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Some connections between the attractors of an IIFS $\mathcal{S}$ and the attractors of the sub-IFSs of $\mathcal{S}$

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Abstract

Based on the results from (Mihail and Miculescu in Math. Rep., Bucur. 11(61)(1):21-32, 2009), where the shift space for an infinite iterated function system (IIFS for short) is defined and the relation between this space and the attractor of the IIFS is described, we give a sufficient condition on a family $(I_j)_{j\in L}$ of nonempty subsets of I, where $\mathcal{S} = (X, (f_i)_{i\in I})$ is an IIFS, in order to have the equality $\overline{\bigcup_{j\in L} A_{I_j}} = A$, where A means the attractor of $\mathcal{S}$ and A_{I_j} means the attractor of the sub-iterated function system $\mathcal{S}_{I_j} = (X, (f_i)_{i\in I_j})$ of $\mathcal{S}$. In addition, we prove that given an arbitrary infinite cardinal number $\mathcal{A}$, if the attractor of an IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$ is of type $\mathcal{A}$ (this means that there exists a dense subset of it having the cardinal less than or equal to $\mathcal{A}$), where (X, d) is a complete metric space, then there exists $\mathcal{S}_J = (X, (f_i)_{i\in J})$ a sub-iterated function system of $\mathcal{S}$, having the property that $\mathrm{card}(J) \le \mathcal{A}$, such that the attractors of $\mathcal{S}$ and $\mathcal{S}_J$ coincide.

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1 Introduction

Iterated function systems (IFSs) were conceived in the present form by John Hutchinson [1] and popularized by Michael Barnsley [2]. The most common and most general way to generate fractals is to use the theory of IFS (which provides a new insight into the modeling of real world phenomena). Because of the variety of their applications (actually one can find fractals almost everywhere in the universe: galaxies, weather, coastlines and borderlines, landscapes, human anatomy, chemical reactions, bacteria cultures, plants, population growth, data compression, economy *etc.*), there is a current effort to extend the classical Hutchinson's framework to more general spaces and to infinite iterated function systems. For example, on the one hand, Gwóźdź-Łukawska and Jachymski [3] discuss the Hutchinson-Barnsley theory for infinite iterated function systems. Łoziński, Życzkowsi and Słomczyński [4] introduce the notion of quantum iterated function systems (QIFS) which is designed to describe certain problems of nonunitary quantum dynamics. Käenmäki [5] constructs a thermodynamical formalism for very general iterated function systems. Leśniak [6] presents a multivalued approach of infinite iterated function systems. In [7–10], and [11] the notion of generalized iterated function system (GIFS), which is a family of functions $f_1, \ldots, f_n : X^m \to X$, (X, d) being a metric space and $m, n \in \mathbb{N}$, is introduced. Under certain conditions, the existence of the attractor of such a GIFS is proved and its properties are explored (among them, an upper bound for the Hausdorff-Pompeiu distance between the attractors of two such GIFSs, an upper bound for the Hausdorff-Pompeiu distance between the attractors of such a GIFS and an arbitrary compact set of X are presented, and its continuous dependence in the f_k's is proved). Moreover, in [12] and [13], the existence of an analogue of Hutchinson's measure associated to certain GIFSs with probabilities (GIFSp for short) is proved. Also, we showed that the support of such a measure is the attractor of the given GIFSp and we construct a sequence of measures converging to this measure. On the other hand, in [14], we provided a general framework where attractors are nonempty closed and bounded subsets of topologically complete metric spaces and where the IFSs may be infinite, in contrast to the classical theory (see [2]), where only attractors that are compact metric spaces and IFSs that are finite are considered. In the paper [15], a generalization of the notion of the shift space associated to an IFS is presented. More precisely, the shift space for an infinite iterated function system (IIFS) is defined and the relation between this space and the attractor of the IIFS is described. A canonical projection π (which turns out to be continuous) from the shift space of an IIFS on its attractor is constructed and sufficient conditions for this function to be onto are provided. While it is possible to approximate any compact subset in the space X by an attractor of some IFS, the question as to which compact can be realized as attractors of IFSs remains elusive. The attractors of IFSs come in so many different forms that their diversity never fails to amaze us. The repertory of attractors of IFSs starts with simple spaces such as an interval, a square, the closure of the unit disc (see [16]) and continues with more exotic sets such as the Cantor ternary set, the Sierpinski gasket, the Menger sponge, the Black Spleenwort fern, the Barnsley fern, the Castle fractal, the Julia sets of quadratic transformations (see [2]), the Koch curve, the Polya's curve, the Levy's curve or the Takagi graph (see [17]). Along the same lines, Arenas and Sancez Granero [18] proved that every graph (*i.e.*, a locally connected continuum with a finite number of end points and ramification points) is the attractor of some iterated function system. Sanders [19] proved that arcs in $\mathbb{R}^n$ of finite length are attractors of some IFS on $\mathbb{R}^n$. In [20] Secelean proved that each compact subset of a metric space can be presented as the attractor of a countable iterated function system. At the same time, it is a natural question to ask whether it is true that any compact set is actually the invariant set of some IFS. The answer is negative. Here are some examples: Kwiecinski [21] constructed a locally connected continuum in the plane which is not an attractor of any iterated function system; Crovisier and Rams [22] constructed an embedded Cantor set in $\mathbb{R}$ and showed that it could not be realized as an attractor of any iterated function system; Stacho and Szabo [23] constructed compact sets in $\mathbb{R}$ that are not invariant sets for any IFS; Sanders [19] constructed an n-cell in $\mathbb{R}^{n+1}$ and showed that this n-cell cannot be the attractor of any IFS on $\mathbb{R}^{n+1}$ for each natural number n.

