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Stability of the Ishikawa iteration scheme with errors for two strictly hemicontractive operators in Banach spaces

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Abstract
The main purpose of this paper is to establish the convergence, almost common-stability and common-stability of the Ishikawa iteration scheme with error terms in the sense of Xu (J. Math. Anal. Appl. 224:91-101, 1998) for two Lipschitz strictly hemicontractive operators in arbitrary Banach spaces.

Keywords: Ishikawa iteration scheme with errors; strictly hemicontractive operators; Lipschitz operators; Banach space

1 Preliminaries

Let K be a nonempty subset of an arbitrary Banach space E and E^* be its dual space. The symbols $D(T)$, $R(T)$ and $F(T)$ stand for the domain, the range and the set of fixed points of T respectively (for a single-valued map $T : X \to X$, $x \in X$ is called a fixed point of T iff $T(x) = x$). We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\right\}.$$

Let T be a self-mapping of K.

Definition 1 Then T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \tag{1.1}$$

for all $x, y \in K$. If $L = 1$, then T is called *non-expansive*, and if $0 \leq L < 1$, T is called *contraction*.

Definition 2 [2, 3]

1. The mapping T is said to be *pseudocontractive* if the inequality

$$\|x - y\| \leq \left\|x - y + t\big((I - T)x - (I - T)y\big)\right\| \tag{1.2}$$

holds for each $x, y \in K$ and for all $t > 0$. As a consequence of a result of Kato [4], it follows from the inequality (1.2) that T is *pseudocontractive* if and only if there exists $j(x - y) \in$

$J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 \tag{1.3}$$

for all $x, y \in K$.

2. T is said to be strongly pseudocontractive if there exists a $t > 1$ such that

$$\|x-y\| \le \big\|(1+r)(x-y) - rt(Tx-Ty)\big\| \tag{1.4}$$

for all $x, y \in D(T)$ and $r > 0$.

3. T is said to be local strongly pseudocontractive if, for each $x \in D(T)$, there exists a $t_x > 1$ such that

$$\|x-y\| \le \big\|(1+r)(x-y) - rt_x(Tx-Ty)\big\| \tag{1.5}$$

for all $y \in D(T)$ and $r > 0$.

4. T is said to be strictly hemicontractive if $F(T) \neq \varphi$ and if there exists a $t > 1$ such that

$$\|x-q\| \le \big\|(1+r)(x-q) - rt(Tx-q)\big\| \tag{1.6}$$

for all $x \in D(T)$, $q \in F(T)$ and $r > 0$.

It is easy to verify that an iteration scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable on K is almost T-stable on K. Osilike [5] proved that an iteration scheme which is almost T-stable on X may fail to be T-stable on X.

Clearly, each strongly pseudocontractive operator is local strongly pseudocontractive.

Chidume [6] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudo-contractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Afterwards, several authors generalized this result of Chidume in various directions. Chidume [7] proved a similar result by removing the restriction $\lim_{n\to\infty} \alpha_n = 0$. Tan and Xu [8] extended that result of Chidume to the Ishikawa iteration scheme in a p-uniformly smooth Banach space. Chidume and Osilike [2] improved the result of Chidume [6] to strictly hemicontractive mappings defined on a real uniformly smooth Banach space.

Recently, some researchers have generalized the results to real smooth Banach spaces, real uniformly smooth Banach spaces, real Banach spaces; or to the Mann iteration method, the Ishikawa iteration method; or to strongly pseudocontractive operators, local strongly pseudocontractive operators, strictly hemicontractive operators [9–19].

The main purpose of this paper is to establish the convergence, almost common-stability and common-stability of the Ishikawa iteration scheme with error terms in the sense of Xu [1] for two Lipschitz strictly hemicontractive operators in arbitrary Banach spaces. Our results extend, improve and unify the corresponding results in [2, 3, 10, 11, 15–18, 20–25].

2 Main results

We need the following results.

Lemma 3 [26] *Let* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ *and* $\{\omega_n\}_{n=0}^{\infty}$ *be nonnegative real sequences such that*

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0,$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$, $\sum_{n=0}^{\infty}\gamma_n < \infty$ *and* $\lim_{n\to\infty}\beta_n = 0$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 4 [27] *Let* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *be sequences of nonnegative real numbers and* $0 \leq \theta < 1$, *so that*

$$a_{n+1} \leq \theta a_n + b_n, \quad \text{for all } n \geq 0.$$

(i) *If* $\lim_{n\to\infty} b_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.
(ii) *If* $\sum_{n=0}^{\infty} b_n < \infty$, *then* $\sum_{n=0}^{\infty} a_n < \infty$.

