Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong convergence for total quasi-ϕ-asymptotically nonexpansive semigroups in Banach spaces

Jing Quan[1*], Shih-sen Chang[2] and Xiongrui Wang[1]

*Correspondence: quanjingcq@163.com
[1]Department of Mathematics, Yibin University, Yibin, Sichuan, 644000, China
Full list of author information is available at the end of the article

Abstract
The purpose of this article is to use the modified Halpern-Mann type iteration algorithm for total quasi-ϕ-asymptotically nonexpansive semigroups to prove strong convergence in Banach spaces. The main results presented in this paper extend and improve the corresponding results of many authors.
MSC: 47H05; 47H09; 49M05

Keywords: strong convergence; total quasi-ϕ-asymptotically nonexpansive semigroups; generalized projection

1 Introduction

Throughout this article, we assume that E is a real Banach space with norm $\|\cdot\|$, E^* is the dual space of E; $\langle\cdot,\cdot\rangle$ is the duality pairing between E and E^*; C is a nonempty closed convex subset of E; $\mathbb{N}$ and $\mathbb{R}$ denote the natural number set and the set of nonnegative real numbers respectively. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \left\{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2; \|f^*\| = \|x\|, x \in E\right\}$$

is called *the normalized duality mapping*. Let $T : C \to C$ be a nonlinear mapping; $F(T)$ denotes the set of fixed points of mapping T.

Alber *et al.* [1] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings and studied the methods of approximation of fixed points. They are defined as follows.

Definition 1.1 Let $T : C \to C$ be a mapping. T is said to be total asymptotically nonexpansive if there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ such that $\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n$ holds for all $x, y \in C$ and all $n \in \mathbb{N}$.

T is said to be *total asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ such that $\|T^n x - p\| \leq \|x - p\| + \mu_n \psi(\|x - p\|) + \nu_n$ holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$.

Chidume and Ofoedu [2] introduced an iterative scheme for approximation of a common fixed point of a finite family of total asymptotically nonexpansive mappings and total asymptotically quasi-nonexpansive mappings in Banach spaces. Chidume *et al.* [3] gave a new iterative sequence and necessary and sufficient conditions for this sequence to converge to a common fixed point of finite total asymptotically nonexpansive mappings. Chang [4] established some new approximation theorems of common fixed points for a countable family of total asymptotically nonexpansive mappings in Banach spaces.

Recently, many researchers have focused on studying the convergence of iterative algorithms for quasi-ϕ-asymptotically nonexpansive (see [5–9]) and total quasi-ϕ-asymptotically nonexpansive (see [10–12]) mappings. Ye *et al.* [13] used a new hybrid projection algorithm to obtain strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces. Kim [14] used hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings to prove the strong convergence theorems. Saewan [15] used the shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-ϕ-nonexpansive mappings.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$; it is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \frac{\|x_n+y_n\|}{2} = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E, then the Banach space E is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|y\|}{t}$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for each $x, y \in U$. It is well known that if E is reflexive and smooth, then the duality mapping J is single valued. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfies that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to x$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property.

In the sequel, we assume that E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. We use $\phi : E \times E \to R^+$ to denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

It is obvious that

$$\bigl(\|x\| - \|y\|\bigr)^2 \leq \phi(x,y) \leq \bigl(\|x\| + \|y\|\bigr)^2, \quad \forall x, y \in E, \tag{1}$$

and

$$\phi\bigl(x, J^{-1}\bigl(\lambda Jy + (1-\lambda)Jz\bigr)\bigr) \leq \lambda\phi(x,y) + (1-\lambda)\phi(x,z). \tag{2}$$

Following Alber [16], the generalized projection $\Pi_C x : E \to C$ is defined by

$$\Pi_C x = \arg\inf_{y\in C} \phi(y,x), \quad \forall x \in E.$$

The quasi-ϕ-asymptotically nonexpansive and total quasi-ϕ-asymptotically nonexpansive mappings are defined as follows.

Definition 1.2 A mapping $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$, there exist sequences $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x)$$

holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$.

A mapping $T : C \to C$ is said to be total quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$, there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \mu_n \psi(\phi(p, x)) + \nu_n$$

holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$.

