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Common fixed point theorems for fuzzy mappings in *G*-metric spaces

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Abstract
In this paper, we introduce the concept of Hausdorff *G*-metric in the space of fuzzy sets induced by the metric d_G and obtain some results on Hausdorff *G*-metric. We also prove common fixed point theorems for a family of fuzzy self-mappings in the space of fuzzy sets on a complete *G*-metric space.

MSC: 47H10; 54H25

Keywords: fuzzy set; $D_{G,\infty}$-convergent; $D_{G,\infty}$-Cauchy; fuzzy self-mapping; fixed point

1 Introduction and preliminaries

Fixed point theory is very important in mathematics and has applications in many fields. A number of authors established fixed point theorems for various mappings in different metric spaces. In 2006, Mustafa and Sims [1] introduced the *G*-metric space as a generalization of metric spaces. We now recall some definitions and results in *G*-metric spaces in [1].

Definition 1.1 Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying:

($G1$) $G(x,y,z) = 0$ if $x = y = z$,
($G2$) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,
($G3$) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$, with $y \neq z$,
($G4$) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables),
($G5$) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specifically, a G-metric on X, and the pair (X,G) is a G-metric space.

Lemma 1.1 *Every G-metric space* (X,G) *defines a metric space* (X,d_G) *by*

$$d_G(x,y) = G(x,y,y) + G(x,x,y), \quad \textit{for all } x,y \in X.$$

Definition 1.2 Let (X,G) be a G-metric space. The sequence $\{x_n\}$ in X is said to be

(i) G-convergent to x if for any $\varepsilon > 0$, there exists $x \in X$ and $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$.
(ii) G-Cauchy if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq N$.

Lemma 1.2 *Let (X, G) be a G-metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$ the following are equivalent:*

(i) $\{x_n\}$ *is G-convergent to x.*
(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$.
(iii) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$.
(iv) $G(x_m, x_n, x) \to 0$ *as* $m, n \to +\infty$.

Lemma 1.3 *Let (X, G) be a G-metric space, then for a sequence $\{x_n\}$ in X, the following are equivalent:*

(i) The sequence $\{x_n\}$ *is G-Cauchy.*
(ii) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.
(iii) $\{x_n\}$ *is a Cauchy sequence in the metric space (X, d_G).*

Definition 1.3 A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Lemma 1.4 *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Based on the notion of G-metric spaces, many authors obtained fixed point theorems for mappings satisfying different contractive-type conditions in G-metric spaces (see, *e.g.*, [2–7]) and in partially ordered G-metric spaces (see, *e.g.*, [8–13]). Recently, Kaewcharoen and Kaewkhao [14] introduced the following concepts. Let X be a G-metric space and $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H(\cdot,\cdot,\cdot)$ be the Hausdorff G-distance on $CB(X)$, *i.e.*,

$$H_G(A, B, C) = \max\Big\{\sup_{x\in A} G(x, B, C), \sup_{x\in B} G(x, C, A), \sup_{x\in C} G(x, A, B)\Big\},$$

where

$$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C),$$
$$d_G(x, B) = \inf_{y\in B} d_G(x, y),$$
$$d_G(A, B) = \inf_{x\in A, y\in B} d_G(x, y).$$

Kaewcharoen and Kaewkhao [14] and Tahat *et al.* [15] obtained some common fixed point theorems for single-valued and multi-valued mappings in G-metric spaces.

The existence of fixed points of fuzzy mappings has been an active area of research interest since Heilpern [16] introduced the concept of fuzzy mappings in 1981. Many results have appeared related to fixed points for fuzzy mappings in ordinary metric spaces (see, *e.g.*, [17–22]). Qiu and Shu [23, 24] proved some fixed point theorems for fuzzy self-mappings in ordinary metric spaces. However, there are very few results on fuzzy self-mappings in G-metric spaces. The purpose of this paper is to introduce the notion of Hausdorff G-metric in the space of fuzzy sets which extends the Hausdorff G-distance in [14]. We also establish common fixed point theorems for a family of fuzzy self-mappings in the space of fuzzy sets on a complete G-metric space.

2 A Hausdorff *G*-metric in the space of fuzzy sets

Let (X, d_G) be a metric space, a fuzzy set in X is a function with domain X and values in $I = [0,1]$. If μ is a fuzzy set and $x \in X$, then the function value $\mu(x)$ is called the grade of membership of x in μ.

The α-level set of μ, denoted by $[\mu]_\alpha$, is defined as

$$[\mu]_\alpha = \{x : \mu(x) \ge \alpha\}, \quad \text{if } \alpha \in (0,1],$$
$$[\mu]_0 = \overline{\{x : \mu(x) > 0\}},$$

where $\overline{B}$ is the closure of the non-fuzzy set B.

Let $C(X)$ be the family of all nonempty compact subsets of X. Denote by $\mathcal{C}(X)$ the totality of fuzzy sets which satisfy that for each $\alpha \in I$, $[\mu]_\alpha \in C(X)$. Let $\mu_1, \mu_2 \in \mathcal{C}(X)$, then μ_1 is said to be more accurate than μ_2, denoted by $\mu_1 \subset \mu_2$, if and only if $\mu_1(x) \le \mu_2(x)$ for each $x \in X$. $\mu_1 = \mu_2$ if and only if $\mu_1 \subset \mu_2$ and $\mu_2 \subset \mu_1$.

Let $\mu_1, \mu_2 \in \mathcal{C}(X)$, define

$$\begin{aligned} D_\infty(\mu_1, \mu_2) &= \sup_{0 \le \alpha \le 1} H\big([\mu_1]_\alpha, [\mu_2]_\alpha\big) \\ &= \sup_{0 \le \alpha \le 1} \max\Big\{ \sup_{x \in [\mu_1]_\alpha} d_G\big(x, [\mu_2]_\alpha\big), \sup_{y \in [\mu_2]_\alpha} d_G\big(y, [\mu_1]_\alpha\big) \Big\}. \end{aligned}$$

Lemma 2.1 [23] *The metric space $(\mathcal{C}(X), D_\infty)$ is complete provided (X, d_G) is complete.*

