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Strong convergence theorems for fixed points of asymptotically strict quasi-ϕ-pseudocontractions

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Abstract

In this paper, the fixed point problem of asymptotically strict quasi-ϕ-pseudocontractions is investigated based on hybrid projection algorithms. Strong convergence theorems of fixed points are established in a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property.

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1 Introduction

In what follows, we always assume that E is a Banach space with the dual space E^*. Let C be a nonempty, closed, and convex subset of E. We use the symbol J to stand for the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E^*.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U_E$ with $x \neq y$. It is said to be uniformly convex if for any $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U_E$.

It is well known that if E^* is strictly convex, then J is single valued; if E^* is reflexive, and smooth, then J is single valued and demicontinuous; for more details, see [1] and the references therein.

It is also well known that if D is a nonempty, closed, and convex subset of a Hilbert space H, and $P_D : H \to D$ is the metric projection from H onto D, then P_D is nonexpansive. This fact actually characterizes Hilbert spaces; and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] introduced a generalized projection operator Π_D in Banach spaces which is an analogue of the metric projection in Hilbert spaces.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E. \tag{1.1}$$

Notice that, in a Hilbert space H, (1.1) is reduced to $\phi(x,y) = \|x-y\|^2$ for all $x,y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x,y)$; that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem:

$$\phi(\bar{x}, x) = \min_{y\in C} \phi(y,x).$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and the strict monotonicity of the mapping J; see, for example, [1–4]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$\left(\|y\| - \|x\|\right)^2 \le \phi(y,x) \le \left(\|y\| + \|x\|\right)^2, \quad \forall x,y \in E, \tag{1.2}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{1.3}$$

Recall the following.

(1) A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

(2) T is said to be relatively nonexpansive if

$$\widetilde{F}(T) = F(T) \ne \emptyset, \quad \text{and} \quad \phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

(3) T is said to be relatively asymptotically nonexpansive if

$$\widetilde{F}(T) = F(T) \ne \emptyset, \quad \text{and} \quad \phi\left(p, T^n x\right) \le (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 1.1 The class of relatively asymptotically nonexpansive mappings was first considered in Su and Qin [5]; see also [6, 7] and the references therein.

(4) T is said to be quasi-ϕ-nonexpansive if

$$F(T) \ne \emptyset, \quad \text{and} \quad \phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

(5) T is said to be asymptotically quasi-ϕ-nonexpansive if there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \quad \text{and} \quad \phi(p, T^n x) \leq (1 + \mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

Remark 1.2 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings were first considered in Zhou, Gao, and Tan [8]; see also [9–12].

Remark 1.3 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require $F(T) = \widetilde{F}(T)$.

Remark 1.4 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces, respectively.

(6) T is said to be a strict quasi-ϕ-pseudocontraction if $F(T) \neq \emptyset$, and a constant $\kappa \in [0,1)$ such that

$$\phi(p, Tx) \leq \phi(p,x) + \kappa\phi(x, Tx), \quad \forall x \in C, p \in F(T).$$

Remark 1.5 The class of strict quasi-ϕ-pseudocontractions was first considered in Zhou and Gao [13]; see also Qin, Wang, and Cho [14].

(7) T is said to be an asymptotically strict quasi-ϕ-pseudocontraction if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ and a constant $\kappa \in [0,1)$ such that

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p,x) + \kappa\phi(x, T^n x), \quad \forall x \in C, p \in F(T).$$

Remark 1.6 The class of asymptotically strict quasi-ϕ-pseudocontractions was first considered in Qin, Wang, and Cho [14].

Remark 1.7 The class of strict quasi-ϕ-pseudocontractions and the class of asymptotically strict quasi-ϕ-pseudocontractions are generalizations of the class of asymptotically strict quasi-pseudocontractions and the class of asymptotically strict quasi-pseudocontractions in Banach spaces, respectively.

The following example can be found in [14].

Let $E = l_2 := \{x = \{x_1, x_2, \ldots\} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ and B_E be the closed unit ball in E. Define a mapping $T : B_E \to B_E$ by

$$T(x_1, x_2, \ldots) = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $\{a_i\}$ is a sequence of real numbers such that $a_2 > 0$, $0 < a_j < 1$, where $i \neq 2$, and $\prod_{i=2}^{\infty} a_j = \frac{1}{2}$. Then T is an asymptotically strict quasi-ϕ-pseudocontraction.