In recent years, many researchers have considered the convergence of asymptotically nonexpansive semigroups [17, 18]. The asymptotically nonexpansive semigroups are defined as follows.

Definition 1.3 [17] One-parameter family $\mathbf{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is said to be an asymptotically nonexpansive semigroup on C, if the following conditions are satisfied:

(a) $T(0)x = x$ for each $x \in C$;
(b) $T(t + s)x = T(s)T(t)$ for any $t, s \in R^+$ and $x \in C$;
(c) For any $x \in C$, the mapping $t \to T(t)x$ is continuous;
(d) There exist sequences $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n(t)x - T^n(t)y\| \leq k_n \|x - y\|$$

holds for all $x, y \in C$, $n \in \mathbb{N}$.

We use $F(T)$ to denote the common fixed point set of the semigroup $\mathbf{T}$, *i.e.*, $F(T) = \bigcap_{t \geq 0} F(T(t))$.

Chang [19] used the modified Halpern-Mann type iteration algorithm for quasi-ϕ-asymptotically nonexpansive semigroups to prove the strong convergence in the Banach space. The quasi-ϕ-asymptotically nonexpansive semigroups are defined as follows.

Definition 1.4 [19] One-parameter family $\mathbf{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is said to be a quasi-ϕ-asymptotically nonexpansive semigroup on C if the conditions (a), (b), (c) in Definition 1.3 and following condition (e) are satisfied:

(e) For all $x, y \in C$, $p \in F((T))$, $t \geq 0$, there exist sequences $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$, such that

$$\phi(p, T^n(t)x) \leq k_n \phi(p, x)$$

holds for all $n \in \mathbb{N}$.

2 Preliminaries

This section contains some definitions and lemmas which will be used in the proofs of our main results in the following section.

Definition 2.1 One-parameter family $\mathbf{T} := \{T(t) : t \geq 0\}$ of mappings from C into itself is said to be a total quasi-ϕ-asymptotically nonexpansive semigroup on C if conditions (a), (b), (c) in Definition 1.3 and following condition (f) are satisfied:

(f) If $F(T) \neq \emptyset$, there exist sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n, \nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ such that

$$\phi\big(p, T^n(t)x\big) \leq \phi(p, x) + \mu_n \psi\big(\phi(p, x)\big) + \nu_n$$

holds for all $x \in C$, $p \in F(T)$ and all $n \in \mathbb{N}$.

A total quasi-ϕ-asymptotically nonexpansive semigroup $\mathbf{T}$ is said to be uniformly Lipschitzian if there exists a bounded measurable function $L : [0, \infty) \to (0, +\infty)$ such that

$$\big\| T^{(n)}(t)x - T^{(n)}(t)y \big\| \leq L(t)\|x - y\|, \quad \forall x, y \in C, t \geq 0, n \in \mathbb{N}.$$

The purpose of this article is to use the modified Halpern-Mann type iteration algorithm for total quasi-ϕ-nonexpansive asymptotically semigroups to prove the strong convergence in Banach spaces. The results presented in the article improve and extend the corresponding results of [5, 6, 9–12, 14, 15, 19] and many others.

In order to prove the results of this paper, we shall need the following lemmas:

Lemma 2.1 (See [16]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ *for all* $x \in C$, $y \in E$;

(ii) If $x \in E$ *and* $z \in C$, *then* $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0$, $\forall y \in C$;

(iii) For $x, y \in E$, $\phi(x, y) = 0$ *if and only if* $x = y$.

Lemma 2.2 [19] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3 [10] *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping defined by Definition 1.2. If $\nu_1 = 0$, then the fixed point set $F(T)$ of T is a closed and convex subset of C.*