For $\mu_1, \mu_2, \mu_3 \in \mathcal{C}(X)$, $\alpha \in I$, we define:

$$\begin{aligned} &G_\alpha(\mu_1, \mu_2, \mu_3) = G\big([\mu_1]_\alpha, [\mu_2]_\alpha, [\mu_3]_\alpha\big) = \sup_{x \in [\mu_1]_\alpha} G\big(x, [\mu_2]_\alpha, [\mu_3]_\alpha\big), \\ &G_\infty(\mu_1, \mu_2, \mu_3) = \sup_{0 \le \alpha \le 1} G_\alpha(\mu_1, \mu_2, \mu_3), \\ &D_{G,\alpha}(\mu_1, \mu_2, \mu_3) \\ &\quad = H_G\big([\mu_1]_\alpha, [\mu_2]_\alpha, [\mu_3]_\alpha\big) \\ &\quad = \max\Big\{ \sup_{x \in [\mu_1]_\alpha} G\big(x, [\mu_2]_\alpha, [\mu_3]_\alpha\big), \sup_{x \in [\mu_2]_\alpha} G\big(x, [\mu_3]_\alpha, [\mu_1]_\alpha\big), \sup_{x \in [\mu_3]_\alpha} G\big(x, [\mu_1]_\alpha, [\mu_2]_\alpha\big) \Big\} \\ &\quad = \max\big\{ G_\alpha(\mu_1, \mu_2, \mu_3), G_\alpha(\mu_2, \mu_3, \mu_1), G_\alpha(\mu_3, \mu_1, \mu_2) \big\}, \\ &D_{G,\infty}(\mu_1, \mu_2, \mu_3) \\ &\quad = \sup_{0 \le \alpha \le 1} D_{G,\alpha}(\mu_1, \mu_2, \mu_3) \\ &\quad = \sup_{0 \le \alpha \le 1} \max\big\{ G_\alpha(\mu_1, \mu_2, \mu_3), G_\alpha(\mu_2, \mu_3, \mu_1), G_\alpha(\mu_3, \mu_1, \mu_2) \big\} \\ &\quad = \max\Big\{ \sup_{0 \le \alpha \le 1} G_\alpha(\mu_1, \mu_2, \mu_3), \sup_{0 \le \alpha \le 1} G_\alpha(\mu_2, \mu_3, \mu_1), \sup_{0 \le \alpha \le 1} G_\alpha(\mu_3, \mu_1, \mu_2) \Big\} \\ &\quad = \max\big\{ G_\infty(\mu_1, \mu_2, \mu_3), G_\infty(\mu_2, \mu_3, \mu_1), G_\infty(\mu_3, \mu_1, \mu_2) \big\}. \end{aligned}$$

Proposition 2.1 *If $A, B \in C(X)$ and $x \in A$, then there exists $y \in B$ such that*

$$2\big[G(x, y, y) + G(y, x, x)\big] \le H_G(A, B, B).$$

Proof For $x \in A$, there exists $y \in B$ such that

$$d_G(x,y) = d_G(x,B) = \frac{1}{2}G(x,B,B),$$

it follows that

$$2\big[G(x,y,y) + G(y,x,x)\big] = G(x,B,B) \le H_G(A,B,B). \qquad \square$$

Proposition 2.2 *If $A, B \in C(X)$ and $A_1 \subseteq A$, then there exists $B_1 \in C(X)$ such that $B_1 \subseteq B$ and*

$$H_G(A_1,B_1,B_1) \le H_G(A,B,B).$$

Proof Let $C = \{y:$ there exists $x \in A_1$, such that $2(G(x,y,y) + G(y,x,x)) \le H_G(A,B,B)\}$ and let $B_1 = C \cap B$. For any $x \in A_1 \subseteq A$ and $B \in C(X)$, Proposition 2.1 implies that B_1 is nonempty. Moreover, for any $x \in A_1$, there exists $y \in B_1$ such that $2[G(x,y,y) + G(y,x,x)] \le H_G(A,B,B)$. It follows that

$$\begin{aligned} G(A_1,B_1,B_1) &= \sup_{x\in A_1} G(x,B_1,B_1) = \sup_{x\in A_1} 2d_G(x,B_1) \\ &= 2\sup_{x\in A_1}\inf_{y\in B_1}\big[G(x,y,y) + G(y,x,x)\big] \le H_G(A,B,B). \end{aligned} \tag{1}$$

On the other hand, for any $y \in B_1$, there exists $x \in A_1$ such that $2[G(x,y,y) + G(y,x,x)] \le H_G(A,B,B)$. Hence,

$$\begin{aligned} G(B_1,B_1,A_1) &= \sup_{y\in B_1} G(y,B_1,A_1) = \sup_{y\in B_1}\big[d_G(y,A_1)\big] + \sup_{y\in B_1}\big[d_G(y,B_1)\big] + d_G(A_1,B_1) \\ &= \sup_{y\in B_1}\inf_{x\in A_1}\big[G(x,y,y) + G(y,x,x)\big] + 0 + \inf_{x\in A_1, y\in B_1}\big[G(x,y,y) + G(y,x,x)\big] \\ &\le \sup_{y\in B_1}\inf_{x\in A_1}\big[G(x,y,y) + G(y,x,x)\big] + \sup_{y\in B_1}\inf_{x\in A_1}\big[G(x,y,y) + G(y,x,x)\big] \\ &= \sup_{y\in B_1}\inf_{x\in A_1} 2\big[G(x,y,y) + G(y,x,x)\big] \le H_G(A,B,B). \end{aligned} \tag{2}$$

From (1) and (2), we have

$$H_G(A_1,B_1,B_1) \le H_G(A,B,B).$$

Finally, we can conclude that $B_1 \in C(X)$ from the closeness of C and the compactness of B. $\square$

Proposition 2.3 *Let $\mu_1, \mu_2 \in \mathcal{C}(X)$ and $\mu_3 \subset \mu_1$, then there exists $\mu_4 \in \mathcal{C}(X)$ such that $\mu_4 \subset \mu_2$ and*

$$D_{G,\infty}(\mu_3,\mu_4,\mu_4) \le D_{G,\infty}(\mu_1,\mu_2,\mu_2).$$

Proof Let $\alpha \in I$, by $\mu_3 \subset \mu_1$, we have $[\mu_3]_\alpha \subseteq [\mu_1]_\alpha$. Let

$$\begin{aligned} C_\alpha &= \bigl\{y: \text{ there exists } x \in [\mu_3]_\alpha \text{ such that } 2\bigl[G(x,y,y) + G(y,x,x)\bigr] \\ &\qquad \le D_{G,\infty}(\mu_1,\mu_2,\mu_2)\bigr\}, \\ D_\alpha &= \bigl\{z : 2d_G\bigl(z,[\mu_3]_\alpha\bigr) \le D_{G,\infty}(\mu_1,\mu_2,\mu_2)\bigr\}, \end{aligned}$$

we can get that $C_\alpha = D_\alpha$. Let $B_\alpha = D_\alpha \cap [\mu_2]_\alpha$, then B_α is nonempty compact and $B_\alpha \subseteq B_\beta$, for $0 \le \beta \le \alpha \le 1$. From the proof of Proposition 2.2, we have

$$H_G\bigl([\mu_3]_\alpha, B_\alpha, B_\alpha\bigr) \le D_{G,\infty}(\mu_1,\mu_2,\mu_2).$$

