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Graphical approximation of common solutions to generalized nonlinear relaxed cocoercive operator equation systems with (A, η)-accretive mappings

Fang Li[1], Heng-you Lan[1*] and Yeol Je Cho[2]

* Correspondence: hengyoulan@163.com
[1]Department of Mathematics, Sichuan University of Science and Engineering, Zigong, 643000, Sichuan, People's Republic of China
Full list of author information is available at the end of the article

Abstract

In this paper, we develop a new perturbed iterative algorithm framework with errors based on the variational graphical convergence of operator sequences with (A, η)-accretive mappings in Banach space. By using the generalized resolvent operator technique associated with (A, η)-accretive mappings, we also prove the existence of solutions for a class of generalized nonlinear relaxed cocoercive operator equation systems and the variational convergence of the sequence generated by the perturbed iterative algorithm in q-uniformly smooth Banach spaces. The obtained results improve and generalize some well-known results in recent literatures.

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1 Introduction

It is well known that standard Yosida regularizations/approximations have been tremendously effective to approximation solvability of general variational inclusion problems in the context of resolvent operators that turned out to be nonexpansive. This class of nonlinear Yosida approximations have been applied to approximation solvability of nonlinear inhomogeneous evolution inclusions of the form

$$f(t) \in u'(t) + Mu(t) - \omega u(t), \quad u(0) = u_0$$

for almost all $t \in [0, T]$, where $T \in (0,1)$ is fixed, $\omega \in R$ (see [1]). For more general details on approximation solvability of general nonlinear inclusion problems, we refer the reader to [2-18] and the references therein.

On the other hand, it is well known that variational inequalities and variational inclusions provide mathematical models to some problems arising in economics, mechanics, and engineering science and have been studied extensively. There are many methods to find solutions of variational inequality and variational inclusion problems. Among these methods, the resolvent operator technique is very important. For some literature, we recommend to the following example, and the reader [2-15,17,18] and the references therein.

Example 1.1. ([19]) Let $V : R^n \to R$ be a local Lipschitz continuous function, and let K be a closed convex set in R^n. If x^* is a solution to the following problem:

$$\min_{x \in K} V(x),$$

then

$$0 \in \partial V(x^*) + \mathcal{N}_K(x^*),$$

where $\partial V(x^*)$ denotes the subdifferential of V at x^* and $\mathcal{N}_K(x^*)$ the normal cone of K at x^*.

In 2006, Lan et al. [7] introduced a new concept of (A, η)-accretive mappings, which provides a unifying framework for maximal monotone operators, m-accretive operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, generalized m-accretive mappings, H-accretive operators, (H, η)-monotone operators, and A-monotone mappings. Recently, by using the concept of (A, η)-accretive mappings and the resolvent operator technique associated with (A, η)-accretive mappings, Jin [5] introduced and studied a new class of nonlinear variational inclusion systems with (A, η)-accretive mappings in q-uniformly smooth Banach spaces and developed some new iterative algorithms to approximate the solutions of the mentioned nonlinear variational inclusion systems. Furthermore, by using the resolvent operator technique, Petrot [14] studied the common solutions for a generalized system of relaxed cocoercive mixed variational inequality problems and fixed point problems for Lipschitz mappings in Hilbert spaces, and Agarwal and Verma [2] introduced and studied a new system of nonlinear (set-valued) variational inclusions involving (A, η)-maximal relaxed monotone and relative (A, η)-maximal monotone mappings in Hilbert spaces and proved its approximation solvability based on the variational graphical convergence of operator sequences. For more literature, we recommend to the reader [9,20] and the references therein.

Motivated and inspired by the above works, the purpose of this paper is to consider and study the following generalized nonlinear operator equation system with (A, η)-accretive mappings in real Banach space $\mathcal{B}_1 \times \mathcal{B}_2$:

Find $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2$ and $u \in S(x), v \in T(y)$ such that

$$\begin{cases} p(x) = R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x)} \left[(1-\lambda_1)A_1(p(x)) + \lambda_1(A_1(f(y)) - \rho N_1(u, y) + a)\right], \\ h(y) = R^{\varrho\lambda_2, A_2}_{\eta_2, M_2(y, \cdot)} \left[(1-\lambda_2)A_2(h(y)) + \lambda_2(A_2(g(x)) - \varrho N_2(x, v) + b)\right], \end{cases} \tag{1.1}$$

where for all $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2, R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x)} = (A_1 + \rho\lambda_1 M_1(\cdot, x))^{-1}$ and $R^{\varrho\lambda_2, A_2}_{\eta_2, M_2(y, \cdot)} = (A_2 + \varrho\lambda_2 M_2(y, \cdot))^{-1}$ are two resolvent operators and two constants $\rho, \varrho > 0, N_1 : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_1, N_2 : \mathcal{B}_1 \times \mathcal{B}_2 \to \mathcal{B}_2, p : \mathcal{B}_1 \to \mathcal{B}_1, h : \mathcal{B}_2 \to \mathcal{B}_2, f : \mathcal{B}_2 \to \mathcal{B}_1, g : \mathcal{B}_1 \to \mathcal{B}_2$ are single-valued operators, $\lambda_1, \lambda_2 > 0$ are two constants, $(a, b) \in \mathcal{B}_1 \times \mathcal{B}_2$ is an any given element, and $S : \mathcal{B}_1 \to 2^{\mathcal{B}_1}, T : \mathcal{B}_2 \to 2^{\mathcal{B}_2}, A_i : \mathcal{B}_i \to \mathcal{B}_i, \eta_i : \mathcal{B}_i \times \mathcal{B}_i \to \mathcal{B}_i, M_i : \mathcal{B}_i \times \mathcal{B}_i \to 2^{\mathcal{B}_i} (i = 1, 2)$ are any nonlinear operators such that for all $x \in \mathcal{B}_1, M_1(\cdot, x) : \mathcal{B}_1 \to 2^{\mathcal{B}_1}$ is an (A_1,η_1)-accretive mapping and $M_2(y, \cdot) : \mathcal{B}_2 \to 2^{\mathcal{B}_1}$ is an (A_2, η_2)-accretive mapping for all $y \in \mathcal{B}_2$, respectively.

