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Total quasi-ϕ-asymptotically nonexpansive semigroups and strong convergence theorems in Banach spaces

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Abstract
The purpose of this article is to modify the Halpern-Mann-type iteration algorithm for total quasi-ϕ-asymptotically nonexpansive semigroups to have the strong convergence under a limit condition only in the framework of Banach spaces. The results presented in the paper improve and extend the corresponding recent results announced by many authors.

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1 Introduction

Throughout this paper, we assume that E is a real Banach space with the dual E^*, C is a nonempty closed convex subset of E, and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\right\}, \quad x \in E.$$

Let $T : C \to E$ be a nonlinear mapping; we denote by $F(T)$ the set of fixed points of T.

Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

$T : C \to C$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\|, \quad \forall x \in C, p \in F(T).$$

$T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C, n \ge 1.$$

$T : C \to C$ is said to be *quasi-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$\|T^n x - p\| \le k_n \|x - p\|, \quad \forall x \in C, p \in F(T), n \ge 1.$$

A one-parameter family $\mathcal{T} := \{T(t) : t \ge 0\}$ of mappings from C into C is said to be *a nonexpansive semigroup* if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$, $\forall s, t \ge 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\|T(t)x - T(t)y\| \le \|x - y\|$, $\forall x, y \in C$.

We use $F(\mathcal{T})$ to denote a common fixed point set of the nonexpansive semigroup $\mathcal{T}$, *i.e.*, $F(\mathcal{T}) := \bigcap_{t \ge 0} F(T(t))$.

A one-parameter family $\mathcal{T} := \{T(t) : t \ge 0\}$ of mappings from C into C is said to be a *quasi-nonexpansive semigroup* if $F(\mathcal{T}) \neq \emptyset$ and the above conditions (i)-(iii) and the following condition (v) are satisfied:

(v) $\|T(t)x - p\| \le \|x - p\|$, $\forall x \in C$, $p \in F(\mathcal{T})$, $t \ge 0$.

A one-parameter family $\mathcal{T} := \{T(t) : t \ge 0\}$ of mappings from C into C is said to be an *asymptotically nonexpansive semigroup* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that the above conditions (i)-(iii) and the following condition (vi) are satisfied:

(vi) $\|T^n(t)x - T^n(t)y\| \le k_n \|x - y\|$, $\forall x, y \in C$, $n \ge 1$, $t \ge 0$.

A one-parameter family $\mathcal{T} := \{T(t) : t \ge 0\}$ of mappings from C into C is said to be a *quasi-asymptotically nonexpansive semigroup* if $F(\mathcal{T}) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that the above conditions (i)-(iii) and the following condition (vii) are satisfied:

(vii) $\|T^n(t)x - p\| \le k_n \|x - p\|$, $\forall x \in C$, $p \in F(\mathcal{T})$, $t \ge 0$, $n \ge 1$.

As is well known, the construction of fixed points of nonexpansive mappings (asymptotically nonexpansive mappings) and of common fixed points of nonexpansive semi-groups (asymptotically nonexpansive semigroups) is an important problem in the theory of nonexpansive mappings and its applications; in particular, in image recovery, convex feasibility problem, and signal processing problem (see, for example, [1–3]).

Iterative approximation of a fixed point for nonexpansive mappings, asymptotically nonexpansive mappings, nonexpansive semigroups, and asymptotically nonexpansive semigroups in Hilbert or Banach spaces has been studied extensively by many authors (see, for example, [4–31] and the references therein).

The purpose of this article is to introduce the concept of *total quasi-ϕ-asymptotically nonexpansive semigroups*; to modify the Halpern and Mann-type iteration algorithm [13, 14] for total quasi-ϕ-asymptotically nonexpansive semigroups; and to have the strong convergence under a limit condition only in the framework of Banach spaces. The results presented in the paper improve and extend the corresponding results of Kim [32], Suzuki [4], Xu [5], Chang *et al.* [6–8, 22, 23, 30, 33], Cho *et al.* [10], Thong [11], Buong [12], Mann [13], Halpern [14], Qin *et al.* [15], Nakajo *et al.* [18] and others.

