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Fixed point theorems for (ξ,α)-expansive mappings in complete metric spaces

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Abstract

In this paper, we introduce a new, simple and unified approach to the theory of expansive mappings. We present a new category of expansive mappings called (ξ,α)-expansive mappings and study various fixed point theorems for such mappings in complete metric spaces. Further, we shall use these theorems to derive coupled fixed point theorems in complete metric spaces. Our new notion complements the concept of α-ψ-contractive type mappings introduced recently by Samet *et al.* (Nonlinear Anal. 2011, doi:10.1016/j.na.2011.10.014). The presented theorems extend, generalize and improve many existing results in the literature. Some comparative examples are constructed which illustrate our results in comparison to some of the existing ones in literature.

Keywords: fixed point; complete metric space; expansive mapping

1 Introduction

The advancement and the rich growth of fixed point theorems in metric spaces have important theoretical and practical applications. The development has been tremendous in the last three decades. Although the concept of a fixed point theorem may appear as an abstract notion in metric spaces, it has remarkable influence on applications such as the theory of differential and integral equations [1], the game theory relevant to the military, sports and medicine as well as economics [2]. Besides, it has applications in physics, engineering, boundary value problems and variational inequalities (see, for instance, Beg [3] and El Naschie [4]). In 1984, Wang *et al.* [5] presented some interesting work on expansion mappings in metric spaces which correspond to some contractive mappings in [6]. Further, Khan *et al.* [7] generalized the result of [5] by using functions. Also, Rhoades [8] and Taniguchi [9] generalized the results of Wang [5] for a pair of mappings. Kang [10] generalized the result of Khan *et al.* [7], Rhoades [8] and Taniguchi [9] for expansion mappings. Recently, Samet *et al.* [11] introduced a new concept of α-ψ-contractive type mappings and established fixed point theorems for such mappings in complete metric spaces.

In this paper, we introduce a new notion of (ξ,α)-expansive mappings and establish various fixed point theorems for such mappings in complete metric spaces. The presented theorems extend, generalize and improve many existing results in the literature. Some examples are considered to illustrate the usability of our obtained results.

2 Preliminaries

We need the following definitions and results, consistent with [5, 11].

Wang *et al.* [5] defined expansion mappings in the form of the following theorem.

Theorem 2.1 [5] *Let (X,d) be a complete metric space. If f is a mapping of X into itself and if there exists a constant $q > 1$ such that*

$$d\big(f(x),f(y)\big) \geq qd(x,y)$$

for each $x,y \in X$ and f is onto, then f has a unique fixed point in X.

Recently, Samet *et al.* [11] considered the following family of functions and presented the new notions of α-ψ-contractive and α-admissible mappings.

Definition 2.1 [11] Let φ denote the family of all functions $\psi : [0,+\infty) \to [0,+\infty)$ which satisfy the following:

(i) $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ;
(ii) ψ is non-decreasing.

Definition 2.2 [11] Let (X,d) be a metric space and $T : X \to X$ be a given self mapping. T is said to be an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0,+\infty)$ and $\psi \in \varphi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\big(d(x,y)\big)$$

for all $x,y \in X$.

Definition 2.3 [11] Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. T is said to be α-admissible if

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Rightarrow \quad \alpha(Tx,Ty) \geq 1.$$

Now, we present some examples of α-admissible mappings.

Example 2.4 Let X be the set of all non-negative real numbers. Let us define the mapping $\alpha : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x \geq y, \\ 0, & \text{if } x < y \end{cases}$$

and define the mapping $T : X \to X$ by $Tx = x^2$ for all $x \in X$. Then T is α-admissible.

Example 2.5 Let X be the set of all non-negative real numbers. Let us define the mapping $\alpha : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} e^{x-y}, & \text{if } x \geq y, \\ 0, & \text{if } x < y \end{cases}$$

and define the mapping $T : X \to X$ by $Tx = e^x$ for all $x \in X$. Then T is α-admissible.

In what follows, we present the main results of Samet *et al.* [11].

Theorem 2.2 [11] *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following condition\s:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 2.3 [11] *Let (X,d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then T has a fixed point.

Samet *et al.* [11] added the following condition (H) to the hypotheses of Theorem 2.2 and Theorem 2.3 to assure the uniqueness of the fixed point:

(H): For all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Further, Samet *et al.* [11] derived coupled fixed point theorems in complete metric spaces using the previously obtained results.