Based on the definition of the resolvent operators associated with (A, η)-accretive mappings, the Equation (1.1) can be written as

$$\begin{cases} a \in A_1(p(x)) - A_1(f(y)) + \rho N_1(u, y) + \rho M_1(p(x), x), \\ b \in A_2(h(y)) - A_2(g(x)) + \varrho N_2(x, v) + \varrho M_2(y, h(y)) \end{cases} \tag{1.2}$$

Remark 1.1. For appropriate and suitable choices of $\mathcal{B}_i, A_i, \eta_i, N_i, M_i(i = 1, 2), p, h, f, g, S, T$, one can obtain a number (systems) of quasi-variational inclusions, generalized (random) quasi- variational inclusions, quasi-variational inequalities, and implicit quasi-variational inequalities as special cases of the Equation (1.1) (or problem (1.2)) include. Below are some special cases of problem.

Example 1.2. If $\mathcal{B}_i = \mathcal{B}(i = 1, 2), p = f = h = g, N_1(x, \cdot) = N_2(\cdot, y) = N(\cdot)$ and $M_1(\cdot,x) = M_1(\cdot)$, $M_2(y,\cdot)$, $= M_2(\cdot)$ for all $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2$ and $a = b = 0$, then the problem (1.2) collapses to the following nonlinear variational inclusion system with (A, η)-accretive mappings:

$$\begin{cases} 0 \in A_1(g(x)) - A_1(g(y)) + \rho N(y) + \rho M_1(g(x)), \\ 0 \in A_2(g(y)) - A_2(g(x)) + \varrho N(x) + \varrho M_2(g(y)). \end{cases} \tag{1.3}$$

The system (1.3) was introduced and studied by Jin [5]. Further, when $A_i = A$, $M_i = M(i = 1,2)$ and $y = x$, the system (1.3) reduces to a nonlinear variational inclusion of find $x \in \mathcal{B}$ such that

$$0 \in N(x) + M(g(x)),$$

which contains the variational inclusions with H-monotone operator, H-accretive mappings, or A-maximal (m)-relaxed monotone (AMRM) mappings in [2,3] as special cases.

Example 1.3. If $\mathcal{B}_i = \mathcal{H}(i = 1, 2)$ is a Hilbert space, $a = b = 0, S : \mathcal{B}_1 \to \mathcal{B}_1$ and $T : \mathcal{B}_2 \to \mathcal{B}_2$ are two single-valued mappings, $p = f = h = g = S = T = I$ is the identity operator and $M_1(\cdot,x) = M_2(y,\cdot) = M(\cdot)$ for all $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2$, then the problem (1.2) is equivalent to solve the following nonlinear variational inclusion system with (A, η)-monotone mappings:

$$\begin{cases} 0 \in A_1(x) - A_1(y) + \rho N(y, x) + \rho M(x), \\ 0 \in A_2(y) - A_2(x) + \varrho N(x, y) + \varrho M(y), \end{cases} \tag{1.4}$$

The system (1.4) was introduced and studied by Wang and Wu [18] and contains the generalized system for mixed variational inequalities with maximal monotone operators in [14] as special cases. Moreover, taking $y = x$, then the system (1.4) reduces to finding an element $x \in \mathcal{H}$ such that

$$0 \in N(x, x) + M(x),$$

which was considered by Verma [17].

Example 1.4. When $\mathcal{B}_i = \mathcal{H}, \lambda_i = 1(i = 1, 2), p = h, A_1 = A_2 = I, N_1(x, \cdot) = N_2(\cdot, y) = N(\cdot)$ and $M_1(\cdot,x) = M_1(\cdot)$,$N_2(y,\cdot) = M_2(\cdot)$ for all $(x, y) \in \mathcal{B}_1 \times \mathcal{B}_2$, the system (1.1) becomes to the following nonlinear operator equation systems: Finding $(x, y) \in \mathcal{H} \times \mathcal{H}$ such that

$$\begin{cases} h(x) = J^{\rho}_{M_1}\left[f(y) - \rho N(y)\right], \\ h(y) = J^{\varrho}_{M_2}\left[g(x) - \varrho N(x)\right], \end{cases} \tag{1.5}$$

where $J^{\rho}_{M_1} = (I + \rho M_1)^{-1}$ and $J^{\varrho}_{M_2} = (I + \varrho M_2)^{-1}$. Based on the definition of the resolvent operators, we know that the system (1.5) is equivalent to solve the following system of general variational inclusions:

$$\begin{cases} 0 \in h(x) - f(y) + \rho N(y) + \rho M_1(h(x)), \\ 0 \in h(y) - g(x) + \varrho N(x) + \varrho M_2(h(y)), \end{cases} \tag{1.6}$$

which was studied by Noor et al. [12] when $M_i = M$ is maximal monotone for $i = 1$, 2. Moreover, some special cases of the problem (1.6) can be found in [4,6] and the references therein.

We also construct a new perturbed iterative algorithm framework with errors based on the variational graphical convergence of operator sequences with (A, η)-accretive mappings in Banach space for approximating the solutions of the nonlinear equation system (1.1) in smooth Banach spaces and prove the existence of solutions and the variational convergence of the sequence generated by the perturbed iterative algorithm in q-uniformly smooth Banach spaces. The results present in this paper improve and generalize the corresponding results of [2,3,5,12,14,17,18] and many other recent works.

2 Preliminaries

Let $\mathcal{B}$ be a real Banach space with dual space $\mathcal{B}^*$, $\langle \cdot, \cdot \rangle$ be the dual pair between $\mathcal{B}$ and $\mathcal{B}^*, CB(\mathcal{B})$ denote the family of all nonempty closed bounded subsets of $\mathcal{B}$, and $2^{\mathcal{B}}$ denote the family of all the nonempty subsets of $\mathcal{B}$. The generalized duality mapping $J_q : \mathcal{B} \to 2^{\mathcal{B}^*}$ is defined by

$$J_q(x) = \left\{f^* \in \mathcal{B}^* : \langle x, f^* \rangle = \|x\|^q, \left\|f^*\right\| = \|x\|^{q-1}\right\}, \quad \forall x \in \mathcal{B},$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = ||x||^{q-2} J_2(x)$ for all $x \neq 0$, and J_q is single-valued if $\mathcal{B}^*$ is strictly convex. In the sequel, we always suppose that $\mathcal{B}$ is a real Banach space such that J_q is single-valued and $\mathcal{H}$ is a Hilbert space. If $\mathcal{B} = \mathcal{H}$, then J_2 becomes the identity mapping on $\mathcal{H}$.

