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Nonexpansive mappings on Abelian Banach algebras and their fixed points

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Abstract

A Banach space X is said to have the fixed point property if for each nonexpansive mapping $T : E \to E$ on a bounded closed convex subset E of X has a fixed point. We show that each infinite dimensional Abelian complex Banach algebra X satisfying: (i) property (A) defined in (Fupinwong and Dhompongsa in Fixed Point Theory Appl. 2010:Article ID 34959, 2010), (ii) $\|x\| \leq \|y\|$ for each $x, y \in X$ such that $|\tau(x)| \leq |\tau(y)|$ for each $\tau \in \Omega(X)$, (iii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$ does not have the fixed point property. This result is a generalization of Theorem 4.3 in (Fupinwong and Dhompongsa in Fixed Point Theory Appl. 2010:Article ID 34959, 2010).

MSC: 46B20; 46J99

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1 Introduction

A Banach space X is said to have the fixed point property (or weak fixed point property) if for each nonexpansive mapping $T : E \to E$ on a bounded closed convex (or weakly compact convex, resp.) subset E of X has a fixed point.

For the weak fixed point property of certain Banach algebras, Lau *et al.* [1] showed that the space $C_0(G)$, where G is a locally compact group, has the weak fixed point property if and only if G is discrete, and a von Neumann algebra has the weak fixed point property if and only if it is finite dimensional. Benavides and Pineda [2] proved that each ω-almost weakly orthogonal closed subspace of $C(K_1)$, where K_1 is a metrizable compact space, has the weak fixed point property and $C(K_2)$, where K_2 is a compact set with $K_2^{(\omega)} = \emptyset$, has the weak fixed point property.

As for the fixed point property, Dhompongsa *et al.* [3] showed that a C^*-algebra has the fixed point property if and only if it is finite dimensional. Fupinwong and Dhompongsa [4] proved that each infinite dimensional unital Abelian Banach algebra X with $\Omega(X) \neq \emptyset$ satisfying: (i) (A) defined in [4], (ii) $\|x\| \leq \|y\|$ for each $x, y \in X$ with $|\tau(x)| \leq |\tau(y)|$ for each $\tau \in \Omega(X)$, (iii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$ does not have the fixed point property. Alimohammadi and Moradi [5] used the above result to obtain sufficient conditions to show that some unital uniformly closed subalgebras of $C(X)$, where X is a compact space, do not have the fixed point property.

In this paper, we show that the unitality in the result proved in [4] can be omitted.

2 Preliminaries and lemmas

We assume that the field of each vector space in this paper is complex.

Let X be a Banach algebra. Define $\widetilde{X} = X \oplus \mathbb{C}$ and a multiplication on $\widetilde{X}$ by

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

We have $\widetilde{X}$ is a unital Banach algebra with the unit $(0, 1)$ and called the unitization of X. $\widetilde{X}$ is also Abelian if X is Abelian.

If $\widetilde{X}$ is the unitization of a Banach algebra X and $\Omega(X)$ is the set of characters on X, then the set $\Omega(\widetilde{X})$ of characters on $\widetilde{X}$ is equal to

$$\{\widetilde{\tau} : \tau \in \Omega(X)\} \cup \{\tau_\infty\},$$

where $\widetilde{\tau}$ is defined from $\tau \in \Omega(X)$ by

$$\widetilde{\tau}\big((a, \lambda)\big) = \tau(a) + \lambda,$$

for each $(a, \lambda) \in \widetilde{X}$, and τ_∞ is the canonical homomorphism defined by

$$\tau_\infty\big((a, \lambda)\big) = \lambda,$$

for each $(a, \lambda) \in \widetilde{X}$.

If X is an Abelian Banach algebra, condition (A) is defined by:

(A) *For each $x \in X$, there exists an element $y \in X$ such that $\tau(y) = \overline{\tau(x)}$, for each $\tau \in \Omega(X)$.*

It can be seen that if X satisfies (A), then so does the unitization $\widetilde{X}$ of X.

Let X be an Abelian Banach algebra. The Gelfand representation $\varphi : X \to C(\Omega(X))$ is defined by $x \mapsto \widehat{x}$, where $\widehat{x}$ is defined by

$$\widehat{x}(\tau) = \tau(x),$$

for each $\tau \in C(\Omega(X))$.