In the present paper, using the results from [15], especially Theorem 4.1, we present a sufficient condition on a family $(I_j)_{j \in L}$ of nonempty subsets of I, where $\mathcal{S} = (X, (f_i)_{i \in I})$ is an IIFS, in order to have the equality $\overline{\bigcup_{j \in L} A_{I_j}} = A$, where A means the attractor of $\mathcal{S}$ and A_{I_j} means the attractor of the sub-iterated function system $\mathcal{S}_{I_j} = (X, (f_i)_{i \in I_j})$ of $\mathcal{S}$. In addition, two examples concerning this result are presented. The first example shows that the above mentioned condition is not necessary, while the second example provides a case for which it is a necessary condition.

Moreover, we prove that given an arbitrary infinite cardinal number $\mathcal{A}$, if the attractor of an IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$ is of type $\mathcal{A}$, where (X, d) is a complete metric space, then there exists $\mathcal{S}_J = (X, (f_i)_{i\in J})$ a sub-iterated function system of $\mathcal{S}$, having the property that $\operatorname{card}(J) \leq \mathcal{A}$, such that the attractors of $\mathcal{S}$ and $\mathcal{S}_J$ coincide.

2 Preliminaries

For the basic facts concerning infinite iterated function systems (IIFSs) and the shift space associated to an IIFS one can consult [15].

Definition 2.1 A metric space (X, d) is said to be of type $\mathcal{A}$, where $\mathcal{A}$ is a cardinal number, if there exists a dense subset A of X having the property that $\operatorname{card} A \leq \mathcal{A}$.

Definition 2.2 Given an IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$ and a subset J of I, the IIFS $\mathcal{S}_J = (X, (f_i)_{i\in J})$ is called a sub-iterated function system of $\mathcal{S}$ (a sub-IFS of $\mathcal{S}$ for short).

The following remark, which actually is Lemma 3.6 from [14], will be extensively used in this paper (see the proofs of Theorems 3.1 and 3.3).

Remark 2.3 Let us consider a complete metric space (X, d), an IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$ and the function $F_{\mathcal{S}} : \mathcal{B}^*(X) \to \mathcal{B}^*(X)$ given by $F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$, for all $B \in \mathcal{B}^*(X)$, where $\mathcal{B}^*(X)$ denotes the family of nonempty bounded closed subsets of X.

Then there exists a unique $A(\mathcal{S}) \in \mathcal{B}^*(X)$ such that

$$F_{\mathcal{S}}\bigl(A(\mathcal{S})\bigr) = A(\mathcal{S}).$$

Moreover, for $T \in \mathcal{B}^*(X)$, we have

$$F_{\mathcal{S}}(T) \subseteq T \quad \Rightarrow \quad A(\mathcal{S}) \subseteq T.$$

The following result is used in the proof of Theorem 3.3.

Proposition 2.4 *Let $\mathcal{S} = (X, (f_i)_{i\in I})$ be an IIFS, where (X, d) is a complete metric space, let $\alpha : \Lambda^* \to \Lambda$ be an arbitrary function, and let us consider the set $M = \{\omega\alpha(\omega) | \omega \in \Lambda^*\}$. Then $\pi(M)$ is dense in $A(\mathcal{S})$.*

Proof Let us consider

$$c := \sup_{i\in I} \operatorname{Lip}(f_i) < 1$$

and $\omega_0 \in \Lambda$.