Theorem 2.4 [11] *Let (X,d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \varphi$ and a function $\alpha : X^2 \times X^2 \to [0,+\infty)$ such that*

$$\alpha\big((x,y),(u,v)\big)d\big(F(x,y),F(u,v)\big) \leq \frac{1}{2}\psi\big(d(x,u)+d(y,v)\big),$$

for all $(x,y),(u,v) \in X \times X$. Suppose also that

(i) *for all $(x,y),(u,v) \in X \times X$, we have*

$$\alpha\big((x,y),(u,v)\big) \geq 1 \quad \Rightarrow \quad \alpha\Big(\big(F(x,y),F(y,x)\big),\big(F(u,v),F(v,u)\big)\Big) \geq 1;$$

(ii) *there exists $(x_0,y_0) \in X \times X$ such that*

$$\alpha\Big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\Big) \geq 1 \quad \text{and} \quad \alpha\Big(\big(F(y_0,x_0),F(x_0,y_0)\big),(y_0,x_0)\Big) \geq 1;$$

(iii) *F is continuous.*

Then F has a coupled fixed point, that is, there exists $(x^*,y^*) \in X \times X$ such that $x^* = F(x^*,y^*)$ and $y^* = F(y^*,x^*)$.

Theorem 2.5 [11] *Let (X,d) be a complete metric space and $F : X \times X \to X$ be a given mapping. Suppose that there exists $\psi \in \varphi$ and a function $\alpha : X^2 \times X^2 \to [0,+\infty)$ such that*

$$\alpha\big((x,y),(u,v)\big)d\big(F(x,y),F(u,v)\big) \leq \frac{1}{2}\psi\big(d(x,u)+d(y,v)\big),$$

for all $(x,y),(u,v) \in X \times X$. Suppose also that

(i) *for all* $(x,y),(u,v) \in X \times X$, *we have*

$$\alpha\big((x,y),(u,v)\big) \geq 1 \quad \Rightarrow \quad \alpha\big(\big(F(x,y),F(y,x)\big),\big(F(u,v),F(v,u)\big)\big) \geq 1;$$

(ii) *there exists* $(x_0,y_0) \in X \times X$ *such that*

$$\alpha\big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\big) \geq 1 \quad \textit{and} \quad \alpha\big(\big(F(y_0,x_0),F(x_0,y_0)\big),(y_0,x_0)\big) \geq 1;$$

(iii) *if* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\alpha\big((x_n,y_n),(x_{n+1},y_{n+1})\big) \geq 1 \quad \textit{and} \quad \alpha\big((y_{n+1},x_{n+1}),(y_n,x_n)\big) \geq 1,$$

$x_n \to x \in X$ *and* $y_n \to y \in X$ *as* $n \to +\infty$, *then*

$$\alpha\big((x_n,y_n),(x,y)\big) \geq 1 \quad \textit{and} \quad \alpha\big((y,x),(y_n,x_n)\big) \geq 1 \quad \textit{for all } n \in \mathbb{N}.$$

Then F *has a coupled fixed point.*

Samet *et al.* [11] added the following condition (H′) to the hypotheses of Theorem 2.4 and Theorem 2.5 to assure the uniqueness of the coupled fixed point:

(H′): For all $(x,y),(u,v) \in X \times X$, there exists $(z_1,z_2) \in X \times X$ such that

$$\alpha\big((x,y),(z_1,z_2)\big) \geq 1, \qquad \alpha\big((z_2,z_1),(y,x)\big) \geq 1$$

and

$$\alpha\big((u,v),(z_1,z_2)\big) \geq 1, \qquad \alpha\big((z_2,z_1),(v,u)\big) \geq 1.$$

3 Main results

We shall make use of the standard notations and terminologies of nonlinear analysis throughout this paper. We introduce here a new notion of (ξ,α)-expansive mappings as follows.

Let χ denote all functions $\xi : [0,+\infty) \to [0,+\infty)$ which satisfy the following properties:

(i) ξ is non-decreasing;
(ii) $\sum_{n=1}^{+\infty} \xi^n(a) < +\infty$ for each $a > 0$, where ξ^n is the nth iterate of ξ;
(iii) $\xi(a+b) = \xi(a) + \xi(b)$, $\forall a,b \in [0,+\infty)$.

Lemma 3.1 [11] *If* $\xi : [0,+\infty) \to [0,+\infty)$ *is a non-decreasing function, then for each* $a > 0$, $\lim_{n\to+\infty} \xi^n(a) = 0$ *implies* $\xi(a) < a$.