Lemma 5 [4] *Let* $x, y \in X$. *Then* $\|x\| \leq \|x + ry\|$ *for every* $r > 0$ *if and only if there is* $f \in J(x)$ *such that* $\operatorname{Re}\langle y, f\rangle \geq 0$.

Lemma 6 [2] *Let* $T : D(T) \subseteq X \to X$ *be an operator with* $F(T) \neq \emptyset$. *Then* T *is strictly hemicontractive if and only if there exists* $t > 1$ *such that for all* $x \in D(T)$ *and* $q \in F(T)$, *there exists* $j \in J(x-q)$ *satisfying*

$$\operatorname{Re}\big\langle x - Tx, j(x-q)\big\rangle \geq \left(1 - \frac{1}{t}\right)\|x-q\|^2.$$

Lemma 7 [24] *Let* X *be an arbitrary normed linear space and* $T : D(T) \subseteq X \to X$ *be an operator.*

(i) *If* T *is a local strongly pseudocontractive operator and* $F(T) \neq \emptyset$, *then* $F(T)$ *is a singleton and* T *is strictly hemicontractive.*
(ii) *If* T *is strictly hemicontractive, then* $F(T)$ *is a singleton.*

In the sequel, let $k = \frac{t-1}{t} \in (0,1)$, where t is the constant appearing in (1.6). Further L denotes the common Lipschitz constant of T and S, and I denotes the identity mapping on an arbitrary Banach space X.

Definition 8 Let K be a nonempty convex subset of X and $T, S : K \to K$ be two operators. Assume that $x_o \in K$ and $x_{n+1} = f(T, S, x_n)$ defines an iteration scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T) \cap F(S) \neq \varphi$. Let $\{y_n\}_{n=0}^{\infty}$ be any bounded sequence in K and put $\varepsilon_n = \|y_{n+1} - f(T, S, y_n)\|$.

(i) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, S, x_n)$ is said to be common-stable on K if $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$.
(ii) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, S, x_n)$ is said to be almost common-stable on K if $\sum_{n=0}^{\infty}\varepsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

We now establish our main results.

Theorem 9 *Let* K *be a nonempty closed convex subset of an arbitrary Banach space* X *and* $T, S : K \to K$ *be two Lipschitz strictly hemicontractive operators. Suppose that* $\{u_n\}_{n=0}^{\infty}$,

$\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K, and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are any sequences in $[0,1]$ satisfying

(i) $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$,

(ii) $c'_n = o(b'_n)$,

(iii) $\lim_{n\to\infty} c_n = 0$,

(iv) $\sum_{n=0}^{\infty} b'_n = \infty$,

(v) $L[(1+L)^2 b'_n + c'_n + (1+L)(b_n + c_n)] + \frac{c'_n}{b'_n} \leq k(k-s)$, $n \geq 0$,

where s is a constant in $(0,k)$. Suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$\begin{aligned} x_{n+1} &= a'_n x_n + b'_n T z_n + c'_n v_n, \\ z_n &= a_n x_n + b_n S x_n + c_n u_n, \quad n \geq 0. \end{aligned} \tag{2.1}$$

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in K and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0,$$

where

$$\begin{aligned} p_n &= a'_n y_n + b'_n T w_n + c'_n v_n, \\ w_n &= a_n y_n + b_n S y_n + c_n u_n, \quad n \geq 0. \end{aligned} \tag{2.2}$$

Then

(a) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point q of T and S. Also,*

$$\begin{aligned} \|x_{n+1} - q\| \leq{}& \bigl(1 - sb'_n\bigr)\|x_n - q\| \\ &+ L(1+L)k^{-1} b'_n c_n \|u_n - q\| + (1+L)k^{-1} c'_n \|v_n - q\|, \quad n \geq 0, \end{aligned}$$

(b)

$$\begin{aligned} \|y_{n+1} - q\| \leq{}& \bigl(1 - sb'_n\bigr)\|y_n - q\| \\ &+ L(1+L)k^{-1} b'_n c_n \|u_n - q\| + (1+L)k^{-1} c'_n \|v_n - q\| + \varepsilon_n, \quad n \geq 0, \end{aligned}$$

(c) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that $\{x_n\}_{n=0}^{\infty}$ is almost common-stable on K,*

