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Strong convergence theorems of quasi-ϕ-asymptotically nonexpansive semi-groups in Banach spaces

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Abstract

The purpose of this article is to modify the Halpern-Mann-type iteration algorithm for quasi-ϕ *47H09; 49J25.*

Keywords: modified Halpern-Mann-type iteration, quasi-ϕ-symptotically nonexpansive semi-groups, quasi-ϕ-nonexpansive semi-groups, weak relatively nonexpansive mapping, relatively nonexpansive semi-groups, generalized projection

1. Introduction

Throughout this article, we assume that E is a real Banach space with the dual E^*, C is a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\}, \ x \in E.$$

Let $T : C \to E$ be a nonlinear mapping, we denote by $F(T)$ the set of fixed points of T.

Recalled that a mapping $T : C \to C$ is said to be *nonexpansive* , if

$$||Tx - Ty|| \le ||x - y||, \ \forall x, \ y \in C.$$

$T : C \to C$ is said to be *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$||Tx - p|| \le ||x - p||, \ \forall x \in C, p \in F(T).$$

$T : C \to C$ is said to be *asymptotically nonexpansive*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$||T^n x - T^n y|| \le k_n ||x - y||, \ \forall x, \ y \in C, \ n \ge 1.$$

$T : C \to C$ is said to be *quasi-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty]$ with $k_n \to 1$ such that

$$||T^n x - p|| \le k_n ||x - p||, \ \forall x \in C, \ p \in F(T), \ n \ge 1.$$

One parameter family $\mathscr{T} := \{T(t) \ : \ t \ge 0\}$ of mappings from C into C is said to be *nonexpansive semi-group*, if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)\ \forall s, t \geq 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $||T(t)x - T(t)y|| \leq ||x - y||, \forall x, y \in C$.

We use $F(\mathscr{J})$ to denote the common fixed point set of the nonexpansive semi-group $\mathscr{J}$, i.e., $F(\mathscr{J}) := \bigcap_{t\geq 0} F(T(t))$.

One parameter family $\mathscr{T} := \{T(t) : t \geq 0\}$ of mappings from C into C is said to be *quasi-nonexpansive semi-group*, if $F(\mathscr{J}) \neq \emptyset$, and the above conditions (i)-(iii) and the following condition (v) are satisfied:

(v) $||T(t)x - p|| \leq ||x - p||, \forall x \in C, p \in F(\mathscr{J}), t \geq 0$.

One parameter family $\mathscr{T} := \{T(t) : t \geq 0\}$ of mappings from C into C is said to be *asymptotically nonexpansive semi-group*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that the above conditions (i)-(iii) and the following condition (vi) are satisfied:

(vi) $||T^n(t)x - T^n(t)y|| \leq k_n||x - y||, \forall x, y \in C, n \geq 1, t \geq 0$.

One parameter family $\mathscr{T} := \{T(t) : t \geq 0\}$ of mappings from C into C is said to be *quasi asymptotically nonexpansive semi-group*, if $F(\mathscr{J}) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that the above conditions (i)-(iii) and the following condition (vii) are satisfied:

(vii) $||T^n(t)x - p|| \leq k_n||x - p||, \forall x \in C, p \in F(\mathscr{J}), t \geq 0, n \geq 1$.

As well known, the construction of fixed points of nonexpansive mappings (asymptotically nonexpansive mappings), and of common fixed points of nonexpansive semi-groups (asymptotically nonexpansive semi-groups) is an important problem in the theory of nonexpansive mappings and its applications, in particular, in image recovery, convex feasibility problem, and signal processing problem (see, for example [1-4]).

Iterative approximation of fixed point for nonexpansive mappings, asymptotically nonexpansive mappings, nonexpansive semi-groups, and asymptotically nonexpansive semi-groups in Hilbert or Banach spaces has been studied extensively by many authors (see, for example, [5-30] and the references therein).

The purpose of this article is to introduce the concept of *quasi-ϕ-asymptotically nonexpansive semi-groups* and to modify the Halpern and Mann-type iteration algorithm [14,15] for quasi-ϕ-asymptotically nonexpansive semi-groups and to have the strong convergence under a limit condition only in the framework of Banach spaces. The results presented in the article improve and extend the corresponding results of Suzuki [5], Xu [6], Chang et al. [7], Zhang [8], Chang et al. [9], Cho et al. [11], Thong [12], Buong [13], Mann [14], Halpern [15], Qin et al. [16], Nakajo and Takahashi [19], Kang et al. [23], Chang et al. [24], and others.