Let us remark that since

$$a_{\omega_0} = \pi(\omega_0) \in \overline{A_{[\omega_0]_m}}$$

and

$$\pi\big([\omega_0]_m\alpha\big([\omega_0]_m\big)\big) = a_{[\omega_0]_m\alpha([\omega_0]_m)} \in \overline{A_{[[\omega_0]_m\alpha([\omega_0]_m)]_m}} = \overline{A_{[\omega_0]_m}}$$

(see point 2 of Theorem 4.1 from [15]), we obtain, using point 1 of the same theorem, that

$$d\big(\pi(\omega_0), \pi\big([\omega_0]_m\alpha\big([\omega_0]_m\big)\big)\big) \leq \operatorname{diam}(\overline{A_{[\omega_0]_m}}) \leq c^m \operatorname{diam}(A),$$

for all $m \in \mathbb{N}$.

Taking into account the fact that $c \in [0,1)$, it follows that

$$\pi(\Lambda) \subseteq \overline{\pi(M)},$$

and therefore, using point 6(ii) of Theorem 4.1 from [15], we conclude that

$$A = \overline{\pi(\Lambda)} \subseteq \overline{\pi(M)} \subseteq A,$$

i.e.,

$$\overline{\pi(M)} = A. \qquad \square$$

3 The main results

Theorem 3.1 *Let $\mathcal{S} = (X, (f_i)_{i\in I})$ be an IIFS, where (X, d) is a complete metric space, $A := A(\mathcal{S})$ be its attractor and $(I_j)_{j\in L}$ be a family of nonempty subsets of I such that $\bigcup_{j\in L} I_j = I$.*

If for every $i_1 \in I_{j_1}, i_2 \in I_{j_2}, \ldots, i_n \in I_{j_n}$, where $\{j_1, j_2, \ldots, j_n\} \subseteq L$, there exists $l \in L$ such that $i_1, i_2, \ldots, i_n \in I_l$, then

$$\overline{\bigcup_{j\in L} A_{I_j}} = A,$$

where A_{I_j} is the attractor of the sub-iterated function system $\mathcal{S}_{I_j} = (X, (f_i)_{i\in I_j})$ of $\mathcal{S}$.

Proof Let us note that on the one hand we have

$$\overline{\bigcup_{j\in L} A_{I_j}} \subseteq A. \qquad (*)$$

Indeed, since

$$F_{\mathcal{S}_{I_j}}(A) = \overline{\bigcup_{i\in I_j} f_i(A)} \subseteq \overline{\bigcup_{i\in I} f_i(A)} = F_{\mathcal{S}_I}(A) = A,$$

using Remark 2.3, we get

$$A_{I_j} \subseteq A,$$

for all $j \in L$, and therefore

$$\bigcup_{j\in L} A_{I_j} \subseteq A.$$

Taking into account the fact that A is a closed set, we get

$$\overline{\bigcup_{j\in L} A_{I_j}} \subseteq A.$$

On the other hand, we have

$$A \subseteq \overline{\bigcup_{j\in L} A_{I_j}}. \qquad (**)$$

Indeed, for an arbitrary $\omega = i_1 i_2 \cdots i_n \in \Lambda^*$, since $\bigcup_{j\in L} I_j = I$, there exist $j_1, j_2, \ldots, j_n \in L$ such that $i_1 \in I_{j_1}, i_2 \in I_{j_2}, \ldots, i_n \in I_{j_n}$, and, according to the hypothesis, there exists $l \in L$ such that $i_1, i_2, \ldots, i_n \in I_l$. Then, using point 5 of Theorem 4.1 from [15], we obtain

$$e_\omega \in A_{I_l} \subseteq \overline{\bigcup_{j\in L} A_{I_j}}.$$

It follows, using again the same point 5 of Theorem 4.1 from [15], that

$$A = \overline{\{e_\omega | \omega \in \Lambda^*\}} \subseteq \overline{\bigcup_{j\in L} A_{I_j}}.$$

From $(*)$ and $(**)$, we obtain that

$$\overline{\bigcup_{j\in L} A_{I_j}} = A.$$

□

Corollary 3.2 *Let $\mathcal{S} = (X, (f_i)_{i\in I})$ be an IIFS, where (X,d) is a complete metric space, and $A := A(\mathcal{S})$ be its attractor.*

Then

$$\overline{\bigcup_{\substack{\emptyset \neq J \subseteq I \\ J \text{ finite}}} A_J} = A,$$

where A_J is the attractor of the sub-iterated function system $\mathcal{S}_J = (X, (f_i)_{i\in J})$ of $\mathcal{S}$.