(8) T is said to be asymptotically regular on C if, for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K}\{\|T^{n+1}x - T^n x\|\} = 0.$$

During the past five decades, many famous existence theorems of fixed points of nonlinear mappings were established. However, from the standpoint of real world applications, it is not only to know the existence of fixed points of nonlinear mappings, but also to be able to construct an iterative process to approximate their fixed points. The simplest and oldest iterative algorithm is the well-known Picard iterative algorithm which generates an iterative sequence from an arbitrary initial x_0 in the following manner:

$$x_{n+1} = Tx_n, \quad n \geq 1,$$

where T is some mapping. The Picard iterative algorithm is a beautiful tool in the study of contractions. A well-known result is the Banach contraction principle. The class of nonexpansive mappings as a class of important nonlinear mappings finds many applications in signal processing, image reconstruction and so on. However, the Picard iterative algorithm fails to converge fixed points of nonexpansive mappings even when the fixed point set is not empty. For overcoming this, a Mann iterative algorithm has been studied extensively recently. The Mann iterative algorithm generates an iterative sequence for an arbitrary initial x_0 in the following manner:

$$x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)x_n, \quad n \geq 0,$$

where T is some mapping and $\{\alpha_n\}$ is some control sequence in $(0,1)$. The classic convergence theorem for fixed points of nonexpansive mappings based on the Mann iterative algorithm was established by Reich [15] in Banach spaces; for more details, see [15] and the reference therein.

It is known that the Mann iterative algorithm only has weak convergence even for nonexpansive mappings in infinite-dimensional Hilbert spaces; for more details, see [16] and the reference therein. To obtain the weak convergence of the Mann iterative algorithm, so-called hybrid projection algorithms have been considered; for more details, see [17–32] and the references therein.

In [24], Marino and Xu established a strong convergence theorem for fixed points of strict pseudocontraction based on hybrid projection algorithms in Hilbert spaces. Recently, Zhou and Gao [13] studied a new projection algorithm for strict quasi-ϕ-pseudocontractions and obtained a strong convergence theorem; for more details, see [13] and the reference therein. Quite recently, Qin, Wang, and Cho [14] proved a strong convergence theorem for fixed points of an asymptotically strict quasi-ϕ-pseudocontraction in a uniformly convex and uniformly smooth Banach space based on the results announced in Zhou and Gao [13]; for more details, see [14] and the reference therein.

In this paper, motivated by the results announced in Zhou and Gao [13] and Qin, Wang, and Cho [14], we consider asymptotically strict quasi-ϕ-pseudocontractions. We establish a strong convergence theorem in a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property to relax the restriction imposed on the space in Qin, Wang, and Cho's results. The results presented in this paper mainly improve the corresponding results announced in Zhou and Gao [13] and Qin, Wang, and Cho [14].

To prove our convergence theorem, we need the following lemmas:

Lemma 1.1 [4] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.2 [4] *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty closed convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 1.3 [21] *Let E be a reflexive, strictly convex, and smooth Banach space. Then we have the following:*

$$\phi(x, y) = 0 \quad \Leftrightarrow \quad x = y, \quad \forall x, y \in E.$$

2 Main results

Theorem 2.1 *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a closed and asymptotically strict quasi-ϕ-pseudocontraction with a sequence $\{\mu_n\} \subset [0,\infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that T is asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ C_{n+1} = \{u \in C_n : \phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa}\langle x_n - u, Jx_n - JT^n x_n\rangle + \mu_n \frac{M_n}{1-\kappa}\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{Υ}$$

where $M_n = \sup\{\phi(p, x_n) : p \in F(T)\}$. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{F(T)} x_0$.

Proof First, we show that $F(T)$ is closed and convex. The closedness of $F(T)$ follows from the closedness of T. Next, we show that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$, and $p_t = tp_1 + (1-t)p_2$, where $t \in (0,1)$. We see that $p_t = Tp_t$. Indeed, we see from the definition of T that

$$\phi\big(p_1, T^n p_t\big) \leq (1 + \mu_n)\phi(p_1, p_t) + \kappa\phi\big(p_t, T^n p_t\big), \tag{2.1}$$

and

$$\phi\big(p_2, T^n p\big) \leq (1 + \mu_n)\phi(p_2, p) + \kappa\phi\big(p_t, T^n p_t\big). \tag{2.2}$$