2.Preliminaries

In the sequel, we assume that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. In what follows, we always use $\phi : E \times E \to \mathscr{R}^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = ||x||^2 + 2\langle x, Jy\rangle + ||y||^2, \ \forall x, y \in E. \tag{2.1}$$

It is obvious from the definition of ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E. \tag{2.2}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz) \leq \lambda\phi(x,\ y) + (1-\lambda)\phi(x,\ z),\ \forall x, y \in E. \tag{2.3}$$

Following Alber [31], the *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = arg \inf_{y\in C} \phi(y,\ x),\ \forall x \in E.$$

Lemma 2.1 [31]. Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$ and $y \in E$;

(b) If $x \in E$ and $z \in C$, then $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0, \forall y \in C$;

(c) For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$;

Remark 2.2. If E is a real Hilbert space H, then $\phi(x, y) = ||x - y||^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Definition 2.3. A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 2.4. (1) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive, if* $F(T) \neq \emptyset$ *and*

$$\phi(p,\ Tx) \leq \phi(p,\ x),\ \forall x \in C,\ p \in F(T).$$

(2) A mapping $T : C \to C$ *is said to be quasi-ϕ-asymptotically nonexpansive, if* $F(T) \neq \emptyset$ *and there exists a real sequence* $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ *such that*

$$\phi(p,\ T^n x) \leq k_n\phi(p,\ x),\ \forall n \geq 1,\ x \in C,\ p \in F(T).$$

Remark 2.5 [23]. (1) From the definitions, it is obvious that a quasi-ϕ-nonexpansive mapping is a quasi-ϕ-asymptotically nonexpansive mapping. However, the converse is not true.

(2) Especially, if E is a real Hilbert space, than a quasi-ϕ-nonexpansive mapping is a quasi-nonexpansive mapping and a quasi-ϕ-asymptotically nonexpansive mapping is a quasi-asymptotically nonexpansive mapping.

Example 2.6 [24]. Let E be a uniformly smooth and strictly convex Banach space and $A : E \to E^*$ be a maximal monotone mapping such that $A^{-1}0 \neq \emptyset$, then $J_r = (J + rA)^{-1}J$ is closed and quasi-ϕ-nonexpansive from E onto $D(A)$;

Example 2.7 [23]. Let Π_C be the generalized projection from a smooth, reflexive, and strictly convex Banach space E onto a nonempty closed convex subset C of E, then Π_C is a closed and quasi-ϕ -nonexpansive from E onto C.

Lemma 2.8 Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $||x_n - y_n|| \to 0$.

Lemma 2.9 [23]. Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping, then $F(T)$ is a closed convex subset of C.

Definition 2.10. (I) Let E be a real Banach space, C be a nonempty closed convex subset of E. $\mathscr{T} := \{T(t)\ :\ t \geq 0\}$ be one parameter family of mappings from C into C. $\mathscr{J}$ is said to be

(1) *quasi-ϕ-nonexpansive semi-group*, if $\mathscr{F} = \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ and the following conditions are satisfied

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ $\forall$ $s, t \geq 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\phi(p, T(t)x) \leq \phi(p, x), \forall t \geq 0, p \in \mathscr{F}, x \in C$.

(2) $\mathscr{J}$ is said to be *quasi-ϕ-asymptotically nonexpansive semi-group*, if the set $\mathscr{F} = \bigcap_{t \geq 0} F(T(t))$ is nonempty, and there exists a sequence $\{k_n\} \subset [1, \infty)$, with $k_n \to 1$ such that the conditions (i)-(iii) and the following conditions (v) are satisfied:

(v) $\phi(p, T^n(t)x) \leq k_n \phi(p, x), \forall t \geq 0, p \in \mathscr{F}, n \geq 1, x \in C$.

(II) A quasi-ϕ-asymptotically nonexpansive semi-group $\mathscr{J}$ is said to be *uniformly Lipschitzian*, if there exists a bounded measurable function $L : [0, \infty) \to (0, \infty)$ such that

$$||T^n(t)x - T^n(t)y|| \leq L(t)||x - y||, \ \forall x, \ y \in C, \ \forall n \geq 1, \ t \geq 0.$$