Definition 3.1 Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an (ξ,α)-expansive mapping if there exist two functions $\xi \in \chi$ and $\alpha : X \times X \to [0,+\infty)$ such that

$$\xi\big(d(Tx,Ty)\big) \geq \alpha(x,y)d(x,y) \tag{1}$$

for all $x,y \in X$.

Remark 3.1 If $T : X \to X$ is an expansion mapping, then T is an (ξ, α)-expansive mapping, where $\alpha(x,y) = 1$ for all $x, y \in X$ and $\xi(a) = ka$ for all $a \geq 0$ and some $k \in [0,1)$.

We now prove our main results.

Theorem 3.2 *Let (X,d) be a complete metric space and $T : X \to X$ be a bijective, (ξ,α)-expansive mapping satisfying the following conditions:*
(i) T^{-1} is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$;
(iii) T is continuous.
Then T has a fixed point, that is, there exists $u \in X$ such that $Tu = u$.

Proof Let us define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n+1}, \quad \text{for all } n \in \mathbb{N},$$

where $x_0 \in X$ is such that $\alpha(x_0, T^{-1}x_0) \geq 1$. Now, if $x_n = x_{n+1}$ for any $n \in \mathbb{N}$, one has that x_n is a fixed point of T from the definition $\{x_n\}$. Without loss of generality, we can suppose $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$.

It is given that $\alpha(x_0,x_1) = \alpha(x_0, T^{-1}x_0) \geq 1$. Recalling that T^{-1} is α-admissible, therefore, we have

$$\alpha\left(T^{-1}x_0, T^{-1}x_1\right) = \alpha(x_1,x_2) \geq 1.$$

Using mathematical induction, we obtain

$$\alpha(x_n,x_{n+1}) \geq 1, \tag{2}$$

for all $n \in \mathbb{N}$. Using (2) and applying the inequality (1) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$d(x_n,x_{n+1}) \leq \alpha(x_n,x_{n+1})d(x_n,x_{n+1}) \leq \xi\left(d(Tx_n, Tx_{n+1})\right) = \xi\left(d(x_{n-1},x_n)\right).$$

Therefore, by repetition of the above inequality, we have that

$$d(x_n,x_{n+1}) \leq \xi^n\left(d(x_0,x_1)\right), \quad \text{for all } n \in \mathbb{N}.$$

For any $n > m \geq 0$, we have

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m+1}) + d(x_{m+1},x_{m+2}) \\ &\quad + d(x_{m+2},x_{m+3}) + \cdots + d(x_{n-1},x_n) \\ &\leq \xi^m\left(d(x_0,x_1)\right) + \cdots + \xi^{n-1}\left(d(x_0,x_1)\right). \end{aligned}$$

From $\sum \xi^n(a) < +\infty$ for each $a > 0$, it follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X,d). So, there exists $u \in X$ such that $x_n \to u$ as $n \to +\infty$. From the continuity of T, it follows that $x_n = Tx_{n+1} \to Tu$ as $n \to +\infty$. By the uniqueness of the limit, we get $u = Tu$, that is, u is a fixed point of T. This completes the proof. □

In what follows, we prove that Theorem 3.2 still holds for T not necessarily continuous, assuming the following condition:

(P): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha\left(T^{-1}x_n, T^{-1}x\right) \geq 1 \tag{3}$$

for all n.

Theorem 3.3 *If in Theorem 3.2 we replace the continuity of T by the condition (P), then the result holds true.*

Proof Following the proof of Theorem 3.2, we know that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to u \in X$ as $n \to +\infty$. Now, from the hypothesis (3), we have

$$\alpha\left(T^{-1}x_n, T^{-1}u\right) \geq 1, \tag{4}$$

for all $n \in \mathbb{N}$. Utilizing the inequalities (1), (4) and the triangular inequality, we obtain

$$\begin{aligned} d\left(T^{-1}u, u\right) &\leq d\left(T^{-1}u, x_{n+1}\right) + d(x_{n+1}, u) \\ &= d\left(T^{-1}x_n, T^{-1}u\right) + d(x_{n+1}, u) \\ &\leq \alpha\left(T^{-1}x_n, T^{-1}u\right)d\left(T^{-1}x_n, T^{-1}u\right) + d(x_{n+1}, u) \\ &\leq \xi\left(d(x_n, u)\right) + d(x_{n+1}, u). \end{aligned}$$

Continuity of ξ at $t = 0$ implies that $d(T^{-1}u, u) = 0$ as $n \to +\infty$. That is, $T^{-1}u = u$. Consider, $Tu = T(T^{-1}u) = (TT^{-1})u = u$. This gives an end to the proof. □

We now present some examples in support of our results.