(d) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof From (ii), we have $c'_n = t_n b'_n$, where $t_n \to 0$ as $n \to \infty$. It follows from Lemma 7 that $F(T) \cap F(S)$ is a singleton; that is, $F(T) \cap F(S) = \{q\}$ for some $q \in K$. Set

$$M = \max\Bigl\{\sup_{n\geq 0}\bigl\{\|u_n - q\|\bigr\}, \sup_{n\geq 0}\bigl\{\|v_n - q\|\bigr\}\Bigr\}.$$

Since T is strictly hemicontractive, it follows form Lemma 6 that

$$\operatorname{Re}\bigl\langle x - Tx, j(x-q)\bigr\rangle \geq k\|x - q\|^2, \quad \forall x \in K,$$

which implies that

$$\operatorname{Re}\big\langle (I-T-kI)x-(I-T-kI)q, j(x-q)\big\rangle \geq 0, \quad \forall x \in K.$$

In view of Lemma 5, we have

$$\|x-q\| \leq \big\|x-q+r\big[(I-T-kI)x-(I-T-kI)q\big]\big\|, \quad \forall x \in K, \forall r > 0. \tag{2.3}$$

Also,

$$\begin{aligned}\big(1-b_n'\big)x_n &= \big(1-(1-k)b_n'\big)x_{n+1}+b_n'(I-T-kI)x_{n+1}\\&\quad+b_n'(Tx_{n+1}-Tz_n)-c_n'(v_n-x_n),\end{aligned} \tag{2.4}$$

and

$$\big(1-b_n'\big)q = \big(1-(1-k)b_n'\big)q+b_n'(I-T-kI)q. \tag{2.5}$$

From (2.4) and (2.5), we infer that for all $n \geq 0$,

$$\begin{aligned}\big(1-b_n'\big)\|x_n-q\| &\geq \big\|\big(1-(1-k)b_n'\big)(x_{n+1}-q)+b_n'(I-T-kI)(x_{n+1}-q)\big\|\\&\quad-b_n'\|Tx_{n+1}-Tz_n\|-c_n'\|v_n-x_n\|\\&=\big(1-(1-k)b_n'\big)\left\|x_{n+1}-q+\frac{b_n'}{1-(1-k)b_n'}(I-T-kI)(x_{n+1}-q)\right\|\\&\quad-b_n'\|Tx_{n+1}-Tz_n\|-c_n'\|v_n-x_n\|\\&\geq\big(1-(1-k)b_n'\big)\|x_{n+1}-q\|-b_n'\|Tx_{n+1}-Tz_n\|\\&\quad-c_n'\|v_n-x_n\|,\end{aligned}$$

which implies that for all $n \geq 0$,

$$\begin{aligned}\|x_{n+1}-q\| &\leq \frac{1-b_n'}{1-(1-k)b_n'}\|x_n-q\|\\&\quad+\frac{b_n'}{1-(1-k)b_n'}\|Tx_{n+1}-Tz_n\|+\frac{c_n'}{1-(1-k)b_n'}\|v_n-x_n\|\\&\leq\big(1-kb_n'\big)\|x_n-q\|+k^{-1}b_n'\|Tx_{n+1}-Tz_n\|+k^{-1}c_n'\|v_n-x_n\|\\&\leq\big(1-kb_n'\big)\|x_n-q\|+k^{-1}Lb_n'\|x_{n+1}-z_n\|+k^{-1}c_n'\|v_n-x_n\|\\&\leq\big(1-kb_n'\big)\|x_n-q\|+k^{-1}Lb_n'\|x_{n+1}-z_n\|\\&\quad+k^{-1}c_n'\big(\|v_n-q\|+\|x_n-q\|\big)\\&=\big(1-kb_n'+k^{-1}c_n'\big)\|x_n-q\|+k^{-1}Lb_n'\|x_{n+1}-z_n\|\\&\quad+k^{-1}c_n'\|v_n-q\|,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\|x_{n+1}-z_n\| &\leq \big\|b_n'(Tz_n-x_n)+c_n'(v_n-x_n)\big\|\\&\quad+\big\|b_n(x_n-Sx_n)-c_n(u_n-x_n)\big\|\end{aligned}$$