2 Preliminaries

In the sequel, we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. In what follows, we always use $\phi : E \times E \to \mathcal{R}^+$

to denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.1}$$

It is obvious from the definition of ϕ that

$$\big(\|x\| - \|y\|\big)^2 \le \phi(x,y) \le \big(\|x\| + \|y\|\big)^2, \quad \forall x, y \in E; \tag{2.2}$$

$$\phi\big(x, J^{-1}\big(\lambda Jy + (1-\lambda)Jz\big)\big) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \quad \forall x, y \in E \tag{2.3}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.4}$$

Following Alber [34], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg\inf_{y\in C} \phi(y,x), \quad \forall x \in E.$$

Lemma 2.1 ([34]) *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold*:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y)$ *for all $x \in C$ and $y \in E$*;

(b) *If $x \in E$ and $z \in C$, then* $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \ge 0, \forall y \in C$;

(c) *For $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.*

Remark 2.2 If E is a real Hilbert space H, then $\phi(x,y) = \|x-y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Definition 2.3 A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, $Tx = y$.

Definition 2.4 (1) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive,* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, p \in F(T).$$

(2) A mapping $T : C \to C$ is said to be $(\{k_n\})$*-quasi-ϕ-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ such that

$$\phi\big(p, T^n x\big) \le k_n\phi(p,x), \quad \forall n \ge 1, x \in C, p \in F(T).$$

(3) A mapping $T : C \to C$ is said to be $(\{\nu_n\}, \{\mu_n\}, \zeta)$*-total quasi-$\phi$-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ such that

$$\phi\big(p, T^n x\big) \le \phi(p,x) + \nu_n\zeta\big(\phi(p,x)\big) + \mu_n, \quad \forall n \ge 1, x \in C, p \in F(T). \tag{2.5}$$

Remark 2.5 ([22]) From the definitions, it is obvious that a quasi-ϕ-nonexpansive mapping is a $(\{k_n = 1\})$-quasi-ϕ-asymptotically nonexpansive mapping and a $(\{k_n\})$-quasi-

ϕ-asymptotically nonexpansive mapping is a $(\{\nu_n\},\{\mu_n\},\zeta)$-total quasi-$\phi$-asymptotically nonexpansive mapping with $\nu_n = k_n - 1$, $\mu_n = 0$, $\zeta(t) = t$, $\forall t \geq 0$. However, the converse is not true.

Example 2.6 ([23]) Let E be a uniformly smooth and strictly convex Banach space and $A : E \to E^*$ be a maximal monotone mapping such that $A^{-1}0 \neq \emptyset$, then $J_r = (J + rA)^{-1}J$ is closed and quasi-ϕ-nonexpansive from E onto $D(A)$.

Example 2.7 ([30]) (1) Let C be a unit ball in a real Hilbert space l^2 and let $T : C \to C$ be a mapping defined by

$$T : (x_1, x_2, \ldots) \to \big(0, x_1^2, a_2x_2, a_3x_3, \ldots\big), (x_1, x_2, \ldots) \in l^2, \tag{2.6}$$

where $\{a_i\}$ is a sequence in $(0,1)$ such that $\Pi_{i=2}^{\infty} a_i = \frac{1}{2}$. It is proved in [30] that T is (single-valued) total quasi-ϕ-asymptotically nonexpansive.

(2) Let $I = [0,1]$, $X = C(I)$ (*the Banach space of continuous functions defined on I with the uniform convergence norm* $\|f\|_C = \sup_{t\in I} |f(t)|$), $D = \{f \in X : f(x) \geq 0, \forall x \in I\}$ and a, b be two constants in $(0,1)$ with $a < b$. Let $T : D \to 2^D$ be a multi-valued mapping defined by

$$T(f) = \begin{cases} \{g \in D : a \leq f(x) - g(x) \leq b, \forall x \in I\}, & \text{if } f(x) > 1, \forall x \in I; \\ \{0\}, & \text{otherwise.} \end{cases} \tag{2.7}$$

It is proved that $T : C \to 2^C$ is a multi-valued total quasi-ϕ-asymptotically nonexpansive mapping.

Example 2.8 Let Π_C be the generalized projection from a smooth, reflexive, and strictly convex Banach space E onto a nonempty closed convex subset C of E, then Π_C is a closed and quasi-ϕ-nonexpansive from E onto C.