The modulus of smoothness of $\mathcal{B}$ is the function $\mathcal{X}_{\mathcal{B}} : [0, \infty) \to [0, \infty)$ defined by

$$\mathcal{X}_{\mathcal{B}}(t) = \sup\left\{\frac{1}{2}\left(\left\|x + y\right\| + \left\|x - y\right\|\right) - 1 : \|x\| \leq 1, \left\|y\right\| \leq t\right\}.$$

A Banach space $\mathcal{B}$ is called uniformly smooth if $\lim_{t\to 0}\dfrac{\mathcal{X}_{\mathcal{B}}(t)}{t} = 0$.

$\mathcal{B}$ is called q-uniformly smooth if there exists a constant $c > 0$ such that $\mathcal{X}_{\mathcal{B}}(t) \leq ct^q, \quad q > 1$. Remark that J_q is single-valued if $\mathcal{B}$ is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [21] proved the following result:

Lemma 2.1. Let $\mathcal{B}$ be a real uniformly smooth Banach space. Then, $\mathcal{B}$ is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in \mathcal{B}$,

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

In the sequel, we give some concept and lemmas needed later.

Definition 2.1. Let $\mathcal{B}$ be a q-uniformly smooth Banach space and $T, A : \mathcal{B} \to \mathcal{B}$ be two single-valued mappings. T is said to be

(i) accretive if

$$\langle T(x) - T(y), J_q(x-y)\rangle \ge 0, \quad \forall x, y \in \mathcal{B};$$

(ii) strictly accretive if T is accretive and

$$\langle T(x) - T(y), J_q(x-y)\rangle = 0$$

if and only if $x = y$;

(iii) r-strongly accretive if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(x-y)\rangle \ge r\|x-y\|^q, \quad \forall x, y \in \mathcal{B};$$

(iv) γ-strongly accretive with respect to A if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \ge \gamma\|x-y\|^q, \quad \forall x, y \in \mathcal{B};$$

(v) m-relaxed cocoercive with respect to A if, there exists a constant $m > 0$ such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \ge -m\|T(x) - T(y)\|^q, \quad \forall x, y \in \mathcal{B};$$

(vi) (π, ι)-relaxed cocoercive with respect to A if, there exist constants $\pi, \iota > 0$ such that

$$\langle T(x) - T(y), J_q(A(x) - A(y))\rangle \ge -\pi\|x-y\|^q + \iota\|T(x) - T(y)\|^q, \quad \forall x, y \in \mathcal{B};$$

(vii) s-Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \le s\|x-y\|, \quad \forall x, y \in \mathcal{B}.$$

In a similar way, we can define (relaxed) cocoercivity and Lipschitz continuity of the operator $N(\cdot,\cdot) : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ in the first and second arguments.

Remark 2.1. (1) The notion of the cocoercivity is applied in several directions, especially to solving variational inequality problems using the auxiliary problem principle and projection methods [16], while the notion of the relaxed cocoercivity is more general than the strong monotonicity as well as cocoercivity. Several classes of relaxed cocoercive variational inequalities and variational inclusions have been studied in [2,5,7-10,12,16-18].

(2) When $\mathcal{B} = \mathcal{H}$, (i)-(iv) of Definition 2.1 reduce to the definitions of monotonicity, strict monotonicity, strong monotonicity, and strong monotonicity with respect to A, respectively (see [3,18]).

Definition 2.2. A single-valued mapping $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \le \tau \|x-y\|, \quad \forall x,y \in \mathcal{B}.$$

Definition 2.3. Let $\mathcal{B}$ be a q-uniformly smooth Banach space, $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $A, H : \mathcal{B} \to \mathcal{B}$ be single-valued mappings. Then set-valued mapping $M : \mathcal{B} \to 2^{\mathcal{B}}$ is said to be

(i) η-accretive if

$$\langle u-v, J_q(\eta(x,y))\rangle \ge 0, \quad \forall x,y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(ii) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \ge r\|x-y\|^q, \quad \forall x,y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(iii) m-relaxed η-accretive if there exists a constant $m > 0$ such that

$$\langle u-v, J_q(\eta(x,y))\rangle \ge -m\|x-y\|^q, \quad \forall x,y \in \mathcal{B}, u \in M(x), v \in M(y);$$

(iv) $\xi - \hat{\mathbf{H}}$-Lipschitz continuous, if there exists a constant $\xi > 0$ such that

$$\hat{\mathbf{H}}(M(x), M(y)) \le \xi \|x-y\|, \quad \forall x,y \in \mathcal{B},$$

where $\hat{\mathbf{H}}$ is the Hausdorff metric on $CB(\mathcal{B})$;

(v) (A, η)-accretive if M is m-relaxed η-accretive and $(A + \rho M)(\mathcal{B}) = \mathcal{B}$ for every $\rho > 0$.

Remark 2.2. The (A, η)-accretivity generalizes the general (H, η)-accretivity, (I, η)-accretivity (so-called generalized m-accretivity), H-accretivity classical m-accretivity (A, η)-monotonicity, A-monotonicity, (H, η)-monotonicity, H-monotonicity, maximal η-monotonicity, and classical maximal monotonicity as special cases (see, for example, [1,7,8,13] and the references therein.)

Definition 2.4. Let $A : \mathcal{B} \to \mathcal{B}$ be a strictly η-accretive mapping and $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. The resolvent operator $R^{\rho,A}_{\eta,M} : \mathcal{B} \to \mathcal{B}$ is defined by:

$$R^{\rho,A}_{\eta,M}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in \mathcal{B}.$$

Remark 2.3. The resolvent operators associated with (A, η)-accretive mappings include as special cases the corresponding resolvent operators associated with (H, η)-accretive mappings, (A, η)-monotone operators [8], (H, η)-monotone operators, H-accretive operators, generalized m-accretive operators, maximal η-monotone operators, H-monotone operators, A-monotone operators, η-subdifferential operators, the classical m-accretive, and maximal monotone operators. See, for example, [1,7,8,13] and the references therein.