3. Main results

Theorem 3.1. Let C be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space E. Let $\mathscr{T} := \{T(t) \ : \ t \geq 0\}$ be a closed, uniformly L-Lipschitz and quasi-ϕ-asymptotically nonexpansive semi-group with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Let $\{\mapsto_n\}$ be a sequence in [0,1] and $\{\beta_n\}$ be a sequence in (0, 1) satisfying the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}; \ C_1 = C, \\ y_{n,t} = J^{-1}[\alpha_n J x_1 + (1 - \alpha_n)(\beta_n J x_n + (1 - \beta_n) J T^n(t) x_n)], \ t \geq 0, \\ C_{n+1} = \{z \in C_n : \sup_{t \geq 0} \phi(z, \ y_{n,t}) \leq \alpha_n \phi(z, \ x_1) + (1 - \alpha_n)\phi(z, \ x_n) + \xi_n\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\mathscr{F} := \bigcap_{t \geq 0}^{\infty} F(T(t)), \xi_n = (k_n - 1)\sup_{p \in \mathscr{F}} \phi(p, \ x_n), \Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. If $\mathscr{F}$ is bounded in C, then $\{x_n\}$ converges strongly to $\Pi_{\mathscr{F}} x_1$.

Proof. (I) First we prove that $\mathscr{F}$ and C_n, $n \geq 1$ all are closed and convex subsets in C.

In fact, it follows from Lemma 2.9 that $F(T(t))$, $t \geq 0$ is a closed and convex subset of C. Therefore, $\mathscr{F}$ is closed and convex in C.

Again by the assumption that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have that

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \sup_{t \geq 0} \phi(z, \ y_{n,t}) \leq \alpha_n \phi(z, \ x_1) + (1 - \alpha_n)\phi(z, \ x_n) + \xi_n\} \\ &= \bigcap_{t \geq 0} \{z \in C : \phi(z, \ y_{n,t}) \leq \alpha_n \phi(z, \ x_1) + (1 - \alpha_n)\phi(z, \ x_n) + \xi_n\} \bigcap C_n \\ &= \bigcap_{t \geq 0} \{z \in C : 2\alpha_n \langle z, \ Jx_1 \rangle + 2(1 - \alpha_n)\langle z, \ Jx_n \rangle - 2\langle z, \ Jy_{n,t} \rangle \\ &\leq \alpha_n ||x_1||^2 + (1 - \alpha_n)||x_n||^2 - ||y_{n,t}||^2\} \bigcap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex. The conclusion is proved.

(II) Now we prove that $\mathscr{F} \subset C_n, \forall n \geq 1$.

In fact, it is obvious that $\mathscr{F} \subset C_1 = C$. Suppose that $\mathscr{F} \subset C_n$, for some $n \geq 2$. Letting

$$w_{n,t} = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT^n(t)x_n), \ t \geq 0,$$

it follows from (2.3) that for any $u \in \mathscr{F} \subset C_n$, we have

$$\begin{aligned} \phi(u, \ y_{n,t}) &= \phi(u, \ J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)Jw_{n,t})) \\ &\leq \alpha_n \phi(u, \ x_1) + (1 - \alpha_n)\phi(u, \ w_{n,t}), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \phi(u, \ w_{n,t}) &= \phi(u, \ J^{-1}(\beta_n Jx_n + (1 - \beta_n)JT^n(t)x_n) \\ &\leq \beta_n \phi(u, \ x_n) + (1 - \beta_n)\phi(u, \ T^n(t)x_n) \\ &\leq \beta_n \phi(u, \ x_n) + (1 - \beta_n)k_n \phi(u, \ x_n) \\ &= \phi(u, \ x_n) + (1 - \beta_n)(k_n - 1)\phi(u, \ x_n). \end{aligned} \tag{3.3}$$

Therefore, we have

$$\begin{aligned} \sup_{t \geq 0} \phi(u, \ y_{n,t}) &\leq \alpha_n \phi(u, \ x_1) + (1 - \alpha_n)\{\phi(u, \ x_n) + (1 - \beta_n)(k_n - 1)\phi(u, \ x_n)\} \\ &\leq \alpha_n \phi(u, \ x_1) + (1 - \alpha_n)\phi(u, \ x_n) + (k_n - 1) \sup_{p \in \mathscr{F}} \phi(p, \ x_n) \\ &= \alpha_n \phi(u, \ x_1) + (1 - \alpha_n)\phi(u, \ x_n) + \xi_n, \end{aligned}$$

where $\xi_n = (k_n - 1)\sup_{p \in \mathscr{F}} \phi(p, x_n)$. This shows that $u \in C_{n+1}$, and so $\mathscr{F} \subset C_{n+1}$. The conclusion is proved.

(III) Next we prove that $\{x_n\}$ is a Cauchy sequence in C.