Lemma 2.9 *Let E be a smooth, reflexive, and strictly convex real Banach space such that both E and E^* have the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and $(\{\nu_n\},\{\mu_n\},\zeta)$-total quasi-$\phi$-asymptotically nonexpansive mapping, then $F(T)$ is a closed convex subset of C.*

Proof Let $\{x_n\}$ be any sequence in $F(T)$ such that $x_n \to x^*$. Now, we prove that $x^* \in F(T)$. In fact, since $Tx_n = x_n \to x^*$ and T is closed, we have $x^* = Tx^*$, *i.e.*, $F(T)$ is a closed subset in C.

Next, we prove that $F(T)$ is convex. In fact, let $x, y \in F(T)$ and $p = tx + (1 - t)y$, where $t \in (0,1)$. Now, we prove that $p = Tp$. Indeed, since T is $(\{\nu_n\},\{\mu_n\},\zeta)$-total quasi-$\phi$-asymptotically nonexpansive, for any $n \geq 1$, we have

$$\phi\big(x, T^n p\big) \leq \phi(x,p) + \nu_n\zeta\big(\phi(x,p)\big) + \mu_n \tag{2.8}$$

and

$$\phi\big(y, T^n p\big) \leq \phi(y,p) + \nu_n\zeta\big(\phi(y,p)\big) + \mu_n. \tag{2.9}$$

On the other hand, it follows from (2.4) that

$$\phi(x, T^n p) = \phi(x,p) + \phi(p, T^n p) + 2\langle x - p, Jp - JT^n p\rangle \tag{2.10}$$

and

$$\phi(y, T^n p) = \phi(y,p) + \phi(p, T^n p) + 2\langle y - p, Jp - JT^n p\rangle. \tag{2.11}$$

It follows from (2.8)-(2.11) that

$$\begin{aligned}\phi(p, T^n p) &= 2\langle p - x, Jp - JT^n p\rangle + \phi(x, T^n p) - \phi(x,p)\\ &\le 2\langle p - x, Jp - JT^n p\rangle + \nu_n \zeta(\phi(x,p)) + \mu_n\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\phi(p, T^n p) &= 2\langle p - y, Jp - JT^n p\rangle + \phi(y, T^n p) - \phi(y,p)\\ &\le 2\langle p - y, Jp - JT^n p\rangle + \nu_n \zeta(\phi(y,p)) + \mu_n.\end{aligned} \tag{2.13}$$

Multiplying t and $(1-t)$ on both sides of (2.12) and (2.13), respectively and then adding up these two inequalities, we have that

$$\phi(p, T^n p) \le t\nu_n \zeta(\phi(x,p)) + (1-t)\nu_n \zeta(\phi(y,p)) + \mu_n.$$

Letting $n \to \infty$, we have that $\phi(p, T^n p) \to 0$. Hence, it follows from (2.2) that

$$\|T^n p\| \to \|p\|, \tag{2.14}$$

and so

$$\|J(T^n p)\| \to \|Jp\|. \tag{2.15}$$

E is reflexive and so is E^*. Without loss of generality, we can assume that $J(T^n p) \rightharpoonup x^*$ (some point in E^*). In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $x \in E$ such that $Jx = x^*$. Hence, we have

$$\begin{aligned}\phi(p, T^n p) &= \|p\|^2 - 2\langle p, J(T^n p)\rangle + \|T^n p\|^2\\ &= \|p\|^2 - 2\langle p, J(T^n p)\rangle + \|J(T^n p)\|^2.\end{aligned}$$

Taking $\lim_{n\to\infty}$ on both sides of the equality above, we obtain that

$$\begin{aligned}0 &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jp\|^2\\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|p\|^2\\ &= 2\{\|p\|^2 - \langle p, Jx\rangle\}\\ &= 2\langle p, Jp - Jx\rangle.\end{aligned}$$

This implies that $Jp - Jx = 0$. Therefore, we have $J(T^n p) \rightharpoonup Jp$. Since E^* has the Kadec-Klee property, this together with (2.15) shows that $J(T^n p) \to Jp$. Since E is reflexive and strictly convex, J^{-1} is norm-weak-continuous, $T^n p \rightharpoonup p$. Again, since E has the Kadec-Klee property, this together with (2.14) shows that $T^n p \to p$ (as $n \to \infty$). Therefore, $TT^n p = T^{n+1} p \to p$. By virtue of the closeness of T, it follows that $p = Tp$, *i.e.*, $p \in F(T)$. The convexity of $F(T)$ is proved.