Lemma 2.2. ([7]) Let $\mathcal{B}$ be a q-uniformly smooth Banach space and $\eta : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be τ-Lipschitz continuous, $A : \mathcal{B} \to \mathcal{B}$ be a r-strongly η-accretive mapping and $M : \mathcal{B} \to 2^{\mathcal{B}}$ be an (A, η)-accretive mapping. Then, the resolvent operator $R^{\rho,A}_{\eta,M} : \mathcal{B} \to \mathcal{B}$ is $\dfrac{\tau^{q-1}}{r - \rho m}$-Lipschitz continuous, i.e.,

$$\left\| R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y) \right\| \le \frac{\tau^{q-1}}{r - \rho m} \|x - y\|, \quad \forall x, y \in \mathcal{B},$$

where $\rho \in \left(0, \dfrac{r}{m}\right)$ is a constant.

Definition 2.5. Let $M^n, M : \mathcal{B} \to 2^{\mathcal{B}}$ be (A, η)-accretive mappings on $\mathcal{B}$ for $n = 0,1,$ $2,\ldots$. Let $A : \mathcal{B} \to \mathcal{B}$ be r-strongly η-monotone and β-Lipschitz continuous. The sequence M^n is graph-convergent to M, denoted $M^n \xrightarrow{A-G} M$, if for every $(x, y) \in$ $graph(M)$, there exists a sequence $(x_n, y_n) \in graph(M^n)$ such that

$$x_n \to x, \quad y_n \to y \quad \text{as} \quad n \to \infty.$$

Based on Definition 2.6 and Theorem 2.1 in [20], we have the following lemma.

Lemma 2.3. Let $M^n, M : \mathcal{B} \to 2^{\mathcal{B}}$ be (A, η)-accretive mappings on $\mathcal{B}$ for $n = 0,1,$ $2,\ldots$. Then, the sequence $M^n \xrightarrow{A-G} M$ if and only if

$$R^{\rho,A}_{\eta,M^n}(x) \to R^{\rho,A}_{\eta,M}(x), \forall x \in \mathcal{B},$$

where $R^{\rho,A}_{\eta,M} = (A + \rho M^n)^{-1}, R^{\rho,A}_{\eta,M} = (A + \rho M)^{-1}, \rho > 0$ is a constant, and $A : \mathcal{B} \to \mathcal{B}$ is r-strongly η-monotone and β-Lipschitz continuous.

3 Algorithms and graphical convergence

In this section, by using resolvent operator technique associated with (A, η)-accretive mappings, we shall develop a new perturbed iterative algorithm framework with errors for solving the nonlinear operator equation system (1.1) with (A, η)-accretive mappings and relaxed cocoercive operators and prove the existence of solutions and the variational convergence of the sequence generated by the perturbed iterative algorithm in q-uniformly smooth Banach spaces.

Above all, we note that the equalities (1.1) can be written as

$$\begin{cases} p(x) = R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x)}(s), \\ s = (1-\lambda_1)A_1(p(x)) + \lambda_1(A_1(f(y)) - \rho N_1(u, y) + a), \\ h(y) = R^{\varrho\lambda_2, A_2}_{\eta_2, M_2(y, \cdot)}(t), \\ t = (1-\lambda_2)A_2(h(y)) + \lambda_2(A_2(g(x)) - \varrho N_2(x, v) + b), \end{cases}$$

where ρ, $\lambda > 0$ are constants. This formulation allows us to construct the following perturbed iterative algorithm framework with errors.

Algorithm 3.1. *Step* 1. For an arbitrary initial point $(x_0, y_0) \in \mathcal{B}_1 \times \mathcal{B}_2$, take $u_0 \in S(x_0)$ and $v_0 \in T(y_0)$.

Step 2. Choose sequences $\{d_n\} \subset \mathcal{B}_1$ and $\{e_n\} \subset \mathcal{B}_2$ are two error sequences to take into account a possible inexact computation of the operator points, which satisfy the following conditions:

$$\lim_{n\to\infty} d^n = \lim_{n\to\infty} e^n = 0, \sum_{n=1}^{\infty} \left(\|d^n - d^{n-1}\| + \|e^n - e^{n-1}\| \right) < \infty.$$

Step 3. Let the sequence $\{(s_n, t_n, x_n, y_n)\} \subset \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_1 \times \mathcal{B}_2$ satisfy

$$\begin{cases} s_n = (1-\lambda_1)A_1(p(x_n)) + \lambda_1(A_1(f(y_n)) - \rho N_1(u_n, y_n) + a), \\ t_n = (1-\lambda_2)A_2(h(y_n)) + \lambda_2(A_2(g(x_n)) - \varrho N_2(x_n, v_n) + b), \\ x_{n+1} = (1-k)x_n + k\{x_n - p(x_n) + R^{\rho\lambda_1, A_1}_{\eta_1, M_1^n(\cdot, x_n)}(s_n)\} + d_n, \\ y_{n+1} = (1-\kappa)y_n + \kappa\{y_n - h(y_n) + R^{\varrho\lambda_2, A_2}_{\eta_2, M_2^n(y_n, \cdot)}(t_n)\} + e_n, \end{cases} \tag{3.1}$$

where $R^{\rho\lambda_1, A_1}_{\eta_1, M_1^n(\cdot, x)} = (A_1 + \rho\lambda_1 M_1^n(\cdot, x))^{-1}, R^{\varrho\lambda_2, A_2}_{\eta_2, M_2^n(y, \cdot)} = (A_2 + \varrho\lambda_2 M_2^n(y, \cdot))^{-1}, \lambda_1, \lambda_2, \rho, \varrho$ are nonnegative constants and k, $\kappa \in (0, 1]$ are size constants.

Step 4. Choose $u_{n+1} \in S(x_{n+1})$ and $v_{n+1} \in T(y_{n+1})$ such that (see [22])

$$\begin{aligned} \|u_n - u_{n+1}\| &\le \left(1 + \frac{1}{n+1}\right) \hat{\mathbf{H}}\left(S(x_n), S(x_{n+1})\right), \\ \|v_n - v_{n+1}\| &\le \left(1 + \frac{1}{n+1}\right) \hat{\mathbf{H}}\left(T(y_n), T(y_{n+1})\right). \end{aligned} \tag{3.2}$$

Step 5. If s_n, t_n, x_n, y_n, d_n, and e_n satisfy (3.1) and (3.2) to sufficient accuracy, stop; otherwise, set $n: = n + 1$ and return to *Step* 2.