Example 3.2 Let $X = [0, +\infty)$ endowed with standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define the mappings $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by

$$T(x) = \begin{cases} 2x - \frac{3}{2}, & x \geq 1, \\ \frac{x}{2}, & x \in [0, 1) \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 0, & x, y \in [0, 1), \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, T is an (ξ, α)-expansive mapping with $\xi(a) = a/2$ for all $a \geq 0$. In fact, for all $x, y \in X$, we have

$$\frac{1}{2}d(Tx, Ty) \geq \alpha(x, y)d(x, y).$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$. In fact, for $x_0 = 1$, we have

$$\alpha\bigl(1, T^{-1}1\bigr) = 1.$$

Obviously, T is continuous, and so it remains to show that T^{-1} is α-admissible. For this, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $x \geq 1$ and $y \geq 1$, and by the definitions of T^{-1} and α, we have

$$T^{-1}x = \frac{x}{2} + \frac{3}{4} \geq 1, \qquad T^{-1}y = \frac{y}{2} + \frac{3}{4} \geq 1 \quad \text{and} \quad \alpha\bigl(T^{-1}x, T^{-1}y\bigr) = 1.$$

Then T^{-1} is α-admissible.

Now, all the hypotheses of Theorem 3.2 are satisfied. Consequently, T has a fixed point. In this example, 0 and 3/2 are two fixed points of T.

Remark 3.2 The expansion mapping theorem proved by Wang *et al.* [5] cannot be applied in the above example since we have

$$d\left(T\left(\frac{1}{2}\right), T(0)\right) = \frac{1}{4} < \frac{1}{2} = d(1/2, 0).$$

Now, we give an example involving a function T that is not continuous.

Example 3.3 Let $X = [0, +\infty)$ endowed with the standard metric $d(x,y) = |x - y|$ for all $x, y \in X$. Define the mappings $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$ by

$$Tx = \begin{cases} x^2, & x \geq 1, \\ \frac{x}{2}, & 0 \leq x < 1 \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 0, & x, y \in [0,1), \\ 1, & \text{otherwise}. \end{cases}$$

Due to the discontinuity of T at 1, Theorem 3.2 is not applicable in this case. Clearly, T is an (ξ, α)-expansive mapping with $\xi(a) = a/2$ for all $a \geq 0$. In fact, for all $x, y \in X$, we have

$$\frac{1}{2} d(Tx, Ty) \geq \alpha(x,y) d(x,y).$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$. In fact, for $x_0 = 1$, we have

$$\alpha\bigl(1, T^{-1}1\bigr) = 1.$$

Now, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $x \geq 1$, $y \geq 1$ and by the definition of T^{-1} and α, we have

$$T^{-1}x = \sqrt{x} \geq 1, \qquad T^{-1}y = \sqrt{y} \geq 1 \quad \text{and} \quad \alpha\bigl(T^{-1}x, T^{-1}y\bigr) = 1$$

that is, T^{-1} is α-admissible.

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n, by the definition of α, we have $x_n \geq 1$ for all n and $x \geq 1$. Then $\alpha(T^{-1}x_n, T^{-1}x) = 1$.

Therefore, all the required hypotheses of Theorem 3.3 are satisfied, and so T has a fixed point. Here, 0 and 1 are two fixed points of T.

Remark 3.3 As in the previous example, the expansion mapping theorem is not applicable in this case either.

To ensure the uniqueness of the fixed point in Theorems 3.2 and 3.3, we consider the condition:

(U): For all $u, v \in X$, there exists $w \in X$ such that $\alpha(u, w) \geq 1$ and $\alpha(v, w) \geq 1$.

Theorem 3.4 *Adding the condition (U) to the hypotheses of Theorem 3.2 (resp. Theorem 3.3), we obtain the uniqueness of the fixed point of T.*

Proof From Theorem 3.2 and Theorem 3.3, the set of fixed points is non-empty. We shall show that if u and v are two fixed points of T, that is, $T(u) = u$ and $T(v) = v$, then $u = v$. From the condition (U), there exists $w \in X$ such that

$$\alpha(u, w) \geq 1 \quad \text{and} \quad \alpha(v, w) \geq 1. \tag{5}$$

Recalling the α-admissible property of T^{-1}, we obtain from (5)

$$\alpha\big(u, T^{-1}w\big) \geq 1 \quad \text{and} \quad \alpha\big(v, T^{-1}w\big) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Therefore, by repeatedly applying the α-admissible property of T^{-1}, we get

$$\alpha\big(u, T^{-n}w\big) \geq 1 \quad \text{and} \quad \alpha\big(v, T^{-n}w\big) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Using the inequalities (1) and (7), we obtain

$$\begin{aligned} d\big(u, T^{-n}w\big) &\leq \alpha\big(u, T^{-n}w\big) d\big(u, T^{-n}w\big) \\ &\leq \xi\big(d\big(u, T^{-n+1}w\big)\big). \end{aligned}$$