In view of (1.3), we obtain that

$$\phi(p_1, T^n p_t) = \phi(p_1, p_t) + \phi(p_t, T^n p_t) + 2\langle p_1 - p_t, Jp_t - J(T^n p_t)\rangle, \tag{2.3}$$

and

$$\phi(p_2, T^n p_t) = \phi(p_2, p_t) + \phi(p_t, T^n p_t) + 2\langle p_2 - p_t, Jp_t - J(T^n p_t)\rangle. \tag{2.4}$$

It follows from (2.1), (2.2), (2.3), and (2.4) that

$$\phi(p_t, T^n p_t) \le \frac{\mu_n}{1-\kappa}\phi(p_1, p_t) + \frac{2}{1-\kappa}\langle p_t - p_1, Jp_t - J(T^n p_t)\rangle, \tag{2.5}$$

and

$$\phi(p_t, T^n p_t) \le \frac{\mu_n}{1-\kappa}\phi(p_2, p_t) + \frac{2}{1-\kappa}\langle p_t - p_2, Jp_t - J(T^n p_t)\rangle. \tag{2.6}$$

Multiplying t and $(1-t)$ on both sides of (2.5) and (2.6) respectively yields that

$$\phi(p_t, T^n p_t) \le \frac{t\mu_n}{1-\kappa}\phi(p_1, p_t) + \frac{(1-t)\mu_n}{1-\kappa}\phi(p_2, p_t).$$

It follows that

$$\lim_{n\to\infty} \phi(p_t, T^n p_t) = 0.$$

In light of (1.2), we arrive at

$$\lim_{n\to\infty} \|T^n p_t\| = \|p_t\|. \tag{2.7}$$

It follows that

$$\lim_{n\to\infty} \|J(T^n p_t)\| = \|Jp_t\|. \tag{2.8}$$

Since E^* is reflexive, we may, without loss of generality, assume that $J(T^n p_t) \rightharpoonup e^* \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $e \in E$ such that $Je = e^*$. It follows that

$$\begin{aligned}\phi(p_t, T^n p_t) &= \|p_t\|^2 - 2\langle p_t, J(T^n p_t)\rangle + \|T^n p_t\|^2 \\ &= \|p_t\|^2 - 2\langle p_t, J(T^n p_t)\rangle + \|J(T^n p_t)\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above, we obtain that

$$\begin{aligned}0 &\ge \|p_t\|^2 - 2\langle p_t, e^*\rangle + \|e^*\|^2 \\ &= \|p_t\|^2 - 2\langle p_t, Je\rangle + \|Je\|^2 \\ &= \|p_t\|^2 - 2\langle p_t, Je\rangle + \|e\|^2 \\ &= \phi(p_t, e).\end{aligned}$$

This implies from Lemma 1.3 that $p_t = e$, that is, $Jp_t = e^*$. It follows that $J(T^n p_t) \rightharpoonup Jp_t \in E^*$. In view of the Kadec-Klee property of E^*, we obtain from (2.8) that

$$\lim_{n\to\infty} \|J(T^n p_t) - Jp_t\| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T^n p_t \rightharpoonup p_t$. By virtue of the Kadec-Klee property of E, we see from (2.7) that $T^n p_t \to p_t$ as $n \to \infty$. Since T is asymptotically regular, we see that

$$TT^n p_t = T^{n+1} p_t \to p_t,$$

as $n \to \infty$. In view of the closedness of T, we can obtain that $p_t \in F(T)$. This shows that $F(T)$ is convex. This completes the proof that $F(T)$ is closed and convex.

Next, we show that C_n is closed and convex for all $n \geq 1$. It is not hard to see that C_n is closed for each $n \geq 1$. Therefore, we only show that C_n is convex for each $n \geq 1$. It is obvious that $C_1 = C$ is convex. Suppose that C_h is convex for some $h \in \mathbb{N}$. Next, we show that C_{h+1} is also convex for the same h. Let $a, b \in C_{h+1}$ and $c = ta + (1-t)b$, where $t \in (0,1)$. It follows that

$$\phi(x_h, T^h x_h) \leq \frac{2}{1-\kappa}\langle x_h - a, Jx_h - JT^h x_h\rangle + \mu_h \frac{M_h}{1-\kappa}$$

and

$$\phi(x_h, T^h x_h) \leq \frac{2}{1-\kappa}\langle x_h - b, Jx_h - JT^h x_h\rangle + \mu_h \frac{M_h}{1-\kappa},$$

where $a, b \in C_h$. From the above two inequalities, we can get that

$$\phi(x_h, T^h x_h) \leq \frac{2}{1-\kappa}\langle x_h - c, Jx_h - JT^h x_h\rangle + \mu_h \frac{M_h}{1-\kappa},$$

where $c \in C_h$. It follows that C_{h+1} is closed and convex. This completes the proof that C_n is closed and convex.