Now, we prove the existence of a solution of problem (1.1) and the convergence of Algorithm 3.1.

Theorem 3.1. For $i = 1, 2$, let $\mathcal{B}_i$ be a q_i-uniformly smooth Banach space with $q_i > 1$, η_i, A_i, M_i, N_i ($i = 1, 2$) and p, h, f, g be the same as in the Equation (1.1). Also suppose that the following conditions hold:

(H_1) η_i is τ_i -Lipschitz continuous, and A_i is r_i -strongly η_i-accretive, and σ_i-Lipschitz continuous for $i = 1, 2$, respectively;

(H_2) p is δ_1-strongly accretive and l_p-Lipschitz continuous, h is δ_2-strongly accretive and l_h-Lipschitz continuous, f is l_f-Lipschitz continuous and g is l_g-Lipschitz continuous, $S : \mathcal{B}_1 \to CB(\mathcal{B}_1)$ is $\xi - \hat{\mathbf{H}}$-Lipschitz continuous and $T : \mathcal{B}_2 \to CB(\mathcal{B}_2)$ is $\zeta - \hat{\mathbf{H}}$-Lipschitz continuous;

(H_3) N_1 is (π_1, ι_1)-relaxed cocoercive with respect to f_1 and ϖ_2-Lipschitz continuous in the second argument, and N_2 is (π_2, ι_2)-relaxed cocoercive with respect to g_2 and ϖ_1-Lipschitz continuous in the first argument, and N_1 is β_1-Lipschitz continuous in the first variable, and N_2 is β_2-Lipschitz continuous in the second variable, where $f_1 : \mathcal{B}_2 \to \mathcal{B}_1$ is defined by $f_1(y) = A_1 \circ f(y) = A_1(f(y))$ for all $y \in \mathcal{B}_2$ and $g_2 : \mathcal{B}_1 \to \mathcal{B}_2$ is defined by $g_2(x) = A_2 \circ g(x) = A_2(g(x))$ for all $x \in \mathcal{B}_1$;

(H_4) for $n = 0, 1, 2, \ldots, M_i^n : \mathcal{B}_i \times \mathcal{B}_i \to 2^{\mathcal{B}_i} (i = 1, 2)$ are any nonlinear operators such that for all $x \in \mathcal{B}_1, M_1^n(\cdot, x) : \mathcal{B}_1 \to 2^{\mathcal{B}_1}$ is an (A_1, η_1)-accretive mapping with $M_1^n(\cdot, x)\underrightarrow{A_1 - G}M_1(\cdot, x)$, and $M_2^n(y, \cdot) : \mathcal{B}_2 \to 2^{\mathcal{B}_2}$ is an (A_2, η_2)-accretive mapping with $M_2^n(y, \cdot)\underrightarrow{A_2 - G}M_2(y, \cdot)$ for all $y \in \mathcal{B}_2$, respectively;

(H_5) there exist constants $\nu_i (i = 1,2)$, $\rho \in (0, r_1/m_1)$ and $\varrho \in (0, r_2/m_2)$ such that

$$\begin{array}{ll} \left\| R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x)}(z) - R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, y)}(z) \right\| \le \nu_2 \left\| x - y \right\|, & \forall x, y, z \in \mathcal{B}_1, \\ \left\| R^{\varrho\lambda_2, A_2}_{\eta_2, M_2(x, \cdot)}(z) - R^{\varrho\lambda_2, A_2}_{\eta_2, M_2(y, \cdot)}(z) \right\| \le \nu_1 \left\| x - y \right\|, & \forall x, y, z \in \mathcal{B}_2, \end{array} \tag{3.3}$$

and

$$\begin{cases} \nu_2 + \sqrt[q_1]{1 - q_1\delta_1 + c_{q_1} l_p^{q_1}} + \dfrac{\tau_1^{q_1 - 1}[(1 - \lambda_1)\sigma_1 l_p + \rho\lambda_1\beta_1\xi]}{r_1 - \rho\lambda_1 m_1} \\ \qquad + \dfrac{\kappa\lambda_2\tau_2^{q_2 - 1} \sqrt[q_2]{\sigma_2^{q_2} l_g^{q_2} - q_2\varrho\iota_2\varpi_1^{q_2} + q_2\varrho\pi_2 + c_{q_2}\varrho^{q_2}\varpi_1^{q_2}}}{k(r_2 - \varrho\lambda_2 m_2)} < 1, \\ \nu_1 + \sqrt[q_2]{1 - q_2\delta_2 + c_{q_2} l_h^{q_2}} + \dfrac{\tau_2^{q_2 - 1}[(1 - \lambda_2)\sigma_2 l_h + \varrho\lambda_2\beta_2\zeta]}{r_2 - \varrho\lambda_2 m_2} \\ \qquad + \dfrac{k\lambda_1\tau_1^{q_1 - 1} \sqrt[q_1]{\sigma_1^{q_1} l_f^{q_1} - q_1\rho\iota_1\varpi_2^{q_1} + q_1\rho\pi_1 + c_{q_1}\rho^{q_1}\varpi_2^{q_1}}}{\kappa(r_1 - \rho\lambda_1 m_1)} < 1 \end{cases} \tag{3.4}$$

where c_{q_1}, c_{q_2} are the constants as in Lemma 2.1 and k, $\kappa \in (0,1]$ are size constants.

Then, there exist $(x^*, y^*) \in \mathcal{B}_1 \times \mathcal{B}_2$ $u^* \in S(x^*), v^* \in T(y^*)$ such that (x^*,y^*,u^*,v^*) is a solution of the Equation (1.1) and

$$x_n \to x^*, y_n \to y^*, u_n \to u^*, v_n \to v^*, \quad \text{as} \quad n \to \infty,$$

where $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are iterative sequences generated by Algorithm 3.1.