Similar to the proof of Theorem 3 in [23], we can conclude that there exists a fuzzy set μ_4 such that $[\mu_4]_\alpha = B_\alpha$ for $\alpha \in I$. By the compactness of B_α, we have $\mu_4 \in \mathcal{C}(X)$. Therefore,

$$D_{G,\infty}(\mu_3,\mu_4,\mu_4) \le D_{G,\infty}(\mu_1,\mu_2,\mu_2). \qquad \square$$

Proposition 2.4 *Let X be a nonempty set. For any $\mu_1,\mu_2,\mu_3 \in \mathcal{C}(X)$, the following properties hold:*

(i) $D_{G,\infty}(\mu_1,\mu_2,\mu_3) = 0$ *if and only if* $\mu_1 = \mu_2 = \mu_3$,

(ii) $0 < D_{G,\infty}(\mu_1,\mu_1,\mu_2)$ *for all* $\mu_1,\mu_2 \in \mathcal{C}(X)$ *with* $\mu_1 \neq \mu_2$,

(iii) $D_{G,\infty}(\mu_1,\mu_1,\mu_2) \le D_{G,\infty}(\mu_1,\mu_2,\mu_3)$ *for all* $\mu_1,\mu_2,\mu_3 \in \mathcal{C}(X)$ *with* $\mu_2 \neq \mu_3$,

(iv) $D_{G,\infty}(\mu_1,\mu_2,\mu_3) = D_{G,\infty}(\mu_1,\mu_3,\mu_2) = D_{G,\infty}(\mu_2,\mu_1,\mu_3) = \cdots$ *(symmetry in all three variables),*

(v) $D_{G,\infty}(\mu_1,\mu_2,\mu_3) \le D_{G,\infty}(\mu_1,\mu,\mu) + D_{G,\infty}(\mu,\mu_2,\mu_3)$.

Proof The properties (i), (ii) and (iv) are readily derived from the definition of $D_{G,\infty}$.

First, we prove the property (iii).

For any $\alpha \in I$ and $x \in [\mu_1]_\alpha$, $y \in [\mu_2]_\alpha$ and $z \in [\mu_3]_\alpha$, we have

$$d_G(x,y) - d_G(x,z) - d_G(z,y) \le 0,$$

it follows that

$$\begin{aligned} &d_G(x,y) - d_G\bigl(x,[\mu_3]_\alpha\bigr) - d_G\bigl([\mu_2]_\alpha,[\mu_3]_\alpha\bigr) \\ &\quad \le \sup_{y\in[\mu_2]_\alpha} d_G(x,y) - \inf_{z\in[\mu_3]_\alpha} d_G(x,z) - \inf_{y\in[\mu_2]_\alpha, z\in[\mu_3]_\alpha} d_G(z,y) \\ &\quad = \sup_{y\in[\mu_2]_\alpha, z\in[\mu_3]_\alpha} \bigl[d_G(x,y) - d_G(x,z) - d_G(z,y)\bigr] \le 0. \end{aligned}$$

This implies that

$$\inf_{x\in[\mu_1]_\alpha, y\in[\mu_2]_\alpha} d_G(x,y) - \sup_{x\in[\mu_1]_\alpha} d_G\bigl(x,[\mu_3]_\alpha\bigr) \le d_G\bigl([\mu_2]_\alpha,[\mu_3]_\alpha\bigr).$$

Then,

$$d_G\bigl([\mu_1]_\alpha,[\mu_2]_\alpha\bigr) \le \sup_{x\in[\mu_1]_\alpha} d_G\bigl(x,[\mu_3]_\alpha\bigr) + d_G\bigl([\mu_2]_\alpha,[\mu_3]_\alpha\bigr).$$

Hence,

$$\begin{aligned}
&G\big([\mu_1]_\alpha, [\mu_1]_\alpha, [\mu_2]_\alpha\big)\\
&\quad = \sup_{x\in[\mu_1]_\alpha} d_G\big(x, [\mu_2]_\alpha\big) + d_G\big([\mu_1]_\alpha, [\mu_2]_\alpha\big)\\
&\quad \le G\big([\mu_1]_\alpha, [\mu_3]_\alpha, [\mu_2]_\alpha\big)\\
&\quad = \sup_{x\in[\mu_1]_\alpha} d_G\big(x, [\mu_2]_\alpha\big) + \sup_{x\in[\mu_1]_\alpha} d_G\big(x, [\mu_3]_\alpha\big) + d_G\big([\mu_2]_\alpha, [\mu_3]_\alpha\big).
\end{aligned} \tag{3}$$

Similarly, we can prove that

$$G\big([\mu_2]_\alpha, [\mu_1]_\alpha, [\mu_1]_\alpha\big) \le G\big([\mu_2]_\alpha, [\mu_1]_\alpha, [\mu_3]_\alpha\big). \tag{4}$$

By (3) and (4), we have

$$\begin{aligned}
&D_{G,\infty}(\mu_1, \mu_1, \mu_2)\\
&\quad = \sup_{\alpha\in I} D_{G,\alpha}(\mu_1, \mu_1, \mu_2)\\
&\quad = \sup_{\alpha\in I} \max\big\{G\big([\mu_1]_\alpha, [\mu_1]_\alpha, [\mu_2]_\alpha\big), G\big([\mu_2]_\alpha, [\mu_1]_\alpha, [\mu_1]_\alpha\big)\big\}\\
&\quad \le D_{G,\infty}(\mu_1, \mu_2, \mu_3) = \sup_{\alpha\in I} D_{G,\alpha}(\mu_1, \mu_2, \mu_3)\\
&\quad = \sup_{\alpha\in I} \max\big\{G\big([\mu_1]_\alpha, [\mu_3]_\alpha, [\mu_2]_\alpha\big), G\big([\mu_2]_\alpha, [\mu_1]_\alpha, [\mu_3]_\alpha\big), G\big([\mu_3]_\alpha, [\mu_1]_\alpha, [\mu_2]_\alpha\big)\big\}.
\end{aligned}$$

Now, we prove the property (v).