3 Main results

Theorem 3.1 *Let E be a real uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\mathbf{T} := \{T(t) : t \geq 0\}$ be a total quasi-ϕ-asymptotically nonexpansive semigroup from C into itself defined by Definition 2.1. Suppose $\mathbf{T} := \{T(t) : t \geq 0\}$ is closed, uniformly L-Lipschitz and $F(T) := \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. Suppose there exists $M^* > 0$ such that $\psi(\eta_n) \leq M^* \eta_n$. Let α_n be a sequence in $[0,1]$ and β_n be* *a sequence in* $(0,1)$ *satisfying the following conditions:* $\lim_{n\to\infty}\alpha_n = 0$, $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$. *Let* x_n *be a sequence generated by*

$$\begin{cases} x_1 \in E, \quad \text{chosen arbitrarily}; \qquad C_1 = C, \\ l_{n,t} = \beta_n Jx_n + (1-\beta_n)JT^n(t)x_n, \\ y_{n,t} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)l_{n,t}], \quad t \geq 0, \\ C_{n+1} = \{z \in C_n : \sup_{t\geq 0}\phi(z,y_{n,t}) \leq \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_1, \quad \forall n \geq 1, \end{cases} \tag{3}$$

where $\xi_n = \mu_n M^* \sup_{p\in F(T)}\phi(p,x_n)$. *If* $\nu_1 = 0$ *and* $F(T)$ *is bounded in* C, *then the iterative sequence* $\{x_n\}$ *converges strongly to a common fixed point* $x^* \in F(T)$ *in* C.

Proof

(I) We prove $F(T)$ and $C_n (n \in \mathbb{N})$ all are closed and convex subsets in C.

It follows from Lemma 2.3 that $F(T(t))$, $t \geq 0$ is a closed and convex subset of C. So $F(T)$ is closed and convex in C. By the assumption we know that $C_1 = C$ is closed and convex. We suppose that C_n is closed and convex for some $n \geq 2$. By the definition of ϕ, we have that

$$\begin{aligned} C_{n+1} &= \Big\{z \in C_n : \sup_{t\geq 0}\phi(z,y_{n,t}) \leq \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\Big\} \\ &= \bigcap_{t\geq 0}\big\{z \in C : \phi(z,y_{n,t}) \leq \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n) + \xi_n\big\} \cap C_n \\ &= \bigcap_{t\geq 0}\big\{z \in C : 2\alpha_n\langle z,Jx_1\rangle + 2(1-\alpha_n)\langle z,Jx_n\rangle - 2\langle z,Jy_{n,t}\rangle \\ &\qquad \leq \alpha_n\|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{n,t}\|^2\big\} \cap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex.

(II) We prove that $F(T) \subset C_n$.

In fact $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_n$, $n \geq 2$. Let

$$\omega_{n,t} = J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n\big), \quad t \geq 0.$$

It follows from (2) that for any $u \in F(T) \subset C_n$, we have

$$\begin{aligned} \phi(u,y_{n,t}) &= \phi\big(u, J^{-1}\big(\alpha_n Jx_1 + (1-\alpha_n)J\omega_{n,t}\big)\big) \\ &\leq \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,\omega_{n,t}), \end{aligned}$$

and

$$\begin{aligned} \phi(u,\omega_{n,t}) &= \phi\big(u, J^{-1}\big(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n\big)\big) \\ &\leq \beta_n\phi(u,x_n) + (1-\beta_n)\phi\big(u, T^n(t)x_n\big) \\ &\leq \beta_n\phi(u,x_n) + (1-\beta_n)\big[\phi(u,x_n) + \mu_n\psi\big(\phi(u,x_n)\big) + \nu_n\big] \\ &\leq \phi(u,x_n) + (1-\beta_n)\big(\mu_n M^*\phi(u,x_n) + \nu_n\big). \end{aligned}$$

Therefore, we have

$$\begin{aligned}\sup_{t\geq 0}\phi(u,y_{n,t}) &\leq \alpha_n\phi(u,x_1)+(1-\alpha_n)\big[\phi(u,x_n)+(1-\beta_n)\big(\mu_nM^*\phi(u,x_n)+\nu_n\big)\big]\\ &\leq \alpha_n\phi(u,x_1)+(1-\alpha_n)\phi(u,x_n)+\mu_nM^*\sup_{p\in F(T)}\phi(p,x_n)+\nu_n\\ &= \alpha_n\phi(u,x_1)+(1-\alpha_n)\phi(u,x_n)+\xi_n.\end{aligned}$$

Where $\xi_n=\mu_nM^*\sup_{p\in F(T)}\phi(p,x_n)+\nu_n$. This shows that $u\in C_{n+1}$, so $F(T)\subset C_{n+1}$.