Proof. Define $||\cdot||_*$ on $\mathcal{B}_1 \times \mathcal{B}_2$ by

$$\left\|(x, y)\right\|_* = \|x\| + \left\|y\right\|, \quad \forall (x, y) \in \mathcal{B}_1 \times \mathcal{B}_2.$$

It is easy to see that $(\mathcal{B}_1 \times \mathcal{B}_2, \|\cdot\|_*)$ is a Banach space. By the assumptions for relaxed cocoercivity and Lipschitz continuity of N with respect to both arguments, strongly accretivity of p and h, and Lipschitz continuity of S, T, p, f, g and h, Lemmas 2.1 and 2.2, and (3.1)-(3.3), now we know that

$$
\begin{aligned}
&\left\|N_1(u_n, y_{n-1}) - N_1(u_{n-1}, y_{n-1})\right\| \le \beta_1 \|u_n - u_{n-1}\| \\
&\quad \le \beta_1(1+n^{-1})\hat{\mathbf{H}}(S(x_n), S(x_{n-1})) \le \beta_1\xi(1+n^{-1})\|x_n - x_{n-1}\|, \\
&\left\|A_1(f(y_n)) - A_1(f(y_{n-1})) - \rho\left[N_1(u_n, y_n) - N_1(u_n, y_n)\right]\right\|^{q_1} \\
&\quad \le \left\|A_1(f(y_n)) - A_1(f(y_{n-1}))\right\|^{q_1} + \rho^{q_1} c_{q_1}\left\|N_1(u_n, y_n) - N_1(u_n, y_{n-1})\right\|^{q_1} \\
&\qquad - q_1\rho\langle N_1(u_n, y_n) - N_1(u_n, y_{n-1}), J_{q_1}(A_1(f(y_n)) - A_1(f(y_{n-1})))\rangle \\
&\quad \le \left(\sigma_1^{q_1} l_f^{q_1} - q_1\rho\iota_1\varpi_2^{q_1} + q_1\rho\pi_1 + c_{q_1}\rho^{q_1}\varpi_2^{q_1}\right)\|y_n - y_{n-1}\|^{q_1}, \\
&\left\|x_n - x_{n-1} - \left[p(x_n) - p(x_{n-1})\right]\right\| \le \sqrt[q_1]{1 - q_1\delta_1 + c_{q_1} l_p^{q_1}}\,\|x_n - x_{n-1}\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|s_n - s_{n-1}\| \\
&= \big\|(1-\lambda_1)A_1(p(x_n)) + \lambda_1(A_1(f(y_n)) - \rho N_1(u_n, y_n) + a) \\
&\quad -(1-\lambda_1)A_1(p(x_{n-1})) - \lambda_1(A_1(f(y_{n-1})) - \rho N_1(u_{n-1}, y_{n-1}) + a)\big\| \\
&\le (1-\lambda_1)\left\|A_1(p(x_n)) - A_1(p(x_{n-1}))\right\| \\
&\quad + \rho\lambda_1\left\|N_1(u_n, y_{n-1}) - N_1(u_{n-1}, y_{n-1})\right\| \\
&\quad + \lambda_1\left\|A_1(f(y_n)) - A_1(f(y_{n-1})) - \rho[N_1(u_n, y_n) - N_1(u_n, y_{n-1})]\right\| \\
&\le \left[(1-\lambda_1)\sigma_1 l_p + \rho\lambda_1\beta_1\xi(1+n^{-1})\right]\|x_n - x_{n-1}\| \\
&\quad + \lambda_1\sqrt[q_1]{\sigma_1^{q_1} l_f^{q_1} - q_1\rho\iota_1\varpi_2^{q_1} + q_1\rho\pi_1 + c_{q_1}\rho^{q_1}\varpi_2^{q_1}}\,\|y_n - y_{n-1}\|,
\end{aligned}
$$

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\le (1-k)\|x_n - x_{n-1}\| + k\left\|x_n - x_{n-1} - \left[p(x_n) - p(x_{n-1})\right]\right\| \\
&\quad + k\left\|R^{\rho\lambda_1, A_1}_{\eta 1, M_1^n(\cdot, x_n)}(s_n) - R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_n)}(s_n)\right\| + \|d_n - d_{n-1}\| \\
&\quad + k\left\|R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_n)}(s_n) - R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_{n-1})}(S_n)\right\| \\
&\quad + k\left\|R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_{n-1})}(s_n) - R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_{n-1})}(s_{n-1})\right\| \\
&\quad + k\left\|R^{\rho\lambda_1, A_1}_{\eta_1, M_1^{n-1}(\cdot, x_{n-1})}(s_{n-1}) - R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_{n-1})}(s_{n-1})\right\| \\
&\le (1-k)\|x_n - x_{n-1}\| + k\left\|x_n - x_{n-1} - [p(x_n) - p(x_{n-1})]\right\| \\
&\quad + k\nu_2\|x_n - x_{n-1}\| + \frac{k\tau_1^{q_1 - 1}}{r_1 - \rho\lambda_1 m_1}\|s_n - s_{n-1}\| \\
&\quad + k(\varepsilon_n + \varepsilon_{n-1}) + \|d_n - d_{n-1}\| \\
&\le [1 - k(1 - \theta_{1n})]\|x_n - x_{n-1}\| + k\vartheta_1\left\|y_n - y_{n-1}\right\| \\
&\quad + k(\varepsilon_n + \varepsilon_{n-1}) + \|d_n - d_{n-1}\|,
\end{aligned}
\tag{3.5}
$$

where $\varepsilon_l = \left\|R^{\rho\lambda_1, A_1}_{\eta_1, M_1^l(\cdot, x_l)}(s_l) - R^{\rho\lambda_1, A_1}_{\eta_1, M_1(\cdot, x_l)}(s_l)\right\|$ for $l = n - 1, n$ and

$$
\begin{aligned}
\theta_{1,n} &= \nu_2 + \sqrt[q_1]{1 - q_1\delta_1 + c_{q_1} l_p^{q_1}} + \frac{\tau_1^{q_1 - 1}(1-\lambda_1)\sigma_1 l_p + \rho\lambda_1\beta_1\xi(1+n^{-1})}{r_1 - \rho\lambda_1 m_1}, \\
\vartheta_1 &= \frac{\lambda_1\tau_1^{q_1 - 1}\sqrt[q_1]{\sigma_1^{q_1} l_f^{q_1} - q_1\rho\iota_1\varpi_2^{q_1} + q_1\rho\pi_1 + c_{q_1}\rho^{q_1}\varpi_2^{q_1}}}{r_1 - \rho\lambda_1 m_1}
\end{aligned}
$$