Repetition of the above inequality implies that

$$d\big(u, T^{-n}w\big) \leq \xi^n\big(d(u, w)\big), \quad \text{for all } n \in \mathbb{N}.$$

Thus, we have $T^{-n}w \to u$ as $n \to +\infty$. Using the similar steps as above, we obtain $T^{-n}w \to v$ as $n \to +\infty$. Now, the uniqueness of the limit of $T^{-n}w$ gives us $u = v$. This completes the proof. □

Now, we shall show that the coupled fixed point theorems in complete metric spaces can also be derived from our results. Before proving the result, we recall the following definition due to Bhaskar and Lakshmikantham [12].

Definition 3.4 [12] Let $F : X \times X \to X$ be a given mapping. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if

$$F(x,y) = x \quad \text{and} \quad F(y,x) = y.$$

We shall require the following lemma due to Samet *et al.* [11] for the proof of our result.

Lemma 3.5 [11] *Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \to X \times X$ by*

$$T(x,y) = \big(F(x,y), F(y,x)\big), \quad \textit{for all } (x,y) \in X \times X. \tag{8}$$

Then (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T.

We, now prove the following results.

Theorem 3.6 *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a given bijective mapping. Suppose that there exists $\xi \in \chi$ and a function $\alpha : X^2 \times X^2 \to [0, +\infty)$ such that*

$$\xi\big(d\big(F(x,y), F(u,v)\big)\big) \geq \frac{1}{2}\alpha\big((x,y),(u,v)\big)\big[d(x,u) + d(y,v)\big] \tag{9}$$

for all $(x, y), (u, v) \in X \times X$. Suppose also that

(i) for all $(x, y), (u, v) \in X \times X$, we have

$$\alpha\big((x,y),(u,v)\big) \geq 1 \quad \Rightarrow \quad \alpha\big(F^{-1}(x), F^{-1}(u)\big) \geq 1;$$

(ii) there exists $(x_0, y_0) \in X \times X$ such that

$$\alpha\big((x_0,y_0),(a,b)\big) \geq 1 \quad \textit{and} \quad \alpha\big((b,a),(y_0,x_0)\big) \geq 1,$$

where $F^{-1}(x_0) = (a, b)$;

(iii) F is continuous.

Then F has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Proof For the proof of our result, we consider the mapping T given by (8) as a bijective mapping such that

$$T^{-1}(x,y) = F^{-1}(x).$$

Also, consider the complete metric space (Y, ρ), where $Y = X \times X$ and $\rho((x, y), (u, v)) = d(x, u) + d(y, v)$ for all $(x, y), (u, v) \in X \times X$. Using the inequality (9), we have

$$\xi\big(d\big(F(x,y), F(u,v)\big)\big) \geq \frac{1}{2}\alpha\big((x,y),(u,v)\big)\big[d(x,u) + d(y,v)\big] \tag{10}$$

and

$$\xi\big(d\big(F(v,u), F(y,x)\big)\big) \geq \frac{1}{2}\alpha\big((v,u),(y,x)\big)\big[d(v,y) + d(u,x)\big]. \tag{11}$$

Define the function $\eta : Y \times Y \to [0, +\infty)$ by

$$\eta\big((\mu_1, \mu_2), (\nu_1, \nu_2)\big) = \min\big\{\alpha\big((\mu_1, \mu_2), (\nu_1, \nu_2)\big), \alpha\big((\nu_2, \nu_1), (\mu_2, \mu_1)\big)\big\} \tag{12}$$

for all $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in Y$.

Summing up the inequalities (10)-(11) and using (12), we get

$$\xi\big(d\big(F(\mu_1, \mu_2), F(\nu_1, \nu_2)\big)\big) + \xi\big(d\big(F(\nu_2, \nu_1), F(\mu_2, \mu_1)\big)\big) \geq \eta(\mu, \nu)\rho(\mu, \nu), \tag{13}$$

for all $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in Y$.