The following lemma was proved in [4].

Lemma 2.1 *Let X be a unital Abelian Banach algebra satisfying (A) and*

$$\inf\{r(x) : x \in X, \|x\| = 1\} > 0.$$

Then:

(i) the Gelfand representation φ is a bounded isomorphism,
(ii) the inverse φ^{-1} is also a bounded isomorphism.

Let X be an Abelian Banach algebra satisfying (A) and $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$. It can be seen that X is embedded in $C(\Omega(\widetilde{X}))$ as the closed subalgebra $Y = \{\widehat{x} \in C(\Omega(\widetilde{X})) : \widehat{x}(\tau_\infty) = 0\}$. Moreover, for each $x \in \widetilde{X}$, x is in X if and only if $\tau_\infty(x) = 0$.

Lemma 2.2 *Let X be an infinite dimensional Abelian Banach algebra satisfying (A) and*

$$\inf\{r(x) : x \in X, \|x\| = 1\} > 0.$$

Then we have:

(i) $\Omega(X)$ *is an infinite set.*

(ii) *If there exists a bounded sequence* $\{x_n\}$ *in* X *which contains no convergent subsequences and such that* $\{\tau(x_n) : \tau \in \Omega(X)\}$ *is finite for each* $n \in \mathbb{N}$, *then there is an element* $x_0 \in X$ *such that* $\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\}$ *is equal to* $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ *or* $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

(iii) *There is an element* $x_0 \in X$ *such that* $\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\}$ *is an infinite set.*

(iv) *There exists a sequence* $\{x_n\}$ *in* X *such that* $\{\omega(x_n) : \omega \in \Omega(\widetilde{X})\} \subset [0,1]$, *for each* $n \in \mathbb{N}$, *and* $\{(\widehat{x_n})^{-1}\{1\}\}$ *is a sequence of nonempty pairwise disjoint subsets of* $\Omega(\widetilde{X})$.

Proof (i) From Lemma 2.10(i) in [4], we have $\Omega(\widetilde{X})$ is infinite. Since

$$\Omega(\widetilde{X}) = \{\widetilde{\tau} : \tau \in \Omega(X)\} \cup \{\tau_\infty\},$$

where $\widetilde{\tau}$ is defined from $\tau \in \Omega(X)$ by $\widetilde{\tau}((a,\lambda)) = \tau(a) + \lambda$, for each $(a,\lambda) \in \widetilde{X}$, and τ_∞ is the canonical homomorphism, so $\Omega(X)$ is also infinite.

(ii) Let $\{x_n\}$ be a bounded sequence in X which has no convergent subsequences and the set $\{\tau(x_n) : \tau \in \Omega(X)\}$ be finite for each $n \in \mathbb{N}$. Consider $\{x_n\}$ a sequence in $\widetilde{X}$, so $\{\omega(x_n) : \omega \in \Omega(\widetilde{X})\}$ is finite for each $n \in \mathbb{N}$. It follows from the proof of Lemma 2.10(ii) in [4] that

$$\Omega(\widetilde{X}) = \left(\bigcup_{n\in\mathbb{N}} G_n\right) \cup F,$$

where F is a closed set in $\Omega(\widetilde{X})$, G_n is closed and open for each $n \in \mathbb{N}$, and $\{F, G_1, G_2, \ldots\}$ is a partition of $\Omega(\widetilde{X})$. There are two cases to be considered. If τ_∞ is in F, defined $\psi : \Omega(\widetilde{X}) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} 1, & \text{if } \tau \in G_1, \\ \frac{1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 0, & \text{if } \tau \in F. \end{cases}$$

If τ_∞ is in G_{n_0}, for some $n_0 \in \mathbb{N}$, we may assume that $n_0 = 1$, defined $\psi : \Omega(\widetilde{X}) \to \mathbb{R}$ by

$$\psi(\tau) = \begin{cases} 0, & \text{if } \tau \in G_1, \\ \frac{n-1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 1, & \text{if } \tau \in F. \end{cases}$$

For each case, we have the inverse image of each closed set in $\psi(\Omega(\widetilde{X}))$ is closed, so $\psi \in C(\Omega(\widetilde{X}))$. Let $\varphi : \widetilde{X} \to C(\Omega(\widetilde{X}))$ be the Gelfand representation. Therefore, $\varphi^{-1}(\psi)$ is an element in $\widetilde{X}$, say x_0, such that $\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. We have $x_0 \in X$ since $\tau_\infty(x_0) = \psi(\tau_\infty) = 0$.