The following example shows that the condition 'for every $i_1 \in I_{j_1}, i_2 \in I_{j_2}, \ldots, i_n \in I_{j_n}$, where $\{j_1, j_2, \ldots, j_n\} \subseteq L$, there exists $l \in L$ such that $i_1, i_2, \ldots, i_n \in I_l$' is not a necessary condition for the equality $\overline{\bigcup_{j\in L} A_{I_j}} = A$.

Example Let us consider the IIFS

$$\mathcal{S} = \big(([0,1], d), (f_c)_{c\in[0,1]}\big),$$

where d is the usual distance on $[0,1]$ and the function $f_c : [0,1] \to [0,1]$ is given by

$$f_c(x) = c,$$

for each $x \in [0,1]$.

Since

$$[0,1] = \overline{\bigcup_{c\in[0,1]} f_c\big([0,1]\big)} = F_{\mathcal{S}}\big([0,1]\big),$$

we infer that

$$A := A(\mathcal{S}) = [0,1]$$

and the equality

$$\{c\} = \overline{f_c\big(\{c\}\big)} = F_{\mathcal{S}_{\{c\}}}\big(\{c\}\big)$$

implies that

$$\{c\} = A_{\{c\}},$$

where $A_{\{c\}}$ is the attractor of the sub-iterated function system

$$\mathcal{S}_{\{c\}} = \big(\big([0,1],d\big),\{f_c\}\big)$$

of $\mathcal{S}$.

Consequently, on the one hand, the equality

$$A = \overline{\bigcup_{c\in[0,1]} A_{\{c\}}},$$

which is equivalent to $[0,1] = \overline{\bigcup_{c\in[0,1]}\{c\}}$, is valid.

On the other hand, the family $(\{c\})_{c\in[0,1]}$ of nonempty subsets of $[0,1]$ has the property that

$$\bigcup_{c\in[0,1]} \{c\} = [0,1],$$

but does not have the property that for every $c_1, c_2, \dots, c_n \in [0,1]$ there exists $c \in [0,1]$ such that $c_1, c_2, \dots, c_n \in \{c\}$.

We will present now an example for which the condition 'for every $i_1 \in I_{j_1}, i_2 \in I_{j_2}, \dots, i_n \in I_{j_n}$, where $\{j_1, j_2, \dots, j_n\} \subseteq L$, there exists $l \in L$ such that $i_1, i_2, \dots, i_n \in I_l$' is a necessary and sufficient condition for the equality $\overline{\bigcup_{j\in L} A_{I_j}} = A$.

Example Let us consider the IIFS

$$\mathcal{S} = \big(\Lambda(I), (F_i)_{i\in I}\big),$$

whose attractor is

$$\Lambda(I) := A$$

(see Remark 3.2, (i) from [15]) and $(I_j)_{j\in L}$ is a family of nonempty subsets of I such that $\bigcup_{j\in L} I_j = I$.

Then the attractor of a sub-iterated function system

$$\mathcal{S}_J = \bigl(\Lambda(I), (F_i)_{i\in J}\bigr)$$

of $\mathcal{S}$, where $J \subseteq I$, is

$$\Lambda(J) := A_J.$$

We claim that $\overline{\bigcup_{j\in L} A_{I_j}} = A$ if and only if for every $i_1 \in I_{j_1}, i_2 \in I_{j_2}, \ldots, i_n \in I_{j_n}$, where $\{j_1, j_2, \ldots, j_n\} \subseteq L$, there exists $l \in L$ such that $i_1, i_2, \ldots, i_n \in I_l$.