(III) We prove that $\{x_n\}$ is a Cauchy sequence in C.

Since $x_n=\Pi_{C_n}x_1$, from Lemma 2.1(ii), we have

$$\langle x_n-y,Jx_1-Jx_n\rangle\geq 0,\quad \forall y\in C_n.$$

Again since $F(T)\subset C_n$, $n\geq 1$, we have

$$\langle x_n-u,Jx_1-Jx_n\rangle\geq 0,\quad \forall u\in F(T).$$

It follows from Lemma 2.1(i) that for each $u\in F(T)$, $n\geq 1$,

$$\phi(x_n,x_1)=\phi(\Pi_{C_n}x_1,x_1)\leq\phi(u,x_1)-\phi(u,x_n)\leq\phi(u,x_1).$$

Therefore, $\phi(x_n,x_1)$ is bounded. By virtue of (1), x_n is also bounded. Since $x_n=\Pi_{C_n}x_1$ and $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}\subset C_n$, we have $\phi(x_n,x_1)\leq\phi(x_{n+1},x_1)$. This implies that $\{\phi(x_n,x_1)\}$ is nondecreasing. Hence, the limit $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. By the construction of C_n, for any positive integer $m\geq n$, we have $C_m\subset C_n$ and $x_m=\Pi_{C_1}x_1\in C_n$. This shows that

$$\begin{aligned}\phi(x_m,x_n) &= \phi(x_m,\Pi_{C_n}x_1)\\ &\leq\phi(x_m,x_1)-\phi(x_n,x_1)\to 0,\quad \text{as } m,n\to\infty.\end{aligned}$$

It follows from Lemma 2.2 that $\lim_{n,m\to\infty}\|x_m-x_n\|=0$. Hence x_n is a Cauchy sequence in C. Since C is complete, without loss of generality, we can assume that $x_n\to p^*$ (some point in C). By the assumption, we have that

$$\lim_{n\to\infty}\xi_n=\lim_{n\to\infty}\Big[\mu_nM^*\sup_{p\in F(T)}\phi(p,x_n)+\nu_n\Big]=0. \tag{4}$$

(IV) Now we prove $p^*\in F(T)$.

Since $x_{n+1}\in C_{n+1}$ and $\alpha_n\to 0$, it follows from (3) and (4) that

$$\sup_{t\geq 0}\phi(x_{n+1},y_{n,t})\leq\alpha_n\phi(x_{n+1},x_1)+(1-\alpha_n)\phi(x_{n+1},x_n)+\xi_n\to 0\quad \text{as } n\to\infty.$$

Since $x_n\to p^*$, by Lemma 2.2, for each $t\geq 0$, we have

$$\lim_{n\to\infty}y_{n,t}=p^*. \tag{5}$$

Since x_n is bounded, and $\mathbf{T} = \{T(t), t \geq 0\}$ is a total quasi-ϕ-asymptotically nonexpansive semigroup with sequence $\mu_n, \nu_n, p \in F(T)$, we have

$$\phi(p, T^n(t)x) \leq \phi(p,x) + \mu_n \psi(\phi(p,x)) + \nu_n \leq \phi(p,x) + \mu_n M^* \phi(p,x) + \nu_n.$$

This implies that $\{T^n(t)x_n\}_{t\geq 0}$ is uniformly bounded. Since for each $t \geq 0$,

$$\begin{aligned} \|\omega_{n,t}\| &= \left\|J^{-1}\bigl(\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n\bigr)\right\| \\ &\leq \beta_n \|x_n\| + (1-\beta_n)\left\|T^n(t)x_n\right\| \\ &\leq \max\bigl\{\|x_n\|, \left\|T^n(t)x_n\right\|\bigr\}. \end{aligned}$$

This implies that $\{\omega_{n,t}\}$, $t \geq 0$ is also uniformly bounded. Since $\alpha_n \to 0$, from (3) we have

$$\lim_{n\to\infty} \|Jy_{n,t} - J\omega_{n,t}\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - J\omega_{n,t}\| = 0, \quad t \geq 0. \tag{6}$$