$$
\begin{aligned}
&\le b_n'\|x_n - Tz_n\| + c_n'\|v_n - x_n\| \\
&\quad + b_n\|x_n - Sx_n\| + c_n\|u_n - x_n\| \\
&\le b_n'\big(\|x_n - q\| + \|q - Tz_n\|\big) + c_n'\big(\|v_n - q\| + \|x_n - q\|\big) \\
&\quad + b_n\big(\|x_n - q\| + \|q - Sx_n\|\big) + c_n\big(\|u_n - q\| + \|x_n - q\|\big) \\
&\le b_n'\big(\|x_n - q\| + L\|z_n - q\|\big) + c_n'\big(\|v_n - q\| + \|x_n - q\|\big) \\
&\quad + b_n\big(\|x_n - q\| + L\|x_n - q\|\big) + c_n\big(\|u_n - q\| + \|x_n - q\|\big) \\
&= \big[b_n' + c_n' + (1+L)b_n + c_n\big]\|x_n - q\| + Lb_n'\|z_n - q\| \\
&\quad + c_n'\|v_n - q\| + c_n\|u_n - q\|,
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
\|z_n - q\| &= \big\|x_n - q - b_n(x_n - Sx_n) + c_n(u_n - x_n)\big\| \\
&\le \|x_n - q\| + b_n\|x_n - Sx_n\| + c_n\|u_n - x_n\| \\
&\le \|x_n - q\| + b_n\big(\|x_n - q\| + \|q - Sx_n\|\big) \\
&\quad + c_n\big(\|u_n - q\| + \|x_n - q\|\big) \\
&\le \|x_n - q\| + b_n\big(\|x_n - q\| + L\|x_n - q\|\big) \\
&\quad + c_n\big(\|u_n - q\| + \|x_n - q\|\big) \\
&= \big[1 + (1+L)b_n + c_n\big]\|x_n - q\| + c_n\|u_n - q\|.
\end{aligned} \tag{2.8}
$$

Substituting (2.8) in (2.7), we have

$$
\begin{aligned}
\|x_{n+1} - z_n\| &\le \big[b_n' + c_n' + (1+L)b_n + c_n\big]\|x_n - q\| \\
&\quad + Lb_n'\big[\big[1 + (1+L)b_n + c_n\big]\|x_n - q\| \\
&\quad + c_n\|u_n - q\|\big] + c_n'\|v_n - q\| + c_n\|u_n - q\| \\
&= \big[(1+L)b_n' + L(1+L)b_nb_n' + (1+L)b_n + c_n' \\
&\quad + \big(1 + Lb_n'\big)c_n\big]\|x_n - q\| \\
&\quad + c_n'\|v_n - q\| + \big(1 + Lb_n'\big)c_n\|u_n - q\|.
\end{aligned} \tag{2.9}
$$

Substituting (2.9) in (2.6), we get

$$
\begin{aligned}
\|x_{n+1} - q\| &\le \big(1 - kb_n' + k^{-1}c_n'\big)\|x_n - q\| + k^{-1}Lb_n'\big[\big[(1+L)b_n' \\
&\quad + L(1+L)b_nb_n' + (1+L)b_n + c_n' + \big(1 + Lb_n'\big)c_n\big]\|x_n - q\| \\
&\quad + c_n'\|v_n - q\| + \big(1 + Lb_n'\big)c_n\|u_n - q\|\big] + k^{-1}c_n'\|v_n - q\| \\
&= \big[1 - b_n'\big[k - k^{-1}L\big((1+L)b_n' + L(1+L)b_nb_n' \\
&\quad + (1+L)b_n + c_n' + \big(1 + Lb_n'\big)c_n\big) - k^{-1}t_n\big]\big]\|x_n - q\| \\
&\quad + k^{-1}Lb_n'\big(1 + Lb_n'\big)c_n\|u_n - q\| + k^{-1}\big(1 + Lb_n'\big)c_n'\|v_n - q\| \\
&\le \big[1 - b_n'\big[k - k^{-1}L\big((1+L)^2b_n' + (1+L)b_n \\
&\quad + c_n' + (1+L)c_n\big) - k^{-1}t_n\big]\big]\|x_n - q\|
\end{aligned}
$$

$$\begin{aligned}
&+ k^{-1}L(1+L)b'_n c_n\|u_n - q\| + k^{-1}(1+L)c'_n\|v_n - q\| \\
&\leq \bigl(1 - sb'_n\bigr)\|x_n - q\| + k^{-1}L(1+L)b'_n c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| \\
&\leq \bigl(1 - sb'_n\bigr)\|x_n - q\| + k^{-1}L(1+L)b'_n c_n M + k^{-1}(1+L)b'_n t_n M \\
&= \bigl(1 - sb'_n\bigr)\|x_n - q\| + k^{-1}(1+L)Mb'_n(Lc_n + t_n).
\end{aligned}$$

Put

$$\begin{aligned}
&\alpha_n = \|x_n - q\|, \\
&\omega_n = sb'_n, \\
&\beta_n = s^{-1}k^{-1}(1+L)M(Lc_n + t_n), \\
&\gamma_n = 0,
\end{aligned}$$

we have

$$\alpha_{n+1} \leq (1 - \omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0.$$

Observe that $\sum_{n=0}^{\infty} \omega_n = \infty$, $\omega_n \in [0,1]$ and $\lim_{n\to\infty} \beta_n = 0$. It follows from Lemma 3 that $\lim_{n\to\infty} \|x_n - q\| = 0$.