Indeed, the above theorem assures us that the implication '$\Leftarrow$' is valid. For the implication '$\Rightarrow$' let us consider $i_1 \in I_{j_1}, i_2 \in I_{j_2}, \ldots, i_n \in I_{j_n}$, where $\{j_1, j_2, \ldots, j_n\} \subseteq L$. Then

$$\omega \overset{\text{def}}{=} i_1 i_2 \cdots i_n i_1 i_2 \cdots i_n \cdots i_1 i_2 \cdots i_n \cdots \in \Lambda(I) = A = \overline{\bigcup_{j\in L} A_{I_j}},$$

which implies that there exist $l \in L$ and $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n \cdots \in A_{I_l}$ such that

$$d_\Lambda(\alpha, \omega) < \frac{1}{3^{n+1}}.$$

Thus

$$\alpha_1 = i_1, \alpha_2 = i_2, \ldots, \alpha_n = i_n$$

i.e.,

$$\{i_1, i_2, \ldots, i_n\} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq I_l.$$

Theorem 3.3 *Given an infinite cardinal number $\mathcal{A}$, let $\mathcal{S} = (X, (f_i)_{i\in I})$ be an IIFS such that its attractor $A(\mathcal{S})$ is of type $\mathcal{A}$, where (X, d) is a complete metric space.*

Then there exists $\mathcal{S}_J = (X, (f_i)_{i\in J})$, a sub-iterated function system of $\mathcal{S}$, such that

$$\operatorname{card}(J) \le \mathcal{A}$$

and

$$A(\mathcal{S}) = A(\mathcal{S}_J).$$

Proof Let us consider

$$P_{\mathcal{A}}(I) = \bigl\{J \subseteq I \mid \operatorname{card}(J) \le \mathcal{A}\bigr\}.$$

For $J \in P^*_{\mathcal{A}}(I) := P_{\mathcal{A}}(I) - \{\emptyset\}$, with the notations $A := A(\mathcal{S})$ and $A_J := A(\mathcal{S}_J)$, where

$$\mathcal{S}_J = \bigl(X, (f_i)_{i\in J}\bigr),$$

we have

$$F_{\mathcal{S}_J}(A) = \overline{\bigcup_{i\in J} f_i(A)} \subseteq \overline{\bigcup_{i\in I} f_i(A)} = F_{\mathcal{S}}(A) = A,$$

so using Remark 2.3, we get

$$A_J \subseteq A,$$

and therefore,

$$d(A_J, A) = 0,$$

which implies that

$$h(A_J, A) = d(A, A_J).$$

Hence

$$A_J = A \quad \text{if and only if } d(A, A_J) = 0. \tag{$*$}$$

Let us consider

$$\beta = \inf\{d(A, A_J) | J \in P^*_{\mathcal{A}}(I)\}.$$

We claim that there exists $J \in P^*_{\mathcal{A}}(I)$ such that

$$d(A, A_J) = \beta. \tag{$**$}$$

Indeed, for each $n \in \mathbb{N}$ there exists $J_n \in P^*_{\mathcal{A}}(I)$ such that

$$d(A, A_{J_n}) \leq \beta + \frac{1}{n}.$$

Then

$$J := \bigcup_{n\in\mathbb{N}} J_n \in P^*_{\mathcal{A}}(I)$$

and

$$F_{\mathcal{S}_{J_n}}(A_J) = \overline{\bigcup_{i\in J_n} f_i(A_J)} \subseteq \overline{\bigcup_{i\in J} f_i(A_J)} = F_{\mathcal{S}_J}(A_J) = A_J.$$

So, using again Remark 2.3, we get that

$$A_{J_n} \subseteq A_J,$$

and therefore,

$$\beta \leq d(A, A_J) \leq d(A, A_{J_n}) \leq \beta + \frac{1}{n},$$

for all $n \in \mathbb{N}$. The last inequality implies, by letting n to tend to ∞, the equality

$$d(A, A_J) = \beta.$$

Our next claim is that

$$\beta = 0. \qquad (***)$$

Indeed, if we suppose that $\beta > 0$, taking into account the previous claim, we can consider $J \in P^*_{\mathcal{A}}(I)$ such that

$$d(A, A_J) = \beta.$$

According to Zorn's lemma, we can consider a maximal subset C of A having the property that

$$d(x, y) > \frac{\beta}{4},$$

for every $x, y \in C$, $x \neq y$. Since A is of type $\mathcal{A}$, there exists a subset M of A such that

$$\overline{M} = A$$

and

$$\operatorname{card}(M) \leq \mathcal{A}.$$

Thus

$$M \cap B\left(c, \frac{\beta}{8}\right) \neq \emptyset,$$

for every $c \in C$. The function $f : C \to M$, given by $f(c) = y_c$, where y_c is a fixed element of $M \cap B(c, \frac{\beta}{8})$, is injective (since, if for $c_1, c_2 \in C$, $c_1 \neq c_2$, we have $f(c_1) = f(c_2)$, then $y_{c_1} = y_{c_2}$, which implies the contradiction $\frac{\beta}{4} < d(c_1, c_2) \leq d(c_1, y_{c_1}) + d(y_{c_2}, c_2) < \frac{\beta}{8} + \frac{\beta}{8} = \frac{\beta}{4}$), and consequently,