This completes the proof of Lemma 2.9. □

Definition 2.10 (I) Let E be a real Banach space, C be a nonempty closed convex subset of E. $\mathcal{T} := \{T(t) : t \geq 0\}$ be a one-parameter family of mappings from C into C. $\mathcal{T}$ is said to be

(1) a *quasi-ϕ-nonexpansive semigroup* if $\mathcal{F} = \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$ and the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\phi(p, T(t)x) \leq \phi(p, x)$, $\forall t \geq 0$, $p \in \mathcal{F}$, $x \in C$.

(2) $\mathcal{T}$ is said to be a ($\{k_n\}$)-*quasi-ϕ-asymptotically nonexpansive semigroup* if the set $\mathcal{F} = \bigcap_{t\geq 0} F(T(t))$ is nonempty, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that the conditions (i)-(iii) and the following condition (v) are satisfied:

(v) $\phi(p, T^n(t)x) \leq k_n \phi(p, x)$, $\forall t \geq 0$, $p \in \mathcal{F}$, $n \geq 1$, $x \in C$.

(3) $\mathcal{T}$ is said to be a ($\{\nu_n\}, \{\mu_n\}, \zeta$)-*total quasi-$\phi$-asymptotically nonexpansive semigroup* if the set $\mathcal{F} = \bigcap_{t\geq 0} F(T(t))$ is nonempty, and there exists nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that the conditions (i)-(iii) and the following condition (vi) are satisfied:

(vi) $\phi(p, T^n(t)x) \leq \phi(p, x) + \nu_n \zeta(\phi(p, x)) + \mu_n$, $\forall n \geq 1$, $x \in C$, $p \in F(T)$.

(II) A total quasi-ϕ-asymptotically nonexpansive semigroup $\mathcal{T}$ is said to be *uniformly Lipschitzian* if there exists a bounded measurable function $L : [0, \infty) \to (0, \infty)$ such that

$$\left\| T^n(t)x - T^n(t)y \right\| \leq L(t)\|x - y\|, \quad \forall x, y \in C, \forall n \geq 1, t \geq 0.$$

3 Main results

Theorem 3.1 *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property, and let C be a nonempty closed convex subset of E. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ be a closed, uniformly L-Lipschitz and ($\{\nu_n\}, \{\mu_n\}, \zeta$)-total quasi-$\phi$-asymptotically nonexpansive semigroup. Let $\{\alpha_n\}$ be a sequence in* $[0, 1]$ *and $\{\beta_n\}$ be a sequence in* $(0, 1)$ *satisfying the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}; \qquad C_1 = C, \\ y_{n,t} = J^{-1}[\alpha_n Jx_1 + (1 - \alpha_n)(\beta_n Jx_n + (1 - \beta_n)JT^n(t)x_n)], \quad t \geq 0, \\ C_{n+1} = \{z \in C_n : \sup_{t\geq 0} \phi(z, y_{n,t}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\mathcal{F} := \bigcap_{t\geq 0}^{\infty} F(T(t))$, $\xi_n = \nu_n \sup_{p\in\mathcal{F}} \zeta(\phi(p,x_n)) + \mu_n$, $\Pi_{C_{n+1}}$ *is the generalized projection of E onto C_{n+1}. If $\mathcal{F}$ is bounded in C, then $\{x_n\}$ converges strongly to* $\Pi_{\mathcal{F}} x_1$.

Proof (I) First, we prove that $\mathcal{F}$ and C_n, $n \geq 1$ all are closed and convex subsets in C.

In fact, it follows from Lemma 2.9 that $F(T(t))$, $t \geq 0$ is a closed and convex subset of C. Therefore, $\mathcal{F}$ is closed and convex in C.

Again, by the assumption that $C_1 = C$ is closed and convex, suppose that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ, we have that

$$\begin{aligned} C_{n+1} &= \Big\{z \in C_n : \sup_{t\geq 0} \phi(z,y_{n,t}) \leq \alpha_n \phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\Big\} \\ &= \bigcap_{t\geq 0} \big\{z \in C : \phi(z,y_{n,t}) \leq \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\big\} \cap C_n \\ &= \bigcap_{t\geq 0} \big\{z \in C : 2\alpha_n\langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_{n,t}\rangle \\ &\quad \leq \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{n,t}\|^2 + \xi_n\big\} \cap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex. The conclusion is proved.

