\leq \|v_m - p\|^2 - \bigl(1 - r_m^2 \alpha^2\bigr) \|v_m - z_m\|^2.
\end{aligned} \tag{2.4}$$

This in turn implies from restriction (a) that

$$\begin{aligned}
\|y_m - p\|^2 &\leq \alpha_m \|x_m - p\|^2 + (1 - \alpha_m) \|S u_m - p\|^2 \\
&\leq \alpha_m \|x_m - p\|^2 + (1 - \alpha_m) \|u_m - p\|^2 \\
&\leq \alpha_m \|x_m - p\|^2 + (1 - \alpha_m) \bigl(\|v_m - p\|^2 - \bigl(1 - r_m^2 \alpha^2\bigr) \|v_m - z_m\|^2 \bigr) \\
&\leq \|x_m - p\|^2 - (1 - \alpha_m)\bigl(1 - r_m^2 \alpha^2\bigr) \|v_m - z_m\|^2 \\
&\leq \|x_m - p\|^2.
\end{aligned} \tag{2.5}$$

This shows that $p \in C_{m+1}$. This proves that $\mathcal{F} \subset C_n$ for each $n \geq 1$. Note $x_n = \mathrm{Proj}_{C_n} x_1$. For each $p \in \mathcal{F} \subset C_n$, we have $\|x_1 - x_n\| \leq \|x_1 - p\|$. Since B is inverse-strongly monotone, we see from Lemma 1.3 that $(B+M)^{-1}(0)$ is closed and convex. Since A is Lipschitz continuous, we find that $\mathrm{VI}(C,A)$ is close and convex. This proves that $\mathcal{F}$ is closed and convex. It follows that

$$\|x_1 - x_n\| \leq \|x_1 - \mathrm{Proj}_{\mathcal{F}} x_1\|. \tag{2.6}$$

This implies that $\{x_n\}$ is bounded. Since $x_n = \mathrm{Proj}_{C_n} x_1$ and $x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\| \|x_1 - x_{n+1}\|. \end{aligned}$$

It follows that

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|.$$

This proves that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Notice that

$$\begin{aligned} &\|x_n - x_{n+1}\|^2 \\ &\quad = \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\quad = \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\quad \leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.7}$$

In view of $x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1 \in C_{n+1}$, we see that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

This implies that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|.$$

From (2.7), we find that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.8}$$

Since B is β-inverse-strongly monotone, we see from restriction (b) that

$$\begin{aligned}\bigl\|(I-s_nB)x-(I-s_nB)y\bigr\|^2 &= \|x-y\|^2-2s_n\langle x-y,Bx-By\rangle+s_n^2\|Bx-By\|^2\\ &\le \|x-y\|^2-s_n(2\beta-s_n)\|Bx-By\|^2\\ &\le \|x-y\|^2,\quad \forall x,y\in C.\end{aligned}$$

This implies from (2.5) that

$$\begin{aligned}\|y_n-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2\\ &= \alpha_n\|x_n-p\|^2+(1-\alpha_n)\bigl\|J_{s_n}(x_n-s_nBx_n)-J_{s_n}(p-s_nBp)\bigr\|^2\\ &\le \|x_n-p\|^2-(1-\alpha_n)s_n(2\beta-s_n)\|Bx_n-Bp\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)s_n(2\beta-s_n)\|Bx_n-Bp\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2\\ &\le \|x_n-y_n\|\bigl(\|x_n-p\|+\|y_n-p\|\bigr).\end{aligned}$$

In view of restrictions (b) and (c), we find from (2.8) that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{2.9}$$