For any $\alpha \in I$ and $x \in [\mu_2]_\alpha$, $y \in [\mu]_\alpha$, we have

$$d_G\big(x, [\mu_1]_\alpha\big) \le d_G(x, y) + d_G\big(y, [\mu_1]_\alpha\big),$$

it follows that

$$\sup_{x\in[\mu_2]_\alpha} d_G\big(x, [\mu_1]_\alpha\big) \le \sup_{x\in[\mu_2]_\alpha} d_G\big(x, [\mu]_\alpha\big) + \sup_{y\in[\mu]_\alpha} d_G\big(y, [\mu_1]_\alpha\big). \tag{5}$$

From (5) and

$$d_G\big([\mu_1]_\alpha, [\mu_3]_\alpha\big) \le d_G\big([\mu]_\alpha, [\mu_3]_\alpha\big) + d_G\big([\mu]_\alpha, [\mu_1]_\alpha\big),$$

we have

$$\begin{aligned}
G_\alpha(\mu_2, \mu_3, \mu_1) &= G\big([\mu_2]_\alpha, [\mu_3]_\alpha, [\mu_1]_\alpha\big)\\
&= \sup_{x\in[\mu_2]_\alpha} \big[d_G\big(x, [\mu_1]_\alpha\big) + d_G\big(x, [\mu_3]_\alpha\big) + d_G\big([\mu_1]_\alpha, [\mu_3]_\alpha\big)\big]\\
&\le \sup_{x\in[\mu_2]_\alpha} \big[d_G\big(x, [\mu]_\alpha\big) + d_G\big(x, [\mu_3]_\alpha\big) + d_G\big([\mu]_\alpha, [\mu_3]_\alpha\big)\big]\\
&\quad + \sup_{y\in[\mu]_\alpha} \big[d_G\big(y, [\mu_1]_\alpha\big) + d_G\big(y, [\mu]_\alpha\big) + d_G\big([\mu]_\alpha, [\mu_1]_\alpha\big)\big]\\
&= G_\alpha(\mu_2, \mu_3, \mu) + G_\alpha(\mu, \mu, \mu_1).
\end{aligned} \tag{6}$$

Similarly, we can obtain that

$$G_\alpha(\mu_3,\mu_2,\mu_1) \le G_\alpha(\mu_3,\mu_2,\mu) + G_\alpha(\mu,\mu,\mu_1), \tag{7}$$

$$G_\alpha(\mu_1,\mu_2,\mu_3) \le G_\alpha(\mu,\mu_2,\mu_3) + G_\alpha(\mu_1,\mu,\mu). \tag{8}$$

By (6), (7) and (8), we have

$$\begin{aligned} D_{G,\infty}(\mu_1,\mu_2,\mu_3) &= \sup_{0\le\alpha\le1} \max\bigl\{G_\alpha(\mu_2,\mu_3,\mu_1), G_\alpha(\mu_3,\mu_2,\mu_1), G_\alpha(\mu_1,\mu_2,\mu_3)\bigr\} \\ &\le \sup_{0\le\alpha\le1} \max\bigl\{G_\alpha(\mu_2,\mu_3,\mu), G_\alpha(\mu_3,\mu_2,\mu), G_\alpha(\mu,\mu_2,\mu_3)\bigr\} \\ &\quad + \sup_{0\le\alpha\le1} \max\bigl\{G_\alpha(\mu,\mu,\mu_1), G_\alpha(\mu,\mu,\mu_1), G_\alpha(\mu_1,\mu,\mu)\bigr\} \\ &= D_{G,\infty}(\mu,\mu_2,\mu_3) + D_{G,\infty}(\mu_1,\mu,\mu). \end{aligned}$$ □

Remark 2.1 Proposition 2.4 implies that $D_{G,\infty}$ is a G-metric in $\mathcal{C}(X)$, or more specially a Hausdorff G-metric in $\mathcal{C}(X)$.

Definition 2.1 Let $(\mathcal{C}(X), D_{G,\infty})$ be a metric space. The sequence $\{\mu_n\}$ in $\mathcal{C}(X)$ is said to be

(i) $D_{G,\infty}$-convergent to μ if for every $\varepsilon > 0$, there exists $\mu \in \mathcal{C}(X)$ and $N \in \mathbb{N}$ such that $D_{G,\infty}(\mu,\mu_n,\mu_m) < \varepsilon$ for all $n,m \ge N$,

(ii) $D_{G,\infty}$-Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $D_{G,\infty}(\mu_n,\mu_m,\mu_l) < \varepsilon$ for all $n,m,l \ge N$.

Proposition 2.5 *Let $(\mathcal{C}(X), D_{G,\infty})$ be a metric space, then for a sequence $\{\mu_n\} \subset \mathcal{C}(X)$ and $\mu \in \mathcal{C}(X)$, the following are equivalent:*

(i) $\{\mu_n\}$ is $D_{G,\infty}$-convergent to μ.

(ii) $D_\infty(\mu,\mu_n) \to 0$ as $n \to +\infty$.

(iii) $D_{G,\infty}(\mu_n,\mu_n,\mu) \to 0$ as $n \to +\infty$.

(iv) $D_{G,\infty}(\mu,\mu,\mu_n) \to 0$ as $n \to +\infty$.

Proof Since $D_{G,\infty}$ is a G-metric, Lemma 1.2 implies that (i), (iii) and (iv) are equivalent. Now, we prove that (ii) is also an equivalent condition.

"(i) $\Longrightarrow$ (ii)" Suppose $D_{G,\infty}(\mu,\mu_n,\mu_m) \to 0$ as $n,m \to +\infty$, then

$$G_\infty(\mu,\mu_n,\mu_m) = \sup_{0\le\alpha\le1} \sup_{x\in[\mu]_\alpha} \bigl[d_G\bigl(x,[\mu_n]_\alpha\bigr) + d_G\bigl(x,[\mu_m]_\alpha\bigr) + d_G\bigl([\mu_n]_\alpha,[\mu_m]_\alpha\bigr)\bigr] \to 0$$

and

$$G_\infty(\mu_n,\mu,\mu_m) = \sup_{0\le\alpha\le1} \sup_{x\in[\mu_n]_\alpha} \bigl[d_G\bigl(x,[\mu]_\alpha\bigr) + d_G\bigl(x,[\mu_m]_\alpha\bigr) + d_G\bigl([\mu]_\alpha,[\mu_m]_\alpha\bigr)\bigr] \to 0.$$

Thus, for any $\alpha \in I$,

$$\sup_{x\in[\mu]_\alpha} \bigl[d_G\bigl(x,[\mu_n]_\alpha\bigr)\bigr] \to 0 \quad \text{as } n \to +\infty, \tag{9}$$

and

$$\sup_{x\in[\mu_n]_\alpha}\bigl[d_G\bigl(x,[\mu]_\alpha\bigr)\bigr]\to 0 \quad \text{as } n\to+\infty. \tag{10}$$