(II) Now, we prove that $\mathcal{F} \subset C_n$, $\forall n \geq 1$.

In fact, it is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 2$. Letting

$$w_{n,t} = J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n\big), \quad t \geq 0,$$

it follows from (2.3) that for any $u \in \mathcal{F} \subset C_n$, we have

$$\begin{aligned} \phi(u,y_{n,t}) &= \phi\big(u, J^{-1}\big(\alpha_n Jx_1 + (1-\alpha_n)Jw_{n,t}\big)\big) \\ &\leq \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,w_{n,t}), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \phi(u,w_{n,t}) &= \phi\big(u, J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n\big)\big) \\ &\leq \beta_n\phi(u,x_n) + (1-\beta_n)\phi\big(u, T^n(t)x_n\big) \\ &\leq \beta_n\phi(u,x_n) + (1-\beta_n)\big[\phi(u,x_n) + \nu_n\zeta\big(\phi(u,x_n)\big) + \mu_n\big] \\ &\leq \phi(u,x_n) + \nu_n\zeta\big(\phi(u,x_n)\big) + \mu_n. \end{aligned} \tag{3.3}$$

Therefore, we have

$$\begin{aligned} \sup_{t\geq 0}\phi(u,y_{n,t}) &\leq \alpha_n\phi(u,x_1) + (1-\alpha_n)\big\{\phi(u,x_n) + \nu_n\zeta\big(\phi(u,x_n)\big) + \mu_n\big\} \\ &\leq \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,x_n) + \nu_n\sup_{p\in\mathcal{F}}\zeta\big(\phi(p,x_n)\big) + \mu_n \\ &= \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,x_n) + \xi_n, \end{aligned}$$

where $\xi_n = \nu_n\sup_{p\in\mathcal{F}}\zeta(\phi(p,x_n)) + \mu_n$. This shows that $u \in C_{n+1}$, and so $\mathcal{F} \subset C_{n+1}$. The conclusion is proved.

(III) Next, we prove that $\{x_n\}$ is bounded and $\{\phi(x_n, x_1)\}$ is convergent.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 2.1(b), we have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in C_n.$$

Again, since $\mathcal{F} \subset C_n$, $\forall n \geq 1$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in \mathcal{F}.$$

It follows from Lemma 2.1(a) that for each $u \in \mathcal{F}$ and for each $n \geq 1$,

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \tag{3.4}$$

Therefore, $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (2.2), $\{x_n\}$ is also bounded.

Again, since $x_n = \Pi_{C_n} x_1$, $x_{n+1} = \Pi_{C_{n+1}} x_1$, and $x_{n+1} \in C_{n+1} \subset C_n$, $\forall n \geq 1$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1.$$

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing and bounded. Hence, $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. The conclusions are proved.

(IV) Next, we prove that $x_n \to p^*$ (some point in C).

In fact, since $\{x_n\}$ is bounded and the space E is reflexive, we may assume that there exists a subsequence of $\{x_{n_i}\}$ such that $x_{n_i} \rightharpoonup p^*$. Since C_n, $\forall n \geq 1$ is closed and convex, we see that $p^* \in C_n$, $\forall n \geq 1$. This implies that $\phi(x_{n_i}, x_1) \leq \phi(p^*, x_1)$, $\forall n_i$. On the other hand, it follows from the weakly lower semicontinuity of the norm that

$$\begin{aligned}
\phi(p^*, x_1) &= \|p^*\|^2 - 2\langle p^*, Jx_1\rangle + \|x_1\|^2 \\
&\leq \liminf_{n_i\to\infty} \{\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_1\rangle + \|x_1\|^2\} \\
&= \liminf_{n_i\to\infty} \phi(x_{n_i}, x_1) \\
&\leq \limsup_{n_i\to\infty} \phi(x_{n_i}, x_1) \\
&\leq \phi(p^*, x_1),
\end{aligned}$$

which implies that $\phi(x_{n_i}, x_1) \to \phi(p^*, x_1)$ (as $n_i \to \infty$). Hence, $\|x_{n_i}\| \to \|p^*\|$ (as $n_i \to \infty$). In view of the Kadec-Klee property of E, we see that $x_{n_i} \to p^*$ (as $n_i \to \infty$).