Since J_{s_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|v_n-p\|^2 &= \bigl\|J_{s_n}(x_n-s_nBx_n)-J_{s_n}(p-s_nBp)\bigr\|^2\\ &\le \bigl\langle v_n-p,(x_n-s_nBx_n)-(p-s_nBp)\bigr\rangle\\ &= \frac{1}{2}\bigl(\|v_n-p\|^2+\bigl\|(x_n-s_nBx_n)-(p-s_nBp)\bigr\|^2\\ &\quad -\bigl\|(v_n-p)-\bigl((x_n-s_nBx_n)-(p-s_nBp)\bigr)\bigr\|^2\bigr)\\ &\le \frac{1}{2}\bigl(\|v_n-p\|^2+\|x_n-p\|^2-\bigl\|v_n-x_n+s_n(Bx_n-Bp)\bigr\|^2\bigr)\\ &= \frac{1}{2}\bigl(\|v_n-p\|^2+\|x_n-p\|^2-\|v_n-x_n\|^2-s_n^2\|Bx_n-Bp\|^2\\ &\quad -2s_n\langle v_n-x_n,Bx_n-Bp\rangle\bigr)\\ &\le \frac{1}{2}\bigl(\|v_n-p\|^2+\|x_n-p\|^2-\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|\bigr).\end{aligned}$$

This in turn implies that

$$\|v_n-p\|^2\le\|x_n-p\|^2-\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|. \tag{2.10}$$

Combining (2.5) with (2.10), we arrive at

$$\begin{aligned}\|y_n-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2\\ &\le \|x_n-p\|^2-(1-\alpha_n)\|v_n-x_n\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\|.\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\|v_n-x_n\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2+2s_n\|v_n-x_n\|\|Bx_n-Bp\| \\ &\le \|x_n-y_n\|\big(\|x_n-p\|+\|y_n-p\|\big)+2s_n\|v_n-x_n\|\|Bx_n-Bp\|.\end{aligned}$$

In view of (2.8) and (2.9), we see from restriction (c) that

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \tag{2.11}$$

On the other hand, we find from (2.5) that

$$\begin{aligned}(1-\alpha_n)\big(1-r_n^2\alpha^2\big)\|v_n-z_n\|^2 &\le \|x_n-p\|^2-\|y_n-p\|^2 \\ &\le \|x_n-y_n\|\big(\|x_n-p\|+\|y_n-p\|\big).\end{aligned}$$

In view of restrictions (a) and (c), we obtain from (2.8) that

$$\lim_{n\to\infty}\|v_n-z_n\|=0. \tag{2.12}$$

Notice that

$$\begin{aligned}\|u_n-z_n\|^2 &= \big\|P_C(v_n-r_nAz_n)-P_C(v_n-r_nAv_n)\big\|^2 \\ &\le \big\|(v_n-r_nAz_n)-(v_n-r_nAv_n)\big\|^2 \\ &\le r_n^2\alpha^2\|z_n-v_n\|^2.\end{aligned}$$

Thanks to (2.12), we arrive at

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{2.13}$$

Notice that

$$\begin{aligned}\|x_n-Sx_n\| &\le \|x_n-Su_n\|+\|Su_n-Sx_n\| \\ &\le \frac{\|x_n-y_n\|}{1-\alpha_n}+\|u_n-x_n\| \\ &\le \frac{\|x_n-y_n\|}{1-\alpha_n}+\|u_n-z_n\|+\|z_n-v_n\|+\|v_n-x_n\|.\end{aligned}$$

In view of (2.8), (2.11), (2.12), and (2.13), we find from restriction (c) that

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

Since $\{x_n\}$ is bounded, we find that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup q\in C$. From Lemma 1.2, we easily conclude that $q\in F(S)$.

Now, we are in a position to show that $x\in \mathrm{VI}(C,A)$. Define

$$Wx=\begin{cases}Ax+N_Cx, & x\in C,\\ \emptyset, & x\notin C.\end{cases}$$

For any given $(x,y) \in G(W)$, we have $y - Ax \in N_C x$. Since $u_n \in C$, we see from the definition of N_C

$$\langle x - u_n, y - Ax \rangle \geq 0. \tag{2.14}$$

In view of $u_n = P_C(v_n - r_n A z_n)$, we obtain that

$$\langle x - u_n, u_n + r_n A z_n - v_n \rangle \geq 0$$

and hence

$$\left\langle x - u_n, \frac{u_n - v_n}{r_n} + A z_n \right\rangle \geq 0. \tag{2.15}$$