(iii) Assume to the contrary that $\{\omega(x) : \omega \in \Omega(\widetilde{X})\}$ is finite for each $x \in X$. Since X is infinite dimensional, so there is a bounded sequence $\{x_n\}$ in X which has no convergent subsequences. Thus $\{\omega(x_n) : \omega \in \Omega(\widetilde{X})\}$ is finite for each $n \in \mathbb{N}$. It follows from (ii) that there exists $x_0 \in X$ such that $\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\}$ is infinite. This leads to a contradiction.

(iv) It follows from (iii) that there exists an element $x_1 \in X$ such that $\{\omega(x_1) : \omega \in \Omega(\widetilde{X})\}$ is infinite. We may assume that there exists a strictly decreasing sequence of real numbers $\{a_n\}$ such that

$$\{a_n\} \subset \widehat{x_1}\big(\Omega(\widetilde{X})\big) \subset [0,1], \quad a_1 < 1,$$

and $\omega(x_1) = 1$ for some $\omega \in \Omega(\widetilde{X})$. Define $g_1 : [0,1] \to [0,1]$ by

$$g_1(t) = \begin{cases} \frac{t}{a_1}, & \text{if } t \in [0, a_1], \\ 1 + \frac{(g_1(a_2)-1)(t-a_1)}{2(1-a_1)}, & \text{if } t \in [a_1, 1]. \end{cases}$$

So g_1 is a continuous function joining the points $(0,0)$ and $(a_1, 1)$, and $g_1(1) \in (g_1(a_2), 1)$. Let $\widehat{x_2} = g_1 \circ \widehat{x_1}$, and define a continuous function $g_2 : [0,1] \to [0,1]$ by

$$g_2(t) = \begin{cases} \frac{t}{g_1(a_2)}, & \text{if } t \in [0, g_1(a_2)], \\ 1 + \frac{(g_2(g_1(a_3))-1)(t-g_1(a_2))}{2(1-g_1(a_2))}, & \text{if } t \in [g_1(a_2), 1]. \end{cases}$$

g_2 is joining the point $(0,0)$ and $(g_1(a_2), 1)$ and $g_2(1) \in (g_2(g_1(a_3)), 1)$. Let $\widehat{x_3} = g_2 \circ \widehat{x_2}$. Continuing in this process, we obtain a sequence of points $\{x_n\}$ in $\widetilde{X}$ with $\{\omega(x_n) : \omega \in \Omega(X)\} \subset [0,1]$, for each $n \in \mathbb{N}$, and $\{(\widehat{x_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(\widetilde{X})$. Since $g_n(0) = 0$, for each $n \in \mathbb{N}$, so

$$\widehat{x_{i+1}}(\tau_\infty) = (g_i \circ \cdots \circ g_1 \circ \widehat{x_1})(\tau_\infty) = (g_i \circ \cdots \circ g_1)(0) = 0,$$

for each $i \in \mathbb{N}$. Then $\tau_\infty(x_n) = 0$, for each $n \in \mathbb{N}$. Thus $\{x_n\}$ is a sequence in X. □

3 Main theorem

Theorem 3.1 *Let X be an infinite dimensional Abelian Banach algebra satisfying (A) and each of the following:*

(i) If $x, y \in X$ is such that $|\tau(x)| \le |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \le \|y\|$,

(ii) $\inf\{r(x) : x \in X, \|x\| = 1\} > 0$.

Then X does not have the fixed point property.