Using the property ($\xi(a + b) = \xi(a) + \xi(b)$) of the function ξ, we obtain

$$\xi\big(\rho(T\mu, T\nu)\big) \geq \eta(\mu, \nu)\rho(\mu, \nu), \tag{14}$$

for all $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in Y$.

Clearly, T is continuous and (ξ, α)-expansive mapping.

Let $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in Y$ such that $\eta(\mu, \nu) \geq 1$. Using the condition (i), we obtain $\eta(T^{-1}\mu, T^{-1}\nu) \geq 1$. Then T^{-1} is η-admissible.

Moreover, from the condition (ii) of the hypothesis of the theorem, we find that there exists $(x_0, y_0) \in Y$ such that

$$\eta\big((x_0, y_0), T^{-1}(x_0, y_0)\big) \geq 1.$$

So, we have transformed the problem to the complete metric space (Y, ρ). Therefore, all the hypotheses of Theorem 3.2 are satisfied, and so we deduce the existence of a fixed point of T as well as T^{-1}. Now, Lemma 3.5 gives us the existence of a coupled fixed point of F. □

In what follows, we prove that Theorem 3.6 remains valid if we replace the continuity condition of F with the following condition:

(P′): if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \quad \text{and} \quad \alpha\big((y_{n+1}, x_{n+1}), (y_n, x_n)\big) \geq 1,$$

$x_n \to x \in X$ and $y_n \to y \in X$ as $n \to +\infty$, then

$$\alpha\big(T^{-1}(x_n, y_n), T^{-1}(x, y)\big) \geq 1 \quad \text{and} \quad \alpha\big(T^{-1}(y, x), T^{-1}(y_n, x_n)\big) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Theorem 3.7 *If we replace the continuity of F in Theorem 3.6 by the condition (P′), then the result holds true.*

Proof We employ the same notations of the proof of Theorem 3.6. Let $\{x_n, y_n\}$ be a sequence in Y such that $\eta((x_n, y_n), (x_{n+1}, y_{n+1})) \geq 1$ and $(x_n, y_n) \to (x, y)$ as $n \to +\infty$. Using the condition (P′), we have

$$\eta\big(T^{-1}(x_n, y_n), T^{-1}(x, y)\big) \geq 1.$$

Then all the hypotheses of Theorem 3.3 are satisfied. Thus, we deduce the existence of a fixed point of T that gives us from Lemma 3.5 the existence of a coupled fixed point of F. □

To ensure the uniqueness of the coupled fixed point, we consider the following condition:

(U′): For all $(x_1, y_1), (x_2, y_2) \in X \times X$, there exists $(x_3, y_3) \in X \times X$ such that

$$\alpha\big((x_1, y_1), (x_3, y_3)\big) \geq 1, \qquad \alpha\big((y_3, x_3), (y_1, x_1)\big) \geq 1$$

and

$$\alpha\big((x_2, y_2), (x_3, y_3)\big) \geq 1, \qquad \alpha\big((y_3, x_3), (y_2, x_2)\big) \geq 1.$$

Theorem 3.8 *Adding the condition (U′) to the hypotheses of Theorem 3.6 (resp. Theorem 3.7) we obtain the uniqueness of the coupled fixed point of F.*

Proof Clearly, under the hypothesis (U′), T and η satisfy the hypothesis (U). Therefore, from Theorem 3.4 and Lemma 3.5, the result follows immediately. □

Example 3.5 Let $X = \mathbb{N}$ equipped with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{N}$. Then (X, d) is a complete metric space. Define the mapping $F : X \times X \to X$ by

$$F(x, y) = 2^{x-1} \cdot (2y - 1).$$

Clearly, F is a continuous and bijective mapping. Define $\alpha : X^2 \times X^2 \to [0, +\infty)$ by

$$\alpha\big((x, y), (u, v)\big) = \begin{cases} 1, & \text{if } x \geq y, u \geq v, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that for all $(x, y), (u, v) \in X \times X$, we have

$$\xi\big(d\big(F(x, y), F(u, v)\big)\big) \geq \frac{1}{2}\alpha\big((x, y), (u, v)\big)\big[d(x, u) + d(y, v)\big].$$

Then (9) is satisfied with $\xi(a) = a/2$ for all $a \geq 0$. On the other hand, the condition (i) of Theorem 3.6 holds and the condition (ii) of the same theorem is also satisfied with $(x_0, y_0) = (1, 1)$. All the required hypotheses of Theorem 3.6 are true and so we deduce the existence of a coupled fixed point of F. Here, $(1, 1)$ is a coupled fixed point of F.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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