Similarly, we get

$$\begin{aligned}\left\|y_{n+1}-y_n\right\| \le \left[1-\kappa\left(1-\theta_{2n}\right)\right]\left\|y_n-y_{n-1}\right\| + \kappa\vartheta_2\left\|x_n-x_{n-1}\right\| \\ + \kappa\left(\varepsilon_n+\varepsilon_{n-1}\right) + \left\|e_n-e_{n-1}\right\|,\end{aligned} \tag{3.6}$$

where $\varepsilon_l = \left\|R^{\varrho\lambda_2,A_2}_{\eta_2,M_2^l(y_l,\cdot)}(t_l) - R^{\varrho\lambda_2,A_2}_{\eta_2,M_2(y_l,\cdot)}(t_l)\right\|$ for $l = n - 1$, n and

$$\theta_{2,n} = \nu + \sqrt[q_2]{1-q_2\delta_2+c_{q_2}l_h^{q_2}} + \frac{\tau_2^{q_2-1}\left[(1-\lambda_2)\sigma_2 l_h + \varrho\lambda_2\beta_2\zeta\left(1+n^{-1}\right)\right]}{r_2-\varrho\lambda_{2m2}},$$

$$\vartheta_2 = \frac{\lambda_2\tau_2^{q_2-1}\sqrt[q_2]{\sigma_2^{q_2}l_g^{q_2} - q_2\varrho\iota_2\varpi_1^{q_2} + q_2\varrho\pi_2 + c_{q_2}\varrho^{q_2}\varpi_1^{q_2}}}{r_2-\varrho\lambda_2 m_2}.$$

follows from (3.5) and (3.6) that

$$\begin{aligned}&\left\|x_{n+1}-x_n\right\| + \left\|y_{n+1}-y_n\right\| \\ &\le \theta_n\left(\left\|x_n-x_{n-1}\right\| + \left\|y_n-y_{n-1}\right\|\right) \\ &\quad + k\left(\varepsilon_n+\varepsilon_{n-1}\right) + \kappa\left(\varepsilon_n+\varepsilon_{n-1}\right) + \left(\left\|d_n-d_{n-1}\right\| + \left\|e_n-e_{n-1}\right\|\right),\end{aligned} \tag{3.7}$$

where

$$\theta_n = \max\{1+\kappa\vartheta_2 - k(1-\theta_{1,n}), 1+k\vartheta_1 - \kappa(1-\theta_{2,n})\}.$$

Let

$$\theta = \max\{1+\kappa\vartheta_2 - k(1-\theta_1), 1+k\vartheta_1 - \kappa(1-\theta_2)\},$$

where

$$\theta_1 = \nu_2 + \sqrt[q_1]{1-q_1\delta_1+c_{q_1}l_p^{q_1}} + \frac{\tau_1^{q_1-1}\left[(1-\lambda_1)\sigma_1 l_p + \rho\lambda_1\beta_1\xi\right]}{r_1-\rho\lambda_1 m_1},$$

$$\theta_2 = \nu_1 + \sqrt[q_2]{1-q_2\delta_2+c_{q_2}l_h^{q_2}} + \frac{\tau_2^{q_2-1}\left[(1-\lambda_2)\sigma_2 l_h + \varrho\lambda_2\beta_2\zeta\right]}{r_2-\varrho\lambda_2 m_2}.$$

Then, we know that $\theta_n \downarrow \theta$ as $n \to \infty$.

From the condition (3.4), we know that $0 < \theta < 1$ and so there exist $n_0 > 0$ and $\theta_0 \in (\theta, 1)$ such that $\theta_n \le \theta_0$ for all $n \ge n_0$. Therefore, by (3.7), we have

$$\begin{aligned}&\left\|(x_{n+1},y_{n+1})-(x_n,y_n)\right\|_* \\ &\le \theta_0\left\|(x_n,y_n)-(x_{n-1},y_{n-1})\right\|_* + \left(\left\|d_n-d_{n-1}\right\| + \left\|e_n-e_{n-1}\right\|\right) \\ &+k\left(\varepsilon_n+\varepsilon_{n-1}\right) + \kappa\left(\varepsilon_n+\varepsilon_{n-1}\right)\end{aligned}$$

$$\le \theta_0^{n-n_0}\left\|(x_{n_0+1},y_{n_0+1})-(x_{n_0},y_{n_0})\right\|_* + \sum_{i=1}^{n-n_0}\theta_0^{i-1}\varsigma_{n-(i-1)} \tag{3.8}$$

where $\varsigma_n = ||d_n - d_{n-1}|| + ||e_n - e_{n-1}|| + k(\epsilon_n + \epsilon_{n-1}) + \kappa(\varepsilon_n + \varepsilon_{n-1})$ for all $n \ge n_0$. By (3.8), for any $m \ge n > n_0$, we have

$$
\begin{aligned}
&\left\|(x_m, y_m) - (x_n, y_n)\right\|_* \\
&\leq \sum_{j=n}^{m-1} \left(\left\|x_{j+1} - x_j\right\| + \left\|y_{j+1} - y_j\right\|\right) \\
&\leq \sum_{j=n}^{m-1} \theta_0^{j-n_0} \left\|(x_{n_0+1}, y_{n_0+1}) - (x_{n_0}, y_{n_0})\right\|_* + \sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0} \theta_0^{i-1} \varsigma_{j-(i-1)}.
\end{aligned}
\tag{3.9}
$$

It follows from the hypothesis of Algorithm 3.1, Lemma 2.3 and (3.9) that

$$\lim_{n\to\infty} \left\|(x^m, y^m) - (x^n, y^n)\right\|_* = 0.$$

Hence, $\{(x^n, y^n)\}$ is a Cauchy sequence, i.e., there exists $(x^*, y^*) \in \mathcal{B}_1 \times \mathcal{B}_2$ such that $(x^n, y^n) \to (x^*, y^*)$ as $n \to \infty$.