If there exists another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q^* \in C$, we have

$$\begin{aligned}
\phi(p^*, q^*) &= \lim_{n_i,n_j\to\infty} \phi(x_{n_i}, x_{n_j}) \\
&= \lim_{n_i,n_j\to\infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_1) \\
&\leq \lim_{n_i,n_j\to\infty} \phi(x_{n_i}, x_1) - \phi(\Pi_{C_{n_j}} x_1, x_1) \\
&= \lim_{n_i,n_j\to\infty} \phi(x_{n_i}, x_1) - \phi(x_{n_j}, x_1) = 0,
\end{aligned}$$

i.e., $p^* = q^*$. This shows that $x_n \to p^*$. Therefore, we have

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \left\{ \nu_n \sup_{p\in\mathcal{F}} \zeta\left(\phi(p, x_n)\right) + \mu_n \right\} = 0. \tag{3.5}$$

(V) Now, we prove that $p^* \in \mathcal{F}$.

In fact, since $x_{n+1} \in C_{n+1}$, $x_n \to p^*$ and $\alpha_n \to 0$, it follows from (3.1) and (3.5) that

$$\sup_{t\geq 0} \phi(x_{n+1}, y_{n,t}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \xi_n \to 0 \quad (\text{as } n \to \infty). \tag{3.6}$$

This implies that for each $t \geq 0$,

$$\lim_{n\to\infty} \left(\|y_{n,t}\| - \|x_{n+1}\|\right)^2 = 0. \tag{3.7}$$

Therefore,

$$\|y_{n,t}\| \to \left\|p^*\right\|, \quad \text{uniformly in } t \geq 0, \tag{3.8}$$

and so

$$\left\|J(y_{n,t})\right\| \to \left\|Jp^*\right\|, \quad \text{uniformly in } t \geq 0. \tag{3.9}$$

This shows that $\{J(y_{n,t})\}$ is uniformly bounded. E is reflexive and so is E^*. Without loss of generality, we can assume that $J(y_{n,t}) \rightharpoonup y^*$ (some point in E^*). Since E is reflexive, $J(E) = E^*$. Hence, there exists $y \in E$ such that $Jy = y^*$. This implies that $J(y_{n,t}) \rightharpoonup Jy$. Since

$$\begin{aligned} \phi(x_{n+1}, y_{n,t}) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{n,t}\rangle + \|y_{n,t}\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_{n,t}\rangle + \|Jy_{n,t}\|^2. \end{aligned}$$

Letting $n \to \infty$, from (3.6), we have

$$\begin{aligned} 0 &= \left\|p^*\right\|^2 - 2\left\langle p^*, Jy\right\rangle + \left\|Jp^*\right\|^2 \\ &= \left\|p^*\right\|^2 - 2\left\langle p^*, Jy\right\rangle + \left\|p^*\right\|^2 \\ &= 2\left\langle p^*, Jp^* - Jy\right\rangle, \end{aligned}$$

which shows that $Jp^* = Jy$, and so

$$J(y_{n,t}) \rightharpoonup Jp^*. \tag{3.10}$$

This together with (3.9) and the Kadec-Klee property of E^* shows that $J(y_{n,t}) \to Jp^*$. Since J^{-1} is norm-weak-continuous, we have

$$y_{n,t} \rightharpoonup p^*. \tag{3.11}$$

It follows from (3.8), (3.11) and the Kadec-Klee property of E, we have

$$y_{n,t} \to p^*, \quad \text{uniformly in } t \geq 0. \tag{3.12}$$

On the other hand, since $\{x_n\}$ is bounded and $\mathcal{T} := \{T(t) : t \geq 0\}$ is a $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total quasi-$\phi$-asymptotically nonexpansive semigroup, for any given $p \in \mathcal{F}$, we have

$$\phi\big(p, T^n(t)x_n\big) \leq \phi(p, x_n) + \nu_n \zeta\big(\phi(p, x_n)\big) + \mu_n, \quad \forall t \geq 0, n \geq 1.$$

This implies that $\{T^n(t)x_n\}_{t\geq 0}$ is uniformly bounded. Again, since

$$\begin{aligned}\|w_{n,t}\| &= \big\|J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n\big)\big\| \\ &\leq \beta_n \|x_n\| + (1-\beta_n)\big\|T^n(t)x_n\big\| \\ &\leq \max\big\{\|x_n\|, \big\|T^n(t)x_n\big\|\big\}, \quad t \geq 0,\end{aligned}$$

it implies that $\{w_{n,t}\}_{t\geq 0}$ is also uniformly bounded.