In fact, since $x_n = \prod_{C_n} x_1$, from Lemma 2.1(b) we have

$$\langle x_n - y, \ Jx_1 - Jx_n \rangle \geq 0, \ \forall y \in C_n.$$

Again since $\mathscr{F} \subset C_n \forall n \geq 1$, we have

$$\langle x_n - u, \ Jx_1 - Jx_n \rangle \geq 0, \ \forall u \in \mathscr{F}.$$

It follows from Lemma 2.1(a) that for each $u \in \mathscr{F}$ and for each $n \geq 1$

$$\phi(x_n, \ x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, \ x_1) - \phi(u, \ x_n) \leq \phi(u, \ x_1). \tag{3.4}$$

Therefore $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (2.2), $\{x_n\}$ is also bounded. Since $x_n = \prod_{C_n} x_1$ and $x_{n+1} = \prod_{C_n} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$, $\forall n \geq 1$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence, the limit $\lim_{n \to \infty} \phi(x_n, x_1)$ exists. By the construction of $\{C_n\}$, for any positive integer $m \geq n$, we have $C_m \subset C_n$ and $x_m = \Pi_{Cm}\, x_1 \in C_n$. This show that

$$\phi(x_m, \ x_n) = \phi(x_m, \Pi_{C_n} x_1), \leq \phi(x_m, \ x_1) - \phi(x_n, \ x_1) \to 0 \ as \, n, \ m \to \infty.$$

It follows from Lemma 2.8 that $\lim_{n,m \to \infty} ||x_m - x_n|| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is complete, without loss of generality, we can assume that $x_n \to p^*$ (some point in C).

By the assumption, it is easy to see that

$$\lim_{n \to \infty} \xi_n = \lim_{n \to \infty} (k_n - 1) \sup_{p \in \mathscr{F}} \phi(p, \ x_n) = 0. \tag{3.5}$$

(IV) Now we prove that $p^* \in \mathscr{F}$.

In fact, since $x_{n+1} \in C_{n+1}$ and $\alpha_n \to 0$, follows from (3.1) and (3.5) that

$$\sup_{t\geq 0} \phi(x_{n+1}, y_{n,t}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1-\alpha_n)\phi(x_{n+1}, x_n) + \xi_n \to 0 \ (as\ n \to \infty).$$

Since $x_n \to p^*$, by virtue of Lemma 2.8 for each $t \geq 0$

$$\lim_{n\to\infty} y_{n,t} = p^*. \tag{3.6}$$

Since $\{x_n\}$ is bounded, and $\mathscr{T} := \{T(t) : t \geq 0\}$ is a quasi-ϕ-asymptotically nonexpansive semi-group with sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$, for any given $p \in \mathscr{F}$, we have

$$\phi(p, T^n(t)x_n) \leq k_n \phi(p, x_n), \ \forall t \geq 0.$$

This implies that $\{T^n(t)x_n\}_{t\geq 0}$ is uniformly bounded. Since for each $t \geq 0$,

$$\begin{aligned} ||w_{n,t}|| &= ||J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n)|| \\ &\leq \beta_n ||x_n|| + (1-\beta_n)||T^n(t)x_n|| \\ &\leq \max\{||x_n||, ||T^n(t)x_n||\}. \end{aligned}$$

This implies that $\{w_{n,t}\}_{t\geq 0}$ is also uniformly bounded.

Since $\alpha_n \to 0$, from (3.1) we have

$$\lim_{n\to\infty} ||Jy_{n,t} - Jw_{n,t}|| = \lim_{n\to\infty} \alpha_n ||Jx_1 - Jw_{n,t}|| = 0, for\ t \geq 0. \tag{3.7}$$

Since E^* is uniformly smooth, J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (3.6) and (3.7) that

$$\lim_{n\to\infty} w_{n,t} = p^* for\ each\ t \geq 0. \tag{3.8}$$

Since $x_n \to p^*$ and J is uniformly continuous on each bounded subset of E, we have $J x_n \to Jp^*$, and so for each $t \geq 0$

$$\begin{aligned} 0 &= \lim_{n\to\infty} ||Jw_{n,t} - Jp^*|| = \lim_{n\to\infty} ||\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n - Jp^*|| \\ &= \lim_{n\to\infty} ||\beta_n(Jx_n - Jp^*) + (1-\beta_n)(JT^n(t)x_n - Jp^*)|| \\ &= \lim_{n\to\infty} (1-\beta_n)||JT^n(t)x_n - Jp^*||. \end{aligned}$$

By condition (ii), we have that

$$\lim_{n\to\infty} ||(JT^n(t)x_n - Jp^*)|| = 0 \text{ uniformly in } t \geq 0.$$

Since J is uniformly continuous, this shows that $\lim_{n\to\infty} T^n(t)x_n = p^*$ uniformly in $t \geq 0$.