We also have

$$\begin{aligned}
\bigl(1 - b'_n\bigr)y_n = {}& \bigl(1 - (1-k)b'_n\bigr)p_n + b'_n(I - T - kI)p_n \\
& + b'_n(Tp_n - Tw_n) - c'_n(v_n - y_n).
\end{aligned} \tag{2.10}$$

From (2.5) and (2.10), it follows that for all $n \geq 0$,

$$\begin{aligned}
\bigl(1 - b'_n\bigr)\|y_n - q\| \geq {}& \bigl\|\bigl(1 - (1-k)b'_n\bigr)(p_n - q) + b'_n(I - T - kI)(p_n - q)\bigr\| \\
& - b'_n\|Tp_n - Tw_n\| - c'_n\|v_n - y_n\| \\
= {}& \bigl(1 - (1-k)b'_n\bigr)\left\|p_n - q \right. \\
& \left. + \frac{b'_n}{1 - (1-k)b'_n}(I - T - kI)(p_n - q)\right\| \\
& - b'_n\|Tp_n - Tw_n\| - c'_n\|v_n - y_n\| \\
\geq {}& \bigl(1 - (1-k)b'_n\bigr)\|p_n - q\| - b'_n\|Tp_n - Tw_n\| \\
& - c'_n\|v_n - y_n\|,
\end{aligned}$$

which implies that for all $n \geq 0$,

$$\begin{aligned}
\|p_n - q\| \leq {}& \frac{1 - b'_n}{1 - (1-k)b'_n}\|y_n - q\| \\
& + \frac{b'_n}{1 - (1-k)b'_n}\|Tp_n - Tw_n\| + \frac{c'_n}{1 - (1-k)b'_n}\|v_n - y_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \bigl(1-kb'_n\bigr)\|y_n-q\| + k^{-1}b'_n\|Tp_n-Tw_n\| + k^{-1}c'_n\|v_n-y_n\| \\
&\leq \bigl(1-kb'_n\bigr)\|y_n-q\| + k^{-1}Lb'_n\|p_n-w_n\| + k^{-1}c'_n\|v_n-y_n\| \\
&\leq \bigl(1-kb'_n\bigr)\|y_n-q\| + k^{-1}Lb'_n\|p_n-w_n\| \\
&\quad + k^{-1}c'_n\bigl(\|v_n-q\| + \|y_n-q\|\bigr) \\
&= \bigl(1-kb'_n+k^{-1}c'_n\bigr)\|y_n-q\| + k^{-1}Lb'_n\|p_n-w_n\| \\
&\quad + k^{-1}c'_n\|v_n-q\|, \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
\|p_n-w_n\| &\leq \bigl\|b'_n(Tw_n-y_n) + c'_n(v_n-y_n)\bigr\| \\
&\quad + \bigl\|b_n(y_n-Sy_n) - c_n(u_n-y_n)\bigr\| \\
&\leq b'_n\|y_n-Tw_n\| + c'_n\|v_n-y_n\| \\
&\quad + b_n\|y_n-Sy_n\| + c_n\|u_n-y_n\| \\
&\leq b'_n\bigl(\|y_n-q\| + \|q-Tw_n\|\bigr) + c'_n\bigl(\|v_n-q\| + \|y_n-q\|\bigr) \\
&\quad + b_n\bigl(\|y_n-q\| + \|q-Sy_n\|\bigr) + c_n\bigl(\|u_n-q\| + \|y_n-q\|\bigr) \\
&\leq b'_n\bigl(\|y_n-q\| + L\|w_n-q\|\bigr) + c'_n\bigl(\|v_n-q\| + \|y_n-q\|\bigr) \\
&\quad + b_n\bigl(\|y_n-q\| + L\|y_n-q\|\bigr) + c_n\bigl(\|u_n-q\| + \|y_n-q\|\bigr) \\
&= \bigl[b'_n + c'_n + (1+L)b_n + c_n\bigr]\|y_n-q\| + Lb'_n\|w_n-q\| \\
&\quad + c'_n\|v_n-q\| + c_n\|u_n-q\|, \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
\|w_n-q\| &= \bigl\|(y_n-q) - b_n(y_n-Sy_n) + c_n(u_n-y_n)\bigr\| \\
&\leq \|y_n-q\| + b_n\|y_n-Sy_n\| + c_n\|u_n-y_n\| \\
&\leq \|y_n-q\| + b_n\bigl(\|y_n-q\| + \|q-Sy_n\|\bigr) \\
&\quad + c_n\bigl(\|u_n-q\| + \|y_n-q\|\bigr) \\
&\leq \|y_n-q\| + b_n\bigl(\|y_n-q\| + L\|y_n-q\|\bigr) \\
&\quad + c_n\bigl(\|u_n-q\| + \|y_n-q\|\bigr) \\
&= \bigl[1 + (1+L)b_n + c_n\bigr]\|y_n-q\| + c_n\|u_n-q\|. \qquad (2.13)
\end{aligned}$$