Since $\alpha_n \to 0$, from (3.1), we have

$$\lim_{n\to\infty} \|Jy_{n,t} - Jw_{n,t}\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - Jw_{n,t}\| = 0, \quad \text{for } t \geq 0. \tag{3.13}$$

It follows from (3.12) that $Jw_{n,t} \to Jp^*$ (as $n \to \infty$), uniformly in $t \geq 0$. Therefore, we have

$$w_{n,t} \rightharpoonup p^*. \tag{3.14}$$

Since

$$\begin{aligned}\lim_{n\to\infty} \big|\|w_{n,t}\| - \|p^*\|\big| &= \lim_{n\to\infty} \big|\|J(w_{n,t})\| - \|J(p^*)\|\big| \\ &\leq \lim_{n\to\infty} \big\|J(w_{n,t}) - J(p^*)\big\| = 0.\end{aligned}$$

This together with (3.14) shows that

$$w_{n,t} \to p^* \quad (\text{as } n \to \infty), \ \text{uniformly in } t \geq 0. \tag{3.15}$$

Since $x_n \to p^*$, we have $Jx_n \to Jp^*$, and so for each $t \geq 0$,

$$\begin{aligned}0 &= \lim_{n\to\infty} \big\|Jw_{n,t} - Jp^*\big\| = \lim_{n\to\infty} \big\|\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n - Jp^*\big\| \\ &= \lim_{n\to\infty} \big\|\beta_n\big(Jx_n - Jp^*\big) + (1-\beta_n)\big(JT^n(t)x_n - Jp^*\big)\big\| \\ &= \lim_{n\to\infty} (1-\beta_n)\big\|JT^n(t)x_n - Jp^*\big\|.\end{aligned}$$

By condition (ii), we have that

$$\lim_{n\to\infty} \big\|\big(JT^n(t)x_n - Jp^*\big)\big\| = 0, \quad \text{uniformly in } t \geq 0. \tag{3.16}$$

Since J^{-1} is norm-weakly-continuous, this implies that

$$T^n(t)x_n \rightharpoonup p^*, \quad \text{for each } t \geq 0. \tag{3.17}$$

It follows from (3.16) that for each $t \geq 0$,

$$\begin{aligned}\lim_{n\to\infty} \big| \|T^n(t)x_n\| - \|p^*\| \big| &= \lim_{n\to\infty} \big| \|J(T^n(t)x_n)\| - \|J(p^*)\| \big| \\ &\leq \|J(T^n(t)x_n) - J(p^*)\| = 0.\end{aligned}$$

This together with (3.17) and the Kadec-Klee property of E shows that

$$T^n(t)x_n \to p^* \quad (\text{as } n \to \infty) \text{ uniformly in } t \geq 0.$$

Again, by the assumptions that the semigroup $\mathcal{T} := \{T(t) : t \geq 0\}$ is closed and uniformly L-Lipschitzian, we have

$$\begin{aligned}\|T^{n+1}(t)x_n - T^n(t)x_n\| &\leq \|T^{n+1}(t)x_n - T^{n+1}(t)x_{n+1}\| + \|T^{n+1}(t)x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T^n(t)x_n\| \\ &\leq (L(t) + 1)\|x_{n+1} - x_n\| + \|T^{n+1}(t)x_{n+1} - x_{n+1}\| \\ &\quad + \|x_n - T^n(t)x_n\|. \end{aligned} \tag{3.18}$$

Since $\lim_{n\to\infty} T^n(t)x_n = p^*$ uniformly in $t \geq 0$, $x_n \to p^*$ and $L(t) : [0,\infty) \to [0,\infty)$ is a bounded and measurable function, these together with (3.9) imply that

$$\lim_{n\to\infty} \|T^{n+1}(t)x_n - T^n(t)x_n\| = 0, \quad \text{uniformly in } t \geq 0,$$

and so

$$\lim_{n\to\infty} T^{n+1}(t)x_n = p^*, \quad \text{uniformly in } t \geq 0,$$

i.e.,

$$\lim_{n\to\infty} T(t)T^n(t)x_n = p^*, \quad \text{uniformly in } t \geq 0.$$

In view of the closeness of the semigroup $\mathcal{T}$, it yields that $T(t)p^* = p^*$, *i.e.*, $p^* \in F(T(t))$. By the arbitrariness of $t \geq 0$, we have $p^* \in \mathcal{F} := \bigcap_{t\geq 0} F(T(t))$.