$$\operatorname{card}(C) \leq \operatorname{card}(M) \leq \mathcal{A}.$$

Let us consider a fixed element $j_0 \in I$. For each $x \in A$ there exists $c_x \in C$ such that

$$d(x, c_x) \leq \frac{\beta}{4} \qquad (1)$$

(since otherwise $d(x,c) > \frac{\beta}{4}$, for each $c \in C$, which implies that $x \notin C$, and therefore, $C \cup \{x\} \neq C$ and $d(u,v) > \frac{\beta}{4}$ for every $u, v \in C \cup \{x\}$, $u \neq v$; this contradicts the fact that C is a maximal subset of A having the property that $d(x,y) > \frac{\beta}{4}$ for every $x, y \in C$, $x \neq y$). Taking into account Proposition 2.4 (for the function $\alpha : \Lambda^* \to \Lambda$ given by $\alpha(\omega') = j_0 j_0 \cdots j_0 \cdots$, for all $\omega' \in \Lambda^*$), there exists $\omega_{c_x} = i_1(c_x) \cdots i_{n(c_x)}(c_x) \in \Lambda^*$ such that

$$d\bigl(\pi(\omega_{c_x} j_0 j_0 \cdots j_0 \cdots), c_x\bigr) \le \frac{\beta}{4}. \qquad (2)$$

Consequently, using (1) and (2), we get

$$d\bigl(x, \pi(\omega_{c_x} j_0 j_0 \cdots j_0 \cdots)\bigr) \le \frac{\beta}{2}.$$

Since the set

$$J_0 := \bigcup_{c_x, x \in A} \bigl\{i_1(c_x), \ldots, i_{n(c_x)}(c_x)\bigr\} \cup \{j_0\} \in P^*_{\mathcal{A}}(I)$$

and

$$\pi(\omega_{c_x} j_0 j_0 \cdots j_0 \cdots) \in A_{J_0}$$

(see points 4 and 6 of Theorem 4.1 from [15]), we obtain

$$d(x, A_{J_0}) \le \frac{\beta}{2},$$

for each $x \in A$, and therefore,

$$d(A, A_{J_0}) \le \frac{\beta}{2}.$$

This contradicts the definition of β.

From $(**)$ and $(***)$, we conclude that there exists $J \in P^*_{\mathcal{A}}(I)$ such that

$$d(A, A_J) = 0,$$

and, consequently, taking into account $(*)$, we get

$$A = A_J. \qquad \square$$

4 Conclusions

In this paper we presented some connections between the attractors of an IIFS $\mathcal{S}$ and the attractors of the sub-IFSs of $\mathcal{S}$. More precisely, we provided a sufficient condition on a family $(I_j)_{j\in L}$ of nonempty subsets of I, where $\mathcal{S} = (X, (f_i)_{i\in I})$ is an IIFS, in order to have the equality $\overline{\bigcup_{j\in L} A_{I_j}} = A$, where A means the attractor of $\mathcal{S}$ and A_{I_j} means the attractor of the sub-iterated function system $\mathcal{S}_{I_j} = (X, (f_i)_{i\in I_j})$ of $\mathcal{S}$. Moreover, we proved that given an arbitrary infinite cardinal number $\mathcal{A}$, if the attractor of an IIFS $\mathcal{S} = (X, (f_i)_{i\in I})$ is of type $\mathcal{A}$, where (X, d) is a complete metric space, then there exists $\mathcal{S}_J = (X, (f_i)_{i\in J})$, a sub-iterated function system of $\mathcal{S}$, having the property that $\mathrm{card}(J) \leq \mathcal{A}$, such that the attractors of $\mathcal{S}$ and $\mathcal{S}_J$ coincide. Two examples illustrating our results are presented. Let us note that the proof of Theorem 3.3 is based on Proposition 2.4 (which used Theorem 4.1 from [15]) and Zorn's lemma. Since we think that there exists a proof which does not use Zorn's lemma, in a future work we plan to present such a proof based only on Theorem 4.1 from [15].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. They read and approved the final manuscript.

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