Again by the assumptions that the semi-group $\mathscr{T} := \{T(t) : t \geq 0\}$ is closed and uniformly L-Lipschitzian, thus we have

$$\begin{aligned} ||T^{n+1}(t)x_n - T^{n+1}(t)x_n \leq\ & ||T^{n+1}(t)x_n - T^{n+1}(t)x_{n+1}|| + ||T^{n+1}(t)x_{n+1} - x_{n+1}|| \\ & + ||x_{n+1} - x_n|| + ||x_n - T^n(t)x_n|| \\ \leq\ & (L(t)+1)||x_{n+1} - x_n|| + ||T^{n+1}(t)x_{n+1} - x_{n+1}|| + ||x_n - T^n(t)x_n||. \end{aligned} \tag{3.9}$$

Since $\lim_{n\to\infty} T^n(t)x_n = p^*$ uniformly in $t \geq 0$, $x_n \to p^*$ and $L(t)$: $[0, \infty) \to [0, \infty)$ is a bounded and measurable function, these together with (3.9) imply that

$$\lim_{n\to\infty} ||T^{n+1}(t)x_n - T^n(t)x_n|| = 0, \text{uniformly in } t \geq 0,$$

and so

$$\lim_{n\to\infty} T^{n+1}(t)x_n = p^*, \text{uniformly in } t \geq 0,$$

i.e.,

$$\lim_{n\to\infty} T(t)T^n(t)x_n = p^*, \text{ uniformly in } t \geq 0.$$

In view of the closeness of the semi-group $\mathscr{J}$, it yields that $T(t)p^* = p^*$, i.e., $p^* \in F(T(t))$. By the arbitrariness of $t \geq 0$, we have $p^* \in \mathscr{F} := \bigcap_{t\geq 0} F(T(t))$.

(V) Finally, we prove that $x_n \to p^* = \Pi_{\mathscr{F}}x_1$.

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Let $w = \Pi_{\mathscr{F}}x_1$. Since $w \in \mathscr{F} \subset C_n$ and $x_n = \prod_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$, $\forall n$
$\geq$
1. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w, x_1). \tag{3.10}$$

In view of the definition of $\Pi_{\mathscr{F}}x_1$, from (3.10) we have $p^* = w$. Therefore, $x_n \to p^* = \Pi_{\mathscr{F}}x_1$. This completes the proof of Theorem 3.1.

Theorem 3.2. Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 3.1. Let $\mathscr{T} := \{T(t) : t \geq 0\}$ be a closed, quasi-ϕ- nonexpansive semi-group such that the set $\mathscr{F} := \bigcap_{t\geq 0} F(T(t))$ is nonempty. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\ y_{n,t} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JT(t)x_n)],\ t \geq 0, \\ C_{n+1} = \{z \in C_n : \sup_{t\geq 0} \phi(z, y_{n,t}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ \forall n \geq 1. \end{cases} \tag{3.11}$$

Then the sequence $\{x_n\}$converges strongly to $\Pi_{\mathscr{F}}x_1$,

Proof. Since $\mathscr{T} := \{T(t) : t \geq 0\}$ is a closed, quasi-ϕ-nonexpansive semi-groups, by Remark 2.5, it is a closed, uniformly Lipschitzian and quasi-ϕ- asymptotically nonexpansive semi-group with sequence$\{k_n = 1\}$.˙Hence $\xi_n = (k_n - 1)\sup_{u\in\mathscr{F}} \phi(u, x_n) = 0$. Therefore the conditions appearing in Theorem 3.1: "$\mathscr{F}$ is a bounded subset in C" and "$\mathscr{T} := \{T(t) : t \geq 0\}$ is uniformly Lipschitzian" are no use here. Therefore all conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately.

Remark 3.3. Theorems 3.1 and 3.2 improve and extend the corresponding results of Suzuki [5], Xu [6], Chang et al. Chang et al. [7], Zhang [8], Chang et al. [9], Cho et al. [11], Thong [12], Buong [13], Mann [14], Halpern [15], Qin et al. [16], Nakajo and Takahashi [19], Kang et al. [23], Chang et al. [24], and others.

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Authors' contributions
All the authors participated in this article's design and coordination. And they read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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