In view of (2.14) and (2.15), we find that

$$\begin{aligned}
\langle x - u_{n_i}, y \rangle &\geq \langle x - u_{n_i}, Ax \rangle \\
&\geq \langle x - u_{n_i}, Ax \rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{r_{n_i}} + A z_{n_i} \right\rangle \\
&= \langle x - u_{n_i}, Ax - Au_{n_i} \rangle + \langle x - u_{n_i}, Au_{n_i} - Az_{n_i} \rangle \\
&\quad - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{r_{n_i}} \right\rangle \\
&\geq \langle x - u_{n_i}, Au_{n_i} - Az_{n_i} \rangle - \left\langle x - u_{n_i}, \frac{u_{n_i} - v_{n_i}}{r_{n_i}} \right\rangle.
\end{aligned} \tag{2.16}$$

Notice that

$$\|x_n - u_n\| \leq \|x_n - v_n\| + \|v_n - z_n\| + \|z_n - u_n\|.$$

It follows from (2.11), (2.12), and (2.13) that

$$\lim_{n \to \infty} \|x_n - u_n\| = 0.$$

Since A is Lipschitz continuous, we find from (2.16) that

$$\langle x - q, y \rangle \geq 0.$$

Since W is maximal monotone, we conclude that $q \in W^{-1}(0)$. This proves that $q \in \mathrm{VI}(C,A)$.

Finally, we prove that $q \in (B+M)^{-1}(0)$. Notice that

$$x_n - s_n B x_n \in v_n + s_n M v_n;$$

that is,

$$\frac{x_n - v_n}{s_n} - B x_n \in M v_n. \tag{2.17}$$

Let $\mu \in M\nu$. Since M is monotone, we find from (2.17)

$$\left\langle \frac{x_n - \nu_n}{s_n} - Bx_n - \mu, \nu_n - \nu \right\rangle \geq 0.$$

In view of the restriction (b), we see that

$$\langle -Bq - \mu, q - \nu \rangle \geq 0.$$

This implies that $-Bq \in Mq$, that is, $q \in (B+M)^{-1}(0)$. This completes $q \in \mathcal{F}$. Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $q' \in \mathcal{F}$. We can easily conclude from Opial's condition that $q = q'$.

Finally, we show that $q = \operatorname{Proj}_{\mathcal{F}} x_1$ and $\{x_n\}$ converges strongly to q. This completes the proof of Theorem 2.1. In view of the weak lower semicontinuity of the norm, we obtain from (2.6) that

$$\begin{aligned} \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\| &\leq \|x_1 - q\| \leq \liminf_{n\to\infty} \|x_1 - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_1 - x_n\| \leq \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\|, \end{aligned}$$

which yields that $\lim_{n\to\infty} \|x_1 - x_n\| = \|x_1 - \operatorname{Proj}_{\mathcal{F}} x_1\| = \|x_1 - q\|$. It follows that $\{x_n\}$ converges strongly to $\operatorname{Proj}_{\mathcal{F}} x_1$. This completes the proof. □

If $B = 0$, then Theorem 2.1 is reduced to the following.

Corollary 2.2 *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set and $A : C \to H$ be an α-Lipschitz continuous and monotone mapping. Let $M : H \rightrightarrows H$ be a maximal monotone operator such that $D(M) \subset C$. Assume that $\mathcal{F} := F(S) \cap M^{-1}(0) \cap \mathrm{VI}(C,A)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \operatorname{Proj}_C(J_{s_n} x_n - r_n A J_{s_n} x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) S \operatorname{Proj}_C(J_{s_n} x_n - r_n A z_n), \\ C_{n+1} = \{\nu \in C_n : \|y_n - \nu\| \leq \|x_n - \nu\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where $J_{s_n} = (I + s_n M)^{-1}$, $\{r_n\}$ is a sequence in $(0, \frac{1}{\alpha})$, $\{s_n\}$ is a sequence in $(0, +\infty)$, and $\{\alpha_n\}$ is a sequence in $(0,1)$. Assume that the following restrictions are satisfied:

(a) $0 < a \leq r_n \leq b < \frac{1}{\alpha}$;

(b) $0 < c \leq s_n < \infty$;

(c) $0 \leq \alpha_n \leq d < 1$,

where a, b, c, and d are real constants. Then the sequence $\{x_n\}$ converges strongly to $\operatorname{Proj}_{\mathcal{F}} x_1$.

If $M = 0$, then $J_{s_n} = I$. Corollary 2.2 is reduced to the following.

Corollary 2.3 *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set and $A : C \to H$ be an α-Lipschitz continuous and monotone mapping. Assume that $\mathcal{F} := F(S) \cap \mathrm{VI}(C, A)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = \mathrm{Proj}_C(x_n - r_n A x_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S \,\mathrm{Proj}_C(x_n - r_n A z_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $\{r_n\}$ is a sequence in $(0, \frac{1}{\alpha})$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Assume that the following restrictions are satisfied:

(a) $0 < a \le r_n \le b < \frac{1}{\alpha}$;

(b) $0 \le \alpha_n \le c < 1$,

where a, b, and c are real constants. Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathcal{F}} x_1$.

If $A = 0$, then Theorem 2.1 is reduced to the following.

Corollary 2.4 *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set and $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $M : H \rightrightarrows H$ be a maximal monotone operator such that $D(M) \subset C$. Assume that $\mathcal{F} := F(S) \cap (B + M)^{-1}(0)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S J_{s_n}(x_n - s_n B x_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $J_{s_n} = (I + s_n M)^{-1}$, $\{s_n\}$ is a sequence in $(0, 2\beta)$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Assume that the following restrictions are satisfied:

(a) $0 < a \le s_n \le b < 2\beta$;

(b) $0 \le \alpha_n \le c < 1$,

where a, b, and c are real constants. Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathcal{F}} x_1$.

Let $f : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \bigl\{ y \in H : f(z) \ge f(x) + \langle z - x, y \rangle, z \in H \bigr\}, \quad \forall x \in H.$$

From Rockafellar [34], we know that ∂f is maximal monotone. It is not hard to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y \in H} f(y)$.

Let I_C be the indicator function of C, *i.e.*,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator. It is clear that $J_s x = P_C x$, $\forall x \in H$. Notice that $(B + \partial I_C)^{-1}(0) = \mathrm{VI}(C, B)$. Indeed,

$$\begin{aligned} x \in (B + \partial I_C)^{-1}(0) \quad &\Longleftrightarrow \quad 0 \in Bx + \partial I_C x \\ &\Longleftrightarrow \quad -Bx \in \partial I_C x \\ &\Longleftrightarrow \quad \langle Bx, y - x\rangle \geq 0 \\ &\Longleftrightarrow \quad x \in \mathrm{VI}(C, B). \end{aligned} \tag{2.18}$$

In the light of the above, the following is not hard to derive from Corollary 2.4.

Corollary 2.5 *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set and $B : C \to H$ be a β-inverse-strongly monotone mapping. Assume that $\mathcal{F} := F(S) \cap \mathrm{VI}(C, B)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - s_n B x_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where $\{s_n\}$ is a sequence in $(0, 2\beta)$, and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Assume that the following restrictions are satisfied:

(a) $0 < a \leq s_n \leq b < 2\beta$;

(b) $0 \leq \alpha_n \leq c < 1$,

where a, b, and c are real constants. Then the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathcal{F}} x_1$.

3 Applications

First, we consider the problem of finding a minimizer of a proper convex lower semicontinuous function.