Since E is uniformly smooth, J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (5) and (6) that

$$\lim_{n\to\infty} \omega_{n,t} = p^*, \quad \forall t \geq 0.$$

Since $x_n \to p^*$ and J is uniformly continuous on each bounded subset of E, we have $Jx_n \to Jp^*$, and for each $t \geq 0$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\|J\omega_{n,t} - Jp^*\right\| = \lim_{n\to\infty} \left\|\beta_n Jx_n + (1-\beta_n)JT^n(t)x_n - Jp^*\right\| \\ &= \lim_{n\to\infty} \left\|\beta_n\bigl(Jx_n - Jp^*\bigr) + (1-\beta_n)\bigl(JT^n(t)x_n - Jp^*\bigr)\right\| \\ &= \lim_{n\to\infty} (1-\beta_n)\left\|\bigl(JT^n(t)x_n - Jp^*\bigr)\right\|. \end{aligned}$$

By condition $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$, we have that

$$\lim_{n\to\infty} \left\|JT^n(t)x_n - Jp^*\right\| = 0, \quad \text{uniformly for } t \geq 0.$$

Since J is uniformly continuous, this shows that $\lim_{n\to\infty} T^n(t)x_n = p^* = 0$ uniformly for $t \geq 0$. Again by the assumptions that the semigroup $\mathbf{T} := \{T(t) : t \geq 0\}$ is closed and uniformly L-Lipschitzian, we have

$$\begin{aligned} &\left\|T^{n+1}(t)x_n - T^n(t)x_n\right\| \\ &\quad \leq \left\|T^{n+1}(t)x_n - T^{n+1}(t)x_{n+1}\right\| + \left\|T^{n+1}(t)x_{n+1} - x_{n+1}\right\| + \|x_{n+1} - x_n\| + \left\|x_n - T^n(t)x_n\right\| \\ &\quad \leq \bigl(L(t) + 1\bigr)\|x_{n+1} - x_n\| + \left\|T^{n+1}(t)x_{n+1} - x_{n+1}\right\| + \left\|x_n - T^n(t)x_n\right\|. \end{aligned} \tag{7}$$

By $\lim_{n\to\infty} T^n(t)x_n = p^*$ uniformly for $t \geq 0$, $x_n \to p^*$ and $L(t)$ is a bounded and measurable function, and from (7) we have that

$$\lim_{n\to\infty} \left\|T^{n+1}(t)x_n - T^n(t)x_n\right\| = 0 \quad \text{uniformly for } t \geq 0,$$

and

$$\lim_{n\to\infty} T^{n+1}(t)x_n = p^* \quad \text{uniformly for } t \geq 0,$$

so we get

$$\lim_{n\to\infty} T(t)T^n(t)x_n = p^* \quad \text{uniformly for } t \geq 0.$$

By virtue of the closeness of semigroup $\mathbf{T}$, we have that $T(t)p^* = p^*$, *i.e.*, $p^* \in F(T(t))$. By the arbitrariness of $t \geq 0$, we have $p^* \in F(T) = \bigcap_{t\geq 0} F(T(t))$.

(V) Finally, we prove $x_n \to p^* = \Pi_{F(T)}x_1$.

Let $\omega = \Pi_{F(T)}x_1$. Since $\omega \in F(T) \subset C_n$ and $x_n = \Pi_{C_n}x_1$, we get $\phi(x_n, x_1) \leq \phi(\omega, x_1)$, $n \geq 1$. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(\omega, x_1). \tag{8}$$

In view of the definition of $\Pi_{F(T)}x_1$, from (8), we have $p^* = \omega$. Therefore, $x_n \to p^* = \Pi_{F(T)}x_1$. This completes the proof of Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the writing of the present article. And they also read and approved the final manuscript.

Author details
[1] Department of Mathematics, Yibin University, Yibin, Sichuan, 644000, China. [2] College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan, 650221, China.

Acknowledgements
This work was supported by National Research Foundation of Yibin University (No. 2011B07) and by Scientific Research Fund Project of Sichuan Provincial Education Department (No. 12ZB345 and No. 11ZA172).