Substituting (2.13) in (2.12), we have

$$\begin{aligned}
\|p_n-w_n\| &\leq \bigl[b'_n + c'_n + (1+L)b_n + c_n\bigr]\|y_n-q\| \\
&\quad + Lb'_n\bigl[\bigl[1 + (1+L)b_n + c_n\bigr]\|y_n-q\| \\
&\quad + c_n\|u_n-q\|\bigr] + c'_n\|v_n-q\| + c_n\|u_n-q\| \\
&= \bigl[(1+L)b'_n + L(1+L)b_nb'_n + (1+L)b_n + c'_n \\
&\quad + \bigl(1+Lb'_n\bigr)c_n\bigr]\|y_n-q\| \\
&\quad + c'_n\|v_n-q\| + \bigl(1+Lb'_n\bigr)c_n\|u_n-q\|. \qquad (2.14)
\end{aligned}$$

Substituting (2.14) in (2.11), we get

$$\begin{aligned}
\|p_n - q\| &\leq \bigl(1 - kb'_n + k^{-1}c'_n\bigr)\|y_n - q\| + k^{-1}Lb'_n\bigl[\bigl[(1+L)b'_n \\
&\quad + L(1+L)b_nb'_n + (1+L)b_n + c'_n + \bigl(1 + Lb'_n\bigr)c_n\bigr]\|y_n - q\| \\
&\quad + c'_n\|v_n - q\| + \bigl(1 + Lb'_n\bigr)c_n\|u_n - q\|\bigr] + k^{-1}c'_n\|v_n - q\| \\
&= \bigl[1 - b'_n\bigl[k - k^{-1}L\bigl((1+L)b'_n + L(1+L)b_nb'_n \\
&\quad + (1+L)b_n + c'_n + \bigl(1 + Lb'_n\bigr)c_n\bigr) - k^{-1}t_n\bigr]\bigr]\|y_n - q\| \\
&\quad + k^{-1}Lb'_n\bigl(1 + Lb'_n\bigr)c_n\|u_n - q\| + k^{-1}\bigl(1 + Lb'_n\bigr)c'_n\|v_n - q\| \\
&\leq \bigl[1 - b'_n\bigl[k - k^{-1}L\bigl((1+L)^2b'_n + (1+L)b_n \\
&\quad + c'_n + (1+L)c_n\bigr) - k^{-1}t_n\bigr]\bigr]\|y_n - q\| \\
&\quad + k^{-1}L(1+L)b'_nc_n\|u_n - q\| + k^{-1}(1+L)c'_n\|v_n - q\| \\
&\leq \bigl(1 - sb'_n\bigr)\|y_n - q\| + k^{-1}L(1+L)b'_nc_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| \qquad (2.15)
\end{aligned}$$

for any $n \geq 0$. Thus (2.15) implies that

$$\begin{aligned}
\|y_{n+1} - q\| &\leq \|y_{n+1} - p_n\| + \|p_n - q\| \\
&\leq \bigl(1 - sb'_n\bigr)\|y_n - q\| + k^{-1}L(1+L)b'_nc_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| + \varepsilon_n \\
&= (1 - \omega_n)\|y_n - q\| + \omega_n\beta_n + \gamma_n. \qquad (2.16)
\end{aligned}$$

With

$$\begin{aligned}
&\alpha_n = \|y_n - q\|, \\
&\omega_n = sb'_n, \\
&\beta_n = s^{-1}k^{-1}(1+L)M(Lc_n + t_n), \\
&\gamma_n = \varepsilon_n, \quad \forall n \geq 0,
\end{aligned}$$

we have

$$\alpha_{n+1} \leq (1 - \omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq 0.$$

Observe that $\sum_{n=0}^{\infty}\omega_n = \infty$, $\omega_n \in [0,1]$ and $\lim_{n\to\infty}\beta_n = 0$. It follows from Lemma 3 that $\lim_{n\to\infty}\|y_n - q\| = 0$.