It follows that

$$D_\infty(\mu,\mu_n)=\sup_{0\le\alpha\le1}H\bigl([\mu]_\alpha,[\mu_n]_\alpha\bigr)\to 0 \quad \text{as } n\to+\infty.$$

"(ii) $\Longrightarrow$ (i)" Suppose $D_\infty(\mu,\mu_n)\to 0$ as $n\to+\infty$, then (9) and (10) hold.
Moreover, $0\le d_G([\mu_n]_\alpha,[\mu]_\alpha)\le \sup_{x\in[\mu_n]_\alpha}[d_G(x,[\mu]_\alpha)]$ implies that as $n\to+\infty$,

$$d_G\bigl([\mu_n]_\alpha,[\mu]_\alpha\bigr)\to 0. \tag{11}$$

From (9), (10) and (11), we have as $n\to+\infty$,

$$\begin{aligned}
&D_{G,\infty}(\mu_n,\mu_n,\mu)\\
&\quad=\sup_{0\le\alpha\le1}\max\bigl\{G\bigl([\mu_n]_\alpha,[\mu_n]_\alpha,[\mu]_\alpha\bigr),G\bigl([\mu]_\alpha,[\mu_n]_\alpha,[\mu_n]_\alpha\bigr)\bigr\}\\
&\quad=\sup_{0\le\alpha\le1}\max\Bigl\{\sup_{x\in[\mu_n]_\alpha}\bigl[d_G\bigl(x,[\mu]_\alpha\bigr)+d_G\bigl([\mu_n]_\alpha,[\mu]_\alpha\bigr)\bigr],\sup_{x\in[\mu]_\alpha}2d_G\bigl(x,[\mu_n]_\alpha\bigr)\Bigr\}\to 0.
\end{aligned}$$

Thus, from

$$\begin{aligned}
0&\le D_{G,\infty}(\mu,\mu_n,\mu_m)=D_{G,\infty}(\mu_n,\mu,\mu_m)\le D_{G,\infty}(\mu_n,\mu,\mu)+D_{G,\infty}(\mu,\mu,\mu_m)\\
&\le D_{G,\infty}(\mu,\mu_n,\mu_n)+D_{G,\infty}(\mu_n,\mu,\mu_n)+D_{G,\infty}(\mu,\mu_m,\mu_m)+D_{G,\infty}(\mu_m,\mu,\mu_m)\\
&=2D_{G,\infty}(\mu_n,\mu_n,\mu)+2D_{G,\infty}(\mu_m,\mu_m,\mu),
\end{aligned}$$

we can conclude that

$$D_{G,\infty}(\mu,\mu_n,\mu_m)\to 0 \quad \text{as } n,m\to+\infty. \qquad \square$$

Proposition 2.6 *Let $(\mathcal{C}(X),D_{G,\infty})$ be a metric space and $\{\mu_n\}$ a sequence in $\mathcal{C}(X)$, then the following are equivalent:*

(i) The sequence $\{\mu_n\}$ is $D_{G,\infty}$-Cauchy.
(ii) For every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $D_{G,\infty}(\mu_n,\mu_m,\mu_m)<\varepsilon$ for all $n,m>N$.
(iii) $\{\mu_n\}$ is a Cauchy sequence in the metric space $(\mathcal{C}(X),D_\infty)$.

Proof "(i) $\Longleftrightarrow$ (ii)" is evidence.
"(ii) $\Longrightarrow$ (iii)" Suppose that for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $D_{G,\infty}(\mu_n,\mu_m,\mu_m)<\varepsilon$, for all $n,m>N$, then as $n,m\to+\infty$,

$$G_\infty(\mu_n,\mu_m,\mu_m)\to 0 \tag{12}$$

and

$$G_\infty(\mu_m,\mu_m,\mu_n)\to 0. \tag{13}$$

It follows that

$$\sup_{0\le\alpha\le 1}\sup_{x\in[\mu_n]_\alpha} d_G\bigl(x,[\mu_m]_\alpha\bigr) = \frac{1}{2}G_\infty(\mu_n,\mu_m,\mu_m)\to 0. \tag{14}$$

From (13) and

$$0\le \sup_{0\le\alpha\le 1}\sup_{x\in[\mu_m]_\alpha} d_G\bigl(x,[\mu_n]_\alpha\bigr)\le G_\infty(\mu_m,\mu_m,\mu_n),$$

we have

$$\sup_{0\le\alpha\le 1}\sup_{x\in[\mu_m]_\alpha} d_G\bigl(x,[\mu_n]_\alpha\bigr)\to 0 \quad \text{as } n,m\to+\infty. \tag{15}$$

By (14) and (15), we have

$$D_\infty(\mu_n,\mu_m)\to 0 \quad \text{as } n,m\to+\infty,$$

that is, $\{\mu_n\}$ is a Cauchy sequence in the metric space $(\mathcal{C}(X),D_\infty)$.

"(iii) $\Longrightarrow$ (ii)" Suppose $D_\infty(\mu_n,\mu_m)\to 0$ as $n,m\to+\infty$, then (14) and (15) hold. Moreover,

$$0\le \sup_{0\le\alpha\le 1} d_G\bigl([\mu_m]_\alpha,[\mu_n]_\alpha\bigr)\le \sup_{0\le\alpha\le 1}\sup_{x\in[\mu_m]_\alpha} d_G\bigl(x,[\mu_n]_\alpha\bigr)$$

and (15) imply that

$$\sup_{0\le\alpha\le 1} d_G\bigl([\mu_m]_\alpha,[\mu_n]_\alpha\bigr)\to 0 \quad \text{as } n,m\to+\infty. \tag{16}$$

From (14), (15) and (16), we have as $n,m\to+\infty$,

$$\begin{aligned} G_\infty(\mu_n,\mu_m,\mu_m) &= \sup_{0\le\alpha\le 1} G\bigl([\mu_n]_\alpha,[\mu_m]_\alpha,[\mu_m]_\alpha\bigr)\\ &= \sup_{0\le\alpha\le 1}\sup_{x\in[\mu_n]_\alpha} 2d_G\bigl(x,[\mu_m]_\alpha\bigr)\to 0 \end{aligned} \tag{17}$$

and

$$\begin{aligned} G_\infty(\mu_m,\mu_m,\mu_n) &= \sup_{0\le\alpha\le 1} G\bigl([\mu_m]_\alpha,[\mu_m]_\alpha,[\mu_n]_\alpha\bigr)\\ &= \sup_{0\le\alpha\le 1}\sup_{x\in[\mu_m]_\alpha} \bigl[d_G\bigl(x,[\mu_n]_\alpha\bigr)+d_G\bigl([\mu_m]_\alpha,[\mu_n]_\alpha\bigr)\bigr]\to 0. \end{aligned} \tag{18}$$

We can get from (17) and (18) that

$$D_{G,\infty}(\mu_n,\mu_m,\mu_m)\to 0 \quad \text{as } n,m\to+\infty. \qquad \square$$

The next proposition follows directly from Lemma 1.4, Lemma 2.1, Proposition 2.5 and Proposition 2.6.