Next, we prove that $u_n \to u^* \in S(x^*)$ and $v_n \to v^* \in T(y^*)$. In fact, because

$$\|u_n - u_{n-1}\| \leq (1 + n^{-1})\hat{\mathbf{H}}\left(S(x_n), S(x_{n-1})\right) \leq \xi(1 + n^{-1})\,\|x_n - x_{n-1}\|,$$

it follows that $\{u_n\}$ is also Cauchy sequence in $\mathcal{B}_1$. Let $u_n \to u^*$. In the sequel, we will show that $u^* \in S(x^*)$. Noting $u_n \in S(x_n)$, from the results in [22], we have

$$
\begin{aligned}
d(u^*, S(x^*)) = \inf\left\{\left\|u_n - y\right\| : y \in S(x^*)\right\} &\leq \left\|u^* - u_n\right\| + d(u_n, S(x^*)) \\
&\leq \left\|u^* - u_n\right\| + \hat{\mathbf{H}}\left(S(x_n), S(x^*)\right) \\
&\leq \left\|u^* - u_n\right\| + \xi\left\|x_n - x^*\right\| \to 0.
\end{aligned}
$$

Hence $d(u^*,S(x^*)) = 0$ and therefore $u^* \in S(x^*)$. Similarly, we have $v_n \to v^* \in T(y^*)$. By continuity and the hypothesis of Algorithm 3.1, we know that (x^*, y^*, u^*, v^*) satisfies the Equation (1.1). This completes the proof.

Remark 3.1. We note that Hilbert space and L_p (or l_p) ($2 \leq p < \infty$) spaces are 2-uniformly smooth Banach spaces and if $\mathcal{B}_i (i = 1, 2)$ is 2-uniformly smooth Banach space, we can choose constants v_i, $\lambda_i (i = 1,2)$, ρ and ϱ such that (3.4) hold. See, for example, [2-18] and the references therein.

Remark 3.2. Condition (3.4) of Theorem 3.1 holds for some suitable value of constants, for example, $q_1 = q_2 = 2$, $c_2 = 1$, $v_1 = v_2 = 0.02$, $\delta_1 = \delta_2 = 0.3$, $l_p = l_h = 0.6$, $\tau_1 = \tau_2 = 0.05$, $\lambda_1 = \lambda_2 = 0.01$, $\sigma_1 = \sigma_2 = 0.5$, $\rho = \varrho = 0.1$, $\beta_1 = \beta_2 = 0.05$, $\xi = 0.7$, $\zeta = 0.4$, $r_1 = r_2 = 0.3$, $m_1 = m_2 = 0.2$, $k = \kappa = 0.5$, $l_f = 0.2$, $l_g = 0.4$, $\iota_1 = \iota_2 = 0.05$, $\varpi_1 = \varpi_2 = 0.05$ and $\pi_1 = \pi_2 = 0.2$.

From Theorem 3.1, we have the following results as an application of Theorem 3.1.

Theorem 3.2. Assume that $\mathcal{H}$ is a real Hilbert space and the following conditions hold:

(H_1) $h : \mathcal{H} \to \mathcal{H}$ is δ-strongly monotone and l_h-Lipschitz continuous, $f : \mathcal{H} \to \mathcal{H}$ is l_f-Lipschitz continuous and $g : \mathcal{H} \to \mathcal{H}$ is l_g-Lipschitz continuous;
(H_2) $N : \mathcal{H} \to \mathcal{H}$ is (π_1, ι_1)-relaxed cocoercive with respect to f and ϖ-Lipschitz continuous, and (π_2, ι_2)-relaxed cocoercive with respect to g;
(H_3) for $i = 1,2$ and $n = 0, 1, 2, \ldots, M_i^n, M_i : \mathcal{H} \to 2^{\mathcal{H}}$ are maximal monotone operators with $M_i^n \underrightarrow{A_1 - G} M_i$;

(H_4) there exist positive constants ρ and ϱ such that

$$\sqrt{1-2\delta+l_h^2} < \min\left\{1-\frac{\kappa}{k}\sqrt{l_g^2-2\varrho\iota_2\varpi^2+2\varrho\pi_2+\varrho^2\varpi^2}\,,\right. \left. 1-\frac{k}{\kappa}\sqrt{l_f^2-2\rho\iota_1\varpi^2+2\rho\pi_1+\rho^2\varpi^2}\right\}.$$

Then, the iterative sequences $\{(x_n, y_n)\}$ generated as follows converges strongly to the common solution (x^*, y^*) of the system (1.5):

For any given $(x_0, y_0) \in \mathcal{H}\times\mathcal{H}$, define an iterative sequence as follows:

$$\begin{cases} x_{n+1} = (1-k)x_n + k\{x_n - h(x_n) + J_{M_1^n}^{\rho}[f(y_n - \rho N(y_n))]\} + d_n, \\ y_{n+1} = (1-\kappa)y_n + \kappa\{y_n - h(y_n) + J_{M_2^n}^{\varrho}[g(x_n) - \varrho N(x_n)]\} + e_n, \end{cases} \tag{3.10}$$

where $J_{M_1^n}^{\rho} = (I+\rho M_1^n)^{-1}, J_{M_2^n}^{\varrho} = (I+\varrho M_2^n)^{-1}, \rho,\varrho > 0, k,\kappa \in (0,1), \{d_n\} \subset \mathcal{H}$ and $\{e_n\} \subset \mathcal{H}$ are two error sequences to take into account a possible inexact computation of the operator points, which satisfy the following conditions:

$$\lim_{n\to\infty} d^n = \lim_{n\to\infty} e^n = 0, \sum_{n=1}^{\infty}\left(\|d^n - d^{n-1}\| + \|e^n - e^{n-1}\|\right) < \infty.$$

Proof. By the nonexpansivity of the resolvent operators associated withmaximal monotone operators and the proof of Theorem 3.1, one can derivethe result.

Remark 3.3. We note that one can obtain the corresponding results of Theorems 3.1-3.2 when there are problems (1.1), (1.3)-(1.5) with (H, η)-accretive mappings, (A, η)-monotone operators, (H, η)-monotone operators, H-accretive operators, generalized m-accretive operators, maximal η-monotone operators, H-monotone operators, A-monotone operators, η-subdifferential operators or the classical m-accretive. The results obtained in this paper improve and generalize the corresponding results of [2,3,5,12,14,17,18] and many other recent works.

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Author details

[1]Department of Mathematics, Sichuan University of Science and Engineering, Zigong, 643000, Sichuan, People's Republic of China [2]Department of Mathematics Education and the RINS, College of Education, Gyeongsang National University, Chinju 660-701, Korea

Authors' contributions

FL carried out the proof of convergence of the theorems and gave some examples to show the main results. H-YL conceived of the study, and participated in its design and coordination. YJC carried out the check of the manuscript and participated in the design of the study. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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