Suppose that $\lim_{n\to\infty} y_n = q$. It follows from equation (2.15) that

$$\begin{aligned}
\varepsilon_n &\leq \|y_{n+1} - q\| + \|p_n - q\| \\
&\leq \bigl(1 - sb'_n\bigr)\|y_n - q\| + k^{-1}L(1+L)b'_nc_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| + \|y_{n+1} - q\| \to 0,
\end{aligned}$$

as $n \to \infty$; that is, $\varepsilon_n \to 0$ as $n \to \infty$. □

Using the techniques in the proof of Theorem 9, we have the following results.

Theorem 10 *Let* $X, K, T, S, s, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}, \{z_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ *and* $\{p_n\}_{n=0}^{\infty}$ *be as in Theorem* 9. *Suppose that* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{a'_n\}_{n=0}^{\infty}, \{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *satisfying conditions* (i), (iii)-(v) *of Theorem* 9 *with*

$$\sum_{n=0}^{\infty} c'_n < \infty.$$

Then the conclusions of Theorem 9 *hold.*

Theorem 11 *Let* $X, K, T, S, s, \{u_n\}_{n=0}^{\infty}, \{v_n\}_{n=0}^{\infty}, \{x_n\}_{n=0}^{\infty}, \{z_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ *and* $\{p_n\}_{n=0}^{\infty}$ *be as in Theorem* 9. *Suppose that* $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{a'_n\}_{n=0}^{\infty}, \{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *satisfying condition* (i), (iii) *and* (v) *of Theorem* 9 *with*

$$\begin{aligned}
&\lim_{n\to\infty} c'_n = 0, \\
&b'_n \geq m > 0, \quad \forall n \geq 0,
\end{aligned}$$

where m is a constant. Then

(a) *the sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the common fixed point q of T and S. Also,*

$$\|x_{n+1} - q\| \leq (1 - sm)\|x_n - q\| + C, \quad \forall n \geq 0,$$

where

$$C = k^{-1}(1+L)\Big[L \sup_{n\geq 0}\big\{c_n\|u_n - q\|\big\} + \sup_{n\geq 0}\big\{c'_n\|v_n - q\|\big\}\Big],$$

(b)

$$\begin{aligned}
\|y_{n+1} - q\| &\leq (1 - sm)\|y_n - q\| + k^{-1}L(1+L)c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| + \varepsilon_n, \quad \forall n \geq 0,
\end{aligned}$$

(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof As in the proof of Theorem 9, we conclude that $F(T) \cap F(S) = \{q\}$ and

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \big(1 - sb'_n\big)\|x_n - q\| + k^{-1}L(1+L)b'_n c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| \\
&\leq (1 - sm)\|x_n - q\| + k^{-1}L(1+L)c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| \\
&\leq (1 - sm)\|x_n - q\| + C, \quad \forall n \geq 0.
\end{aligned}$$

Let

$$a_n = \|x_n - q\|,$$

$$\theta = sm,$$

$$b_n = (sm)^{-1}k^{-1}(1+L)\big[Lc_n\big\|u_n - x^*\big\| + c'_n\big\|v_n - x^*\big\|\big], \quad \forall n \geq 0.$$

Observe that $0 \leq \theta < 1$ and $\lim_{n\to\infty} b_n = 0$. It follows from Lemma 4 that $\lim_{n\to\infty} \|x_n - q\| = 0$.

Also, from (2.15), we have

$$\begin{aligned}
\|y_{n+1} - q\| &\leq \big(1 - sb'_n\big)\|y_n - q\| + k^{-1}L(1+L)b'_n c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| + \varepsilon_n \\
&\leq (1 - sm)\|y_n - q\| + k^{-1}L(1+L)c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| + \varepsilon_n.
\end{aligned}$$

Suppose that $\lim_{n\to\infty} y_n = q$. It follows from equation (2.15) that

$$\begin{aligned}
\varepsilon_n &\leq \|y_{n+1} - q\| + \|p_n - q\| \\
&\leq (1 - sm)\|y_n - q\| + k^{-1}L(1+L)c_n\|u_n - q\| \\
&\quad + k^{-1}(1+L)c'_n\|v_n - q\| + \|y_{n+1} - q\| \to 0,
\end{aligned}$$

as $n \to \infty$; that is, $\varepsilon_n \to 0$ as $n \to \infty$.