Proposition 2.7 *The metric space $(\mathcal{C}(X), D_{G,\infty})$ is complete provided (X, G) is G-complete.*

From the definitions of G_∞ and $D_{G,\infty}$, we can get the next proposition readily.

Proposition 2.8 *If $\mu, \mu_1, \mu_2 \in \mathcal{C}(X)$ and $\mu_1 \subset \mu_2$, then*
(i) $G_\infty(\mu_1, \mu, \mu) \le G_\infty(\mu_2, \mu, \mu)$,
(ii) $G_\infty(\mu, \mu_2, \mu_2) \le G_\infty(\mu, \mu_1, \mu_1) \le D_{G,\infty}(\mu, \mu_1, \mu_1)$,
(iii) $G_\infty(\mu_1, \mu_2, \mu_2) = 0$.

3 Fixed point theorems for fuzzy self-mappings

In this section, we establish two fixed point theorems for fuzzy self-mappings. First, we recall the concept of a fuzzy self-mapping in [23].

Definition 3.1 [23] Let X be a metric space. A mapping F is said to be a fuzzy self-mapping if and only if F is a mapping from the space $\mathcal{C}(X)$ into $\mathcal{C}(X)$, *i.e.*, $F(\mu) \in \mathcal{C}(X)$ for each $\mu \in \mathcal{C}(X)$. $\mu_0 \in \mathcal{C}(X)$ is said to be a fixed point of a fuzzy self-mapping F of $\mathcal{C}(X)$ if and only if $\mu_0 \subset F(\mu_0)$.

Let Φ denote all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying:
(i) ϕ is non-decreasing and continuous from the right,
(ii) $\sum_{n=1}^{\infty} \phi^n(t) < +\infty$, for all $t > 0$, where ϕ^n denotes the nth iterative function of ϕ.

Remark 3.1 It can be directly verified that for any $\phi \in \Phi$ and all $t > 0$, $\phi(t) < t$.

Theorem 3.1 *Let (X, G) be a G-complete metric space and $\{T_i\}_{i=1}^{\infty}$ a sequence of fuzzy self-mappings of $\mathcal{C}(X)$. Suppose that for each $\mu_1, \mu_2 \in \mathcal{C}(X)$ and for arbitrary positive integers i and j, $i \neq j$,*

$$\begin{aligned} &D_{G,\infty}(T_i\mu_1, T_j\mu_2, T_j\mu_2) \\ &\quad \le \phi\Bigg(\max\Bigg\{D_{G,\infty}(\mu_1, \mu_2, \mu_2), G_\infty(\mu_1, T_i\mu_1, T_i\mu_1), G_\infty(\mu_2, T_j\mu_2, T_j\mu_2), \\ &\qquad \frac{1}{2}\big[G_\infty(\mu_1, T_j\mu_2, T_j\mu_2) + G_\infty(\mu_2, T_i\mu_1, T_i\mu_1)\big]\Bigg\}\Bigg), \end{aligned} \tag{19}$$

where $\phi \in \Phi$. Then there exists at least one $\mu_* \in \mathcal{C}(X)$ such that $\mu_* \subset T_i\mu_*$ for all $i \in \mathbb{Z}^+$.

Proof Let $\mu_0 \in \mathcal{C}(X)$ and $\mu_1 \subset T_1\mu_0$, by Proposition 2.3, there exists $\mu_2 \in \mathcal{C}(X)$ such that $\mu_2 \subset T_2\mu_1$ and

$$D_{G,\infty}(\mu_1, \mu_2, \mu_2) \le D_{G,\infty}(T_1\mu_0, T_2\mu_1, T_2\mu_1).$$

Again by Proposition 2.3, we can find $\mu_3 \in \mathcal{C}(X)$ such that $\mu_3 \subset T_3\mu_2$ and

$$D_{G,\infty}(\mu_2, \mu_3, \mu_3) \le D_{G,\infty}(T_2\mu_1, T_3\mu_2, T_3\mu_2).$$

Continuing this process, we can construct a sequence $\{\mu_n\}$ in $\mathcal{C}(X)$ such that

$$\mu_{n+1} \subset T_{n+1}\mu_n, \quad n = 0, 1, 2, \ldots$$

and

$$D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}) \le D_{G,\infty}(T_n\mu_{n-1},T_{n+1}\mu_n,T_{n+1}\mu_n), \quad n=1,2,\ldots. \tag{20}$$

By (19), (20), Proposition 2.8 and (v) in Proposition 2.4, we have

$$\begin{aligned}
&D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}) \\
&\quad\le D_{G,\infty}(T_n\mu_{n-1},T_{n+1}\mu_n,T_{n+1}\mu_n) \\
&\quad\le \phi\biggl(\max\biggl\{D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n),G_\infty(\mu_{n-1},T_n\mu_{n-1},T_n\mu_{n-1}),G_\infty(\mu_n,T_{n+1}\mu_n,T_{n+1}\mu_n), \\
&\qquad \frac{1}{2}\bigl[G_\infty(\mu_{n-1},T_{n+1}\mu_n,T_{n+1}\mu_n)+G_\infty(\mu_n,T_n\mu_{n-1},T_n\mu_{n-1})\bigr]\biggr\}\biggr) \\
&\quad\le \phi\biggl(\max\biggl\{D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n),D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n),D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}), \\
&\qquad \frac{1}{2}\bigl[D_{G,\infty}(\mu_{n-1},\mu_{n+1},\mu_{n+1})+0\bigr]\biggr\}\biggr) \\
&\quad\le \phi\biggl(\max\biggl\{D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n),D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}), \\
&\qquad \frac{1}{2}\bigl[D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n)+D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1})\bigr]\biggr\}\biggr) \\
&\quad= \phi(\max\{D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n),D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}), \quad n=1,2,3,\ldots.
\end{aligned} \tag{21}$$

Suppose that $0 \le D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n) < D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1})$, then

$$D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}) \le \phi\bigl(D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1})\bigr) < D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}),$$

which is a contradiction since $D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}) > 0$.