Proof Assume to the contrary that X has the fixed point property. From Lemma 2.2(iv), there exists a sequence $\{x_n\}$ in X such that $\{\omega(x_n) : \omega \in \Omega(\widetilde{X})\} \subset [0,1]$ for each $n \in \mathbb{N}$, and $\{(\widehat{x_n})^{-1}\{1\}\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(\widetilde{X})$. Let $A_n = (\widehat{x_n})^{-1}\{1\}$, and define $T_n : E_n \to E_n$ by

$$x \mapsto x_n x,$$

where

$$E_n = \big\{x \in X : 0 \le \omega(x) \le 1 \text{ for each } \omega \in \Omega(\widetilde{X}), \text{ and } \omega(x) = 1 \text{ if } \omega \in A_n\big\}.$$

From (i) and (ii), $T_n : E_n \to E_n$ is a nonexpansive mapping on the bounded closed convex set E_n for each $n \in \mathbb{N}$. Indeed, E_n is bounded since

$$\inf\big\{r(x) : x \in X, \|x\| = 1\big\} \le r\left(\frac{x}{\|x\|}\right) = \sup_{\omega \in \Omega(\widetilde{X})} \left|\omega\left(\frac{x}{\|x\|}\right)\right| = \frac{1}{\|x\|} \sup_{\omega \in \Omega(\widetilde{X})} \big|\omega(x)\big|$$

for each $x \in X$. So T_n has a fixed point, say y_n, for each $n \in \mathbb{N}$. We have $y_n = x_n y_n$, hence $\widehat{y_n} = \widehat{x_n}\widehat{y_n}$, and then

$$\widehat{y_n}(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is not in } A_n, \\ 1, & \text{if } \omega \text{ is in } A_n, \end{cases}$$

for each $n \in \mathbb{N}$. We have $\|\widehat{y_m} - \widehat{y_n}\| = 1$, if $m \neq n$, since $A_1, A_2, A_3, \ldots$ are pairwise disjoint. Therefore, $\{\widehat{y_n}\}$ has no convergent subsequences. From Lemma 2.1, $\widetilde{X}$ and $C(\Omega(\widetilde{X}))$ are homeomorphic. So $\{y_n\}$ has no convergent subsequences. From Lemma 2.2(ii), there exists an element x_0 in X such that $\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Let $A_0 = (\widehat{x_0})^{-1}\{1\}$. Define $T_0 : E_0 \to E_0$ by

$$x \mapsto x_0 x,$$

where

$$E_0 = \left\{x \in X : 0 \le \omega(x) \le 1 \text{ for each } \omega \in \Omega(\widetilde{X}), \text{ and } \omega(x) = 1 \text{ if } \omega \in A_0\right\}.$$

From (i) and (ii), T_0 is a nonexpansive mapping on the bounded closed convex set E_0. Hence T_0 has a fixed point, say y_0, *i.e.*, $y_0 = x_0 y_0$. Therefore, $\widehat{y_0} = \widehat{x_0}\widehat{y_0}$. Then

$$\widehat{y_0}(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is not in } A_0, \\ 1, & \text{if } \omega \text{ is in } A_0. \end{cases}$$

Since $\widehat{y_0} = \widehat{x_0}\widehat{y_0}$, so we have $A_0 = (\widehat{y_0})^{-1}\{1\}$ and $\Omega(\widetilde{X}) \backslash A_0 = (\widehat{y_0})^{-1}\{0\}$. Then $\Omega(\widetilde{X})$ is a disjoint union of two compact sets A_0 and $\Omega(\widetilde{X}) \backslash A_0$. If

$$\left\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\right\} = \left\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\},$$

then $\{(\widehat{x_0})^{-1}\{\frac{n}{n+1}\} : n \in \mathbb{N}\} \cup \{(\widehat{x_0})^{-1}\{0\}\}$ is a pairwise disjoint open covering of the compact set $\Omega(\widetilde{X}) \backslash A_0$. This leads to a contradiction. Similarly, if

$$\left\{\omega(x_0) : \omega \in \Omega(\widetilde{X})\right\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\},$$

then A_0 has a pairwise disjoint open covering, which is a contradiction. So we conclude that X does not have the fixed point property. □

The following question is interesting.

Question 3.2 Does the Fourier algebra $A(G)$ or the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G have property (A) when G is an infinite group?

Note that $A(G)$ or $B(G)$ are both commutative semigroup Banach algebras with the fixed point property if and only if G is finite (see Theorem 5.7 and Corollary 5.8 of [6]). It is well known that $A(G)$ is norm dense in $C_0(G)$ with spectrum G.

Competing interests
The author declares that they have no competing interests.

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