Conversely, suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. Put

$$a_n = \|y_n - q\|,$$

$$\theta = sm,$$

$$b_n = (sm)^{-1}k^{-1}(1+L)\big[Lc_n\big\|u_n - x^*\big\| + c'_n\big\|v_n - x^*\big\|\big] + \varepsilon_n, \quad \forall n \geq 0,$$

$$\gamma_n = \varepsilon_n, \quad \forall n \geq 0.$$

Observe that $0 \leq \theta < 1$ and $\lim_{n\to\infty} b_n = 0$. It follows from Lemma 4 that $\lim_{n\to\infty} \|y_n - q\| = 0$. □

As an immediate consequence of Theorems 9 and 11, we have the following:

Corollary 12 *Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T, S : K \to K$ be two Lipschitz strictly hemicontractive operators. Suppose that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are any sequences in $[0,1]$ satisfying*

(vi) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(vii) $L[(1+L)^2\alpha_n + (1+L)\beta_n] \leq k(k-s)$, $n \geq 0$,

where s is a constant in $(0,k)$. Suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tz_n,$$

$$z_n = (1 - \beta_n)x_n + \beta_n Sx_n, \quad n \geq 0.$$

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in K and define $\{\varepsilon_n\}_{n=0}^{\infty}$ by

$$\varepsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0,$$

where

$$p_n = (1 - \alpha_n)y_n + \alpha_n Tw_n,$$

and

$$w_n = (1 - \beta_n)y_n + \beta_n Sy_n, \quad n \geq 0.$$

Then

(a) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point q of T and S,*
(b) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that $\{x_n\}_{n=0}^{\infty}$ is almost common-stable on K,*
(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Corollary 13 *Let X, K, T, S, s, $\{x_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ be as in Theorem* 9. *Suppose that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are sequences in* $[0,1]$ *satisfying conditions* (vi)-(vii) *and* (iii) *of Theorem* 9 *with*

$$\alpha_n \geq m > 0, \quad \forall n \geq 0,$$

where m is a constant. Then

(a) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point q of T and S. Also,*

$$\|x_{n+1} - q\| \leq (1 - sm)\|x_n - q\|, \quad \forall n \geq 0,$$

(b)

$$\|y_{n+1} - q\| \leq (1 - sm)\|y_n - q\| + \varepsilon_n, \quad \forall n \geq 0,$$

(c) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \varepsilon_n = 0$.

Example 14 Let $\mathbb{R}$ denote the set of real numbers with the usual norm, $K = \mathbb{R}$, and define $T, S : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \frac{2}{5}\sin^2 x, \quad \text{and} \quad Sx = \frac{4}{5}x.$$

Set $L = \frac{4}{5}$, $t = \frac{5}{4}$, $s = \frac{1}{400}$. Clearly, $F(T) \cap F(S) = \{0\}$ and

$$|Tx - Ty| \leq \frac{2}{5}|\sin x - \sin y||\sin x + \sin y| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}.$$

Clearly both T and S are Lipschitz operators on $\mathbb{R}$.

Also, it follows from (1.1) that

$$\begin{aligned}\left|(1+r)(x-y)-rt(Tx-Ty)\right| &\geq (1+r)|x-y|-rt|Tx-Ty| \\ &= |x-y|+r\big(|x-y|-t|Tx-Ty|\big) \\ &\geq |x-y|\end{aligned}$$

for any $x, y \in \mathbb{R}$ and $r > 0$. Thus T is strongly pseudocontractive and Lemma 7 ensures that T is strictly hemicontractive. Put

$$\begin{aligned}b'_n &= \frac{25}{81}\frac{1}{\sqrt{n}+100}, \\ c'_n &= \frac{1}{(\sqrt{n}+100)^2}, \\ a'_n &= 1-\big(b'_n+c'_n\big), \\ b_n = c_n &= \frac{5}{9}\frac{1}{n+100}, \\ a_n &= 1-(b_n+c_n), \quad \forall n \geq 0,\end{aligned}$$

then it can be easily seen that

$$L\big[(1+L)^2 b'_n + c'_n + (1+L)(b_n+c_n)\big] + \frac{c'_n}{b'_n} \leq 0.456 \leq 0.049375, \quad \forall n \geq 0.$$

It follows from Theorem 9 that the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.1) converges strongly to the common fixed point 0 of T and S in K and the iterative scheme defined by (2.1) is T-stable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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