Hence,

$$D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}) \le D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n) \tag{22}$$

and

$$D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1}) \le \phi\bigl(D_{G,\infty}(\mu_{n-1},\mu_n,\mu_n)\bigr) \le \cdots \le \phi^n\bigl(D_{G,\infty}(\mu_0,\mu_1,\mu_1)\bigr). \tag{23}$$

Now, we prove that $\{\mu_n\}$ is a $D_{G,\infty}$-Cauchy sequence. For positive integers m, n, we distinguish the following two cases.

Case 1. If $m > n$, then

$$\begin{aligned}
&D_{G,\infty}(\mu_n,\mu_m,\mu_m) \\
&\quad\le D_{G,\infty}(\mu_n,\mu_{n+1},\mu_{n+1})+D_{G,\infty}(\mu_{n+1},\mu_{n+2},\mu_{n+2})+\cdots+D_{G,\infty}(\mu_{m-1},\mu_m,\mu_m) \\
&\quad= \sum_{i=n}^{m-1} D_{G,\infty}(\mu_i,\mu_{i+1},\mu_{i+1}) \le \sum_{i=n}^{m-1} \phi^i\bigl(D_{G,\infty}(\mu_0,\mu_1,\mu_1)\bigr).
\end{aligned} \tag{24}$$

Assume that $D_{G,\infty}(\mu_0,\mu_1,\mu_1) = 0$, then $\mu_0 = \mu_1$. Inequality (21) implies that

$$\begin{aligned} D_{G,\infty}(\mu_1,\mu_2,\mu_2) &\le \phi\bigl(\max\bigl\{D_{G,\infty}(\mu_0,\mu_1,\mu_1), D_{G,\infty}(\mu_1,\mu_2,\mu_2)\bigr\}\bigr) \\ &= \phi\bigl(D_{G,\infty}(\mu_1,\mu_2,\mu_2)\bigr). \end{aligned}$$

It follows from $\phi(t) < t$ that $D_{G,\infty}(\mu_1,\mu_2,\mu_2) = 0$, that is, $\mu_1 = \mu_2$. By induction, we have $\mu_0 = \mu_1 = \cdots = \mu_k = \cdots$. Thus, $\mu_0 = \mu_k \subset T_k\mu_{k-1} = T_k\mu_0$, $k = 1,2,\ldots$.

Suppose that $D_{G,\infty}(\mu_0,\mu_1,\mu_1) > 0$. $\sum_{i=1}^{\infty}\phi^i(t) < \infty$ and (24) yield that

$$D_{G,\infty}(\mu_n,\mu_m,\mu_m) \to 0, \quad \text{as } n,m \to +\infty. \tag{25}$$

Case 2. If $m < n$, from (25) and

$$\begin{aligned} 0 &\le D_{G,\infty}(\mu_n,\mu_m,\mu_m) = D_{G,\infty}(\mu_m,\mu_m,\mu_n) \\ &\le D_{G,\infty}(\mu_m,\mu_n,\mu_n) + D_{G,\infty}(\mu_n,\mu_m,\mu_n) = 2D_{G,\infty}(\mu_m,\mu_n,\mu_n), \end{aligned}$$

we can get that

$$D_{G,\infty}(\mu_n,\mu_m,\mu_m) \to 0 \quad \text{as } n,m \to +\infty. \tag{26}$$

Thus, (25) and (26) imply that $\{\mu_n\}$ is a $D_{G,\infty}$-Cauchy sequence. As (X,G) is G-complete, by Proposition 2.7, we conclude that $(\mathcal{C}(X), D_{G,\infty})$ is complete. There exists $\mu_* \in \mathcal{C}(X)$ such that $D_{G,\infty}(\mu_*,\mu_m,\mu_m) \to 0$ as $m \to +\infty$.

Now, Proposition 2.8 and (19) imply that

$$\begin{aligned} &G_\infty(\mu_*, T_i\mu_*, T_i\mu_*) \\ &\quad\le G_\infty(\mu_*,\mu_j,\mu_j) + G_\infty(\mu_j, T_i\mu_*, T_i\mu_*) \\ &\quad\le G_\infty(\mu_*,\mu_j,\mu_j) + G_\infty(T_j\mu_{j-1}, T_i\mu_*, T_i\mu_*) \\ &\quad\le G_\infty(\mu_*,\mu_j,\mu_j) + D_{G,\infty}(T_j\mu_{j-1}, T_i\mu_*, T_i\mu_*) \\ &\quad\le G_\infty(\mu_*,\mu_j,\mu_j) \\ &\qquad + \phi\biggl(\max\biggl\{D_{G,\infty}(\mu_{j-1},\mu_*,\mu_*), G_\infty(\mu_{j-1}, T_j\mu_{j-1}, T_j\mu_{j-1}), G_\infty(\mu_*, T_i\mu_*, T_i\mu_*), \\ &\qquad \frac{1}{2}\bigl[G_\infty(\mu_{j-1}, T_i\mu_*, T_i\mu_*) + G_\infty(\mu_*, T_j\mu_{j-1}, T_j\mu_{j-1})\bigr]\biggr\}\biggr) \\ &\quad\le D_{G,\infty}(\mu_*,\mu_j,\mu_j) \\ &\qquad + \phi\biggl(\max\biggl\{D_{G,\infty}(\mu_{j-1},\mu_*,\mu_*), D_{G,\infty}(\mu_{j-1},\mu_j,\mu_j), G_\infty(\mu_*, T_i\mu_*, T_i\mu_*), \\ &\qquad \frac{1}{2}\bigl[G_\infty(\mu_{j-1},\mu_*,\mu_*) + G_\infty(\mu_*, T_i\mu_*, T_i\mu_*) \\ &\qquad + G_\infty(\mu_*,\mu_j,\mu_j) + G_\infty(\mu_j, T_j\mu_{j-1}, T_j\mu_{j-1})\bigr]\biggr\}\biggr) \\ &\quad\le D_{G,\infty}(\mu_*,\mu_j,\mu_j) \end{aligned}$$

$$\begin{aligned}&+\phi\Bigg(\max\Bigg\{D_{G,\infty}(\mu_{j-1},\mu_*,\mu_*),D_{G,\infty}(\mu_{j-1},\mu_*,\mu_*)+D_{G,\infty}(\mu_*,\mu_j,\mu_j),\\&G_\infty(\mu_*,T_i\mu_*,T_i\mu_*),\frac{1}{2}\big[G_\infty(\mu_{j-1},\mu_*,\mu_*)\\&+G_\infty(\mu_*,T_i\mu_*,T_i\mu_*)+G_\infty(\mu_*,\mu_j,\mu_j)+0\big]\Bigg\}\Bigg).\end{aligned}\tag{27}$$

Letting $j \to +\infty$, we can see from (27) and Proposition 2.5 that

$$G_\infty(\mu_*,T_i\mu_*,T_i\mu_*)\le\phi\big(G_\infty(\mu_*,T_i\mu_*,T_i\mu_*)\big).$$

It implies that $G_\infty(\mu_*,T_i\mu_*,T_i\mu_*)=0$, that is, $\mu_*\subset T_i\mu_*$.