(VI) Finally, we prove that $x_n \to p^* = \Pi_{\mathcal{F}} x_1$.

Let $w = \Pi_{\mathcal{F}} x_1$. Since $w \in \mathcal{F} \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$, $\forall n \geq 1$. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w, x_1). \tag{3.19}$$

In view of the definition of $\Pi_{\mathcal{F}} x_1$, from (3.10) we have $p^* = w$. Therefore, $x_n \to p^* = \Pi_{\mathcal{F}} x_1$. This completes the proof of Theorem 3.1. □

From Theorem 3.1, we can obtain the following.

Theorem 3.2 *Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 3.1. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ be a closed, uniformly L-Lipschitz and $(\{k_n\})$-quasi-ϕ-asymptotically nonexpansive semigroup with $\{k_n\} \subset [1,\infty)$, $k_n \to 1$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}; \qquad C_1 = C, \\ y_{n,t} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n)], \quad t \geq 0, \\ C_{n+1} = \{z \in C_n : \sup_{t\geq 0} \phi(z, y_{n,t}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.20}$$

where $\mathcal{F} := \bigcap_{t\geq 0}^{\infty} F(T(t))$, $\xi_n = (k_n - 1)\sup_{p\in\mathcal{F}} \phi(p, x_n)$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. If $\mathcal{F}$ is bounded in C, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof It follows from Definition 2.10 that if $\mathcal{T} := \{T(t) : t \geq 0\}$ is a closed, uniformly L-Lipschitz and $(\{k_n\})$-quasi-ϕ-asymptotically nonexpansive semigroup, then it must be a closed, uniformly L-Lipschitz $(\{\nu_n\}, \{\mu_n\}, \zeta)$-total quasi-$\phi$-asymptotically nonexpansive semigroup with $\nu_n = k_n - 1$, $\mu_n = 0$, $\forall n \geq 1$ and $\zeta(t) = t$, $t \geq 0$. Therefore, all the conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately. □

Theorem 3.3 *Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 3.1. Let $\mathcal{T} := \{T(t) : t \geq 0\}$ be a closed, quasi-ϕ-nonexpansive semigroup such that the set $\mathcal{F} := \bigcap_{t\geq 0} F(T(t))$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}; \qquad C_1 = C, \\ y_{n,t} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT(t)x_n)], \quad t \geq 0, \\ C_{n+1} = \{z \in C_n : \sup_{t\geq 0} \phi(z, y_{n,t}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{cases} \tag{3.21}$$

Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof Since $\mathcal{T} := \{T(t) : t \geq 0\}$ is a closed, quasi-ϕ-nonexpansive semigroup, by Remark 2.5, it is a closed, uniformly Lipschitzian and quasi-ϕ-asymptotically nonexpansive semigroup with the sequence $\{k_n = 1\}$. Hence, $\xi_n = (k_n - 1)\sup_{u\in\mathcal{F}} \phi(u, x_n) = 0$. Therefore, the conditions appearing in Theorem 3.1: '$\mathcal{F}$ is a bounded subset in C' and '$\mathcal{T} := \{T(t) : t \geq 0\}$ is uniformly Lipschitzian' are of no use here. Therefore, all conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.3 can be obtained from Theorem 3.2 immediately. □

Remark 3.4 Theorems 3.1, 3.2 and 3.3 improve and extend the corresponding results of Suzuki [4], Xu [5], Chang *et al.* [6–8, 22, 23, 30], Cho *et al.* [10], Thong [11], Buong [12], Mann [13], Halpern [14], Qin *et al.* [15], Nakajo *et al.* [18] and others.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

S-sC and JKK conceived the study and participated in its design and coordination. JKK and LW suggested many good ideas that are useful for achievement this paper and made the revision. JKK and S-sC prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

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