Received: 8 May 2012 Accepted: 22 August 2012 Published: 5 September 2012

References
1. Alber, YI, Chidume, CE, Zegeye, H: Approximating fixed point of total asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2006**, Article ID 10673 (2006)
2. Chidume, CE, Ofoedu, EU: Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings. J. Math. Anal. Appl. **333**(1), 128-141 (2007)
3. Chidume, CE, Ofoedu, EU: A new iteration process for approximation of common fixed points for finite families of total asymptotically nonexpansive mappings. Int. J. Math. Math. Sci. **2009**, Article ID 615107 (2009). doi:10.1155/2009/615107
4. Chang, S, Kim, JK, Joseph Lee, HW, Chan, CK: A generalization and improvement of Chidume theorems for total asymptotically nonexpansive mappings in Banach spaces. J. Inequal. Appl. **2012**, 37 (2012). doi:10.1186/1029-242X-2012-37
5. Chang, S, Chan, CK, Joseph Lee, HW: Modified block iterative algorithm for Quasi-ϕ-asymptotically nonexpansive mappings and equilibrium problem in Banach spaces. Appl. Math. Comput. **217**, 7520-7530 (2011)
6. Qin, X, Cho, SY, Kang, SM: On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings. Appl. Math. Comput. **215**(11), 3874-3883 (2010)
7. Saewan, S, Kumam, P: Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi-ϕ-asymptotically nonexpansive mappings. Abstr. Appl. Anal. **2010**, 22 (2010). doi:10.1155/2010/357120
8. Chang, S, Wanga, L, Tang, Y-K, Wanga, B, Qin, L-J: Strong convergence theorems for a countable family of quasi-ϕ-asymptotically nonexpansive nonself mappings. Appl. Math. Comput. **218**, 7864-7870 (2012)
9. Qin, X, Huang, S, Wang, T: On the convergence of hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mappings. Comput. Math. Appl. **61**(4), 851-859 (2011)

10. Chang, S, Joseph Lee, HW, Chan, CK, Zhang, WB: A modified Halpern type iterative algorithm for total quasi-ϕ-asymptotically nonexpansive mappings with applications. Appl. Math. Comput. **218**(11), 6489-6497 (2012)
11. Wang, X, Chang, S, Wang, L, Tang, Y-K, Xu, YG: Strong convergence theorems for nonlinear operator equations with total quasi-ϕ-asymptotically nonexpansive mappings and applications. Fixed Point Theory Appl. **2012**, 34 (2012). doi:10.1186/1687-1812-2012-34
12. Zuo, P, Chang, S, Liu, M: On a hybrid algorithm for a family of total quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2012**, 70 (2012). doi:10.1186/1687-1812-2012-70
13. Ye, J, Huang, J: Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces. J. Math. Comput. Sci. **1**, 1-18 (2011)
14. Kim, JK: Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings. Fixed Point Theory Appl. **2011**, 10 (2011)
15. Saewan, S, Kumam, P: The shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-ϕ-nonexpansive mappings. Fixed Point Theory Appl. **2011**, 9 (2011)
16. Alber, YI: Metric and generalized projection operators in Banach spaces: properties and applications. In: Kartosator, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15-50. Dekker, New York (1996)
17. Chang, SS, Joseph Lee, HW, Chan, CK: Convergence theorem of common fixed point for asymptotically nonexpansive semigroups in Banach spaces. Appl. Math. Comput. **212**, 60-65 (2009)
18. Zegeye, H, Shahzadb, N, Damana, OA: Strong convergence theorems for a semigroup of asymptotically nonexpansive mappings. Math. Comput. Model. **54**, 2077-2086 (2011)
19. Chang, S, Wang, L, Tang, Y-K, Zao, Y-H, Wang, B: Strong convergence theorems of quasi-ϕ-asymptotically nonexpansive semi-groups in Banach spaces. Fixed Point Theory Appl. **2012**, 15 (2012). doi:10.1186/1687-1812-2012-15

doi:10.1186/1687-1812-2012-142
Cite this article as: Quan et al.: Strong convergence for total quasi-ϕ-asymptotically nonexpansive semigroups in Banach spaces. *Fixed Point Theory and Applications* 2012 **2012**:142.