If in Theorem 3.1 we choose $\phi(t)=kt$, where $k\in(0,1)$ is a constant, we obtain the following corollary. □

Corollary 3.1 *Let (X,G) be a G-complete metric space and $\{T_i\}_{i=1}^\infty$ a sequence of fuzzy self-mappings of $\mathcal{C}(X)$. Suppose that for each $\mu_1,\mu_2\in\mathcal{C}(X)$ and for arbitrary positive integers i and j, $i\neq j$,*

$$\begin{aligned}&D_{G,\infty}(T_i\mu_1,T_j\mu_2,T_j\mu_2)\\&\quad\le k\Bigg(\max\Bigg\{D_{G,\infty}(\mu_1,\mu_2,\mu_2),G_\infty(\mu_1,T_i\mu_1,T_i\mu_1),G_\infty(\mu_2,T_j\mu_2,T_j\mu_2),\\&\qquad\frac{1}{2}\big[G_\infty(\mu_1,T_j\mu_2,T_j\mu_2)+G_\infty(\mu_2,T_i\mu_1,T_i\mu_1)\big]\Bigg\}\Bigg),\end{aligned}$$

where $k\in(0,1)$. Then there exists at least one $\mu_*\in\mathcal{C}(X)$ such that $\mu_*\subset T_i\mu_*$ for all $i\in\mathbb{Z}^+$.

The following example illustrates Theorem 3.1.

Example 3.1 Let $X=\{0,1,2,3,\ldots\}$. Define $G: X\times X\times X\to X$ by

$$G(x,y,z)=\begin{cases}x+y+z, & \text{if } x,y,z \text{ are all distinct and different from zero;}\\ x+z, & \text{if } x=y\neq z \text{ and all are different from zero;}\\ y+z+1, & \text{if } x=0, y\neq z \text{ and } y,z \text{ are different from zero;}\\ y+2, & \text{if } x=0, y=z\neq 0;\\ z+1, & \text{if } x=0, y=0, z\neq 0;\\ 0, & \text{if } x=y=z.\end{cases}$$

Then X is a complete nonsymmetric G-metric space [5].

For $\mu,\nu\in\mathcal{C}(X)$, $y\in X$ and $\lambda>0$, owing to Zadeh's extension principle [25], scalar multiplication and addition are defined by

$$(\lambda\mu)(y)=\mu\left(\frac{y}{\lambda}\right)$$

and

$$(\mu+\nu)(x)=\sup_{x_1,x_2:x_1+x_2=x}\min\big\{\mu(x_1),\nu(x_2)\big\}.$$

For any $0 < a < 1$ and $\mu, \nu, \omega \in \mathcal{C}(X)$, we can get easily from the definition of $G(x,y,z)$ that

$$D_{G,\infty}(a\mu, a\nu, a\omega) = aD_{G,\infty}(\mu, \nu, \omega) \tag{28}$$

and

$$D_{G,\infty}(\mu, a\nu, a\nu) \le D_{G,\infty}(\mu, \nu, \nu). \tag{29}$$

Now, suppose $0 < p \le 1$, define $\mu_0 : X \to \mathcal{C}(X)$ by

$$\mu_0(x) = \begin{cases} p, & \text{if } x = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $0 < q < 1$, define $\{T_i\}_{i=1}^{\infty}$ a sequence of fuzzy self-mappings of $\mathcal{C}(X)$ as

$$T_i(\mu) = q^i \mu + \mu_0 \quad \text{for any } \mu \in \mathcal{C}(X).$$

For any $i, j \in Z^+$, without loss of generality, suppose $i < j$. For each $\mu_1, \mu_2 \in \mathcal{C}(X)$, by (28), (29) and the definition of α-level set, we have

$$\begin{aligned} &D_{G,\infty}(T_i\mu_1, T_j\mu_2, T_j\mu_2) \\ &\quad = D_{G,\infty}\bigl(q^i\mu_1 + \mu_0, q^j\mu_2 + \mu_0, q^j\mu_2 + \mu_0\bigr) \\ &\quad \le q^i D_{G,\infty}\bigl(\mu_1, q^{j-i}\mu_2, q^{j-i}\mu_2\bigr) \le q^i D_{G,\infty}(\mu_1, \mu_2, \mu_2) \\ &\quad \le q^i \biggl(\max\biggl\{D_{G,\infty}(\mu_1, \mu_2, \mu_2), G_\infty(\mu_1, T_i\mu_1, T_i\mu_1), G_\infty(\mu_2, T_j\mu_2, T_j\mu_2), \\ &\qquad \frac{1}{2}\bigl[G_\infty(\mu_1, T_j\mu_2, T_j\mu_2) + G_\infty(\mu_2, T_i\mu_1, T_i\mu_1)\bigr]\biggr\}\biggr). \end{aligned}$$

Therefore, $\{T_i\}_{i=1}^{\infty}$ satisfy the conditions of Theorem 3.1 with $\phi(t) = q^i t$. Moreover, for each $0 < b \le p$,

$$\mu_b(x) = \begin{cases} b, & \text{if } x = 0; \\ 0, & \text{otherwise} \end{cases}$$

is a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

4 Conclusion

In this work, by using the new concept of Hausdorff G-metric in the space of fuzzy sets, we establish some common fixed point theorems for a family of fuzzy self-mappings in the space of fuzzy sets on a complete G-metric space. These results are useful in fractal. An iterated function system (*i.e.*, IFS) is the significant content in fractal, and the attractor of the IFS plays a very important role in the fractal graphics. On account of the fuzziness of parameters in fractal, by Zadeh's extension principle [25], we can get an iterated fuzzy function system (*i.e.*, IFFS) corresponding the IFS [24]. For example, $\{T_i\}_{i=1}^n$ in Example 3.1 is an IFFS and $\{\mu_b : 0 < b \leq p\} \subseteq A$, where A is the set of attractors of IFFS. Moreover, we can estimate the area of attractors basing on the fixed points of $\{T_i\}_{i=1}^n$. Our results are also useful in fuzzy differential equation. As we all know, the existence of a solution for a fuzzy differential equation can be established via the fixed point analysis approach (see [26–28]). Therefore, our results provide a new method for studying the fuzzy differential equation in G-metric spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the work. All authors read and approved the final manuscript.

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