Theorem 3.1 *Let $f : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that $(\partial f)^{-1}(0)$ is not empty. Let $\{x_n\}$ be a sequence generated by the following iterative* *process*:

$$\begin{cases} x_1 \in H, \\ C_1 = H, \\ y_n = \arg\min_{s\in H}\{f(z) + \frac{\|z-x_n\|^2}{2s_n}\}, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $\{s_n\}$ *is a positive sequence such that* $0 < a \le s_n$, *where* a *is a real constant. Then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{Proj}_{(\partial f)^{-1}(0)} x_1$.

Proof Putting $A = B = 0$, $S = I$, and $\alpha_n \equiv 0$, we can immediately draw the desired conclusion from Theorem 2.1. □

Second, we consider the problem of approximating a common fixed point of a pair of nonexpansive mappings.

Theorem 3.2 *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $S : C \to C$ *and* $T : C \to C$ *be a pair of nonexpansive mappings with a nonempty common fixed point set. Let* $\{x_n\}$ *be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = (1 - s_n)x_n + s_n Tx_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sz_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \ge 0, \end{cases}$$

where $\{s_n\}$, *and* $\{\alpha_n\}$ *are sequences in* $(0,1)$. *Assume that the following restrictions are satisfied*:

(a) $0 < a \le s_n \le b < 1$;

(b) $0 \le \alpha_n \le c < 1$,

where a, b, *and* c *are real constants. Then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{Proj}_{F(S)\cap F(T)} x_1$.

Proof Putting $A = 0$, $M = \partial I_C$, and $B = I - T$, we see that B is $\frac{1}{2}$-inverse-strongly monotone. We also have $F(T)=\mathrm{VI}(C,B)$ and $P_C(x_n - s_nBx_n) = (1 - s_n)x_n + s_nTx_n$. In view of (2.18), we can immediately obtain the desired result. □

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem in the terminology of Blum and Oettli [35] (see also Fan [36]):

$$\text{Find } x \in C \text{ such that } F(x,y) \ge 0, \quad \forall y \in C. \tag{3.1}$$

To study the equilibrium problem (3.1), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;

(A3) for each $x,y,z \in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Putting $F(x,y) = \langle Ax, y-x\rangle$ for every $x,y \in C$, we see that the equilibrium problem (3.3) is reduced to the variational inequality (1.1).

The following lemma can be found in [35] and [37].

Lemma 3.3 *Let C be a nonempty, closed, and convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $s > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{s}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_s x = \left\{ z \in C : F(z,y) + \frac{1}{s}\langle y-z, z-x\rangle \geq 0, \forall y \in C \right\} \tag{3.2}$$

for all $s > 0$ and $x \in H$. Then, the following hold:

(a) *T_s is single-valued*;

(b) *T_s is firmly nonexpansive*; *that is,*

$$\|T_s x - T_s y\|^2 \leq \langle T_s x - T_s y, x-y\rangle, \quad \forall x,y \in H;$$

(c) $F(T_s) = EP(F)$;

(d) *$EP(F)$ is closed and convex.*

Lemma 3.4 [30] *Let C be a nonempty, closed, and convex subset of H, F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4), *and A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x,y) \geq \langle y-x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{3.3}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1}(0)$, where $FP(F)$ stands for the solution set of (3.1), *and*

$$T_s x = (I + sA_F)^{-1}x, \quad \forall x \in H, r > 0,$$

where T_s is defined as in (3.3).

Finally, we consider finding a solution of the equilibrium problem.

Theorem 3.5 *Let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4) *such that $EP(F) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative process*:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = (I + s_n A_F)^{-1} x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = \operatorname{Proj}_{C_{n+1}} x_1, \quad n \geq 0, \end{cases}$$

where A_F is defined as (3.3), *and $\{s_n\}$ is a positive sequence such that $0 < a \leq s_n < \infty$, where a is a real constant Then the sequence $\{x_n\}$ converges strongly to $\operatorname{Proj}_{FP(F)} x_1$.*

Proof Putting $A = B = 0$, $S = I$ and $\alpha_n \equiv 0$, we immediately reach the desired conclusion from Lemma 3.4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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