Next, we show that $F(T) \subset C_n$. It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_h$ for some $h \in \mathbb{N}$. For any $z \in F(T) \subset C_h$, we see that

$$\phi(z, T^h x_h) \leq (1 + \mu_h)\phi(z, x_h) + \kappa\phi(x_h, T^h x_h). \tag{2.9}$$

On the other hand, we obtain from (1.3) that

$$\phi(z, T^h x_h) = \phi(z, x_h) + \phi(x_h, T^h x_h) + 2\langle z - x_h, Jx_h - JT^h x_h\rangle. \tag{2.10}$$

Combining (2.9) with (2.10), we arrive at

$$\begin{aligned}\phi(x_h, T^h x_h) &\leq \frac{\mu_h}{1-\kappa}\phi(z, x_h) + \frac{2}{1-\kappa}\langle x_h - z, Jx_h - JT^h x_h\rangle \\ &\leq \mu_h \frac{M_h}{1-\kappa} + \frac{2}{1-\kappa}\langle x_h - z, Jx_h - JT^h x_h\rangle,\end{aligned}$$

which implies that $z \in C_{h+1}$. This shows that $F(T) \subset C_{h+1}$. This completes the proof that $F(T) \subset C_n$.

Next, we show that $\{x_n\}$ is a convergent sequence which strongly converges to $\bar{x}$, where $\bar{x} \in F(T)$. Since $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z \in C_n.$$

It follows from $F(T) \subset C_n$ that

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall z' \in F(T). \tag{2.11}$$

By virtue of Lemma 1.2, we obtain that

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(\Pi_{F(T)} x_0, x_0) - \phi(\Pi_{F(T)} x_0, x_n) \\ &\leq \phi(\Pi_{F(T)} x_0, x_0). \end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (1.2) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed, and convex, we see that $\bar{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned} \phi(\bar{x}, x_0) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_0 \rangle + \|x_0\|^2 \\ &\leq \liminf_{n\to\infty} \bigl(\|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2\bigr) \\ &= \liminf_{n\to\infty} \phi(x_n, x_0) \\ &\leq \limsup_{n\to\infty} \phi(x_n, x_0) \\ &\leq \phi(\bar{x}, x_0), \end{aligned}$$

which implies that $\phi(x_n, x_0) \to \phi(\bar{x}, x_0)$ as $n \to \infty$. Hence, $\|x_n\| \to \|\bar{x}\|$ as $n \to \infty$. In view of the Kadec-Klee property of E, we see that $x_n \to \bar{x}$ as $n \to \infty$. Notice that $x_{n+1} = \Pi_{\bigcap_{i\in\Delta} F(T_i)} x_0 \in C_{n+1} \subset C_n$. It follows that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \end{aligned}$$

Since $x_n = \Pi_{C_n} x_0$, and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we arrive at $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$, $\forall n \geq 1$. This shows that $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty} \phi(x, x_0)$ exists. It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{2.12}$$

By virtue of $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we find that

$$\phi\left(x_n, T^n x_n\right) \leq \frac{2}{1-\kappa}\left\langle x_n - x_{n+1}, Jx_n - JT^n x_n\right\rangle + \mu_n \frac{M_n}{1-\kappa}.$$

It follows that

$$\lim_{n\to\infty} \phi\left(x_n, T^n x_n\right) = 0. \tag{2.13}$$

In view of (1.2), we see that

$$\lim_{n\to\infty} \left(\|x_n\| - \left\|T^n x_n\right\|\right) = 0.$$

Since $x_n \to \bar{x}$, we find that

$$\lim_{n\to\infty} \left\|T^n x_n\right\| = \|\bar{x}\|. \tag{2.14}$$

It follows that

$$\lim_{n\to\infty} \left\|J\left(T^n x_n\right)\right\| = \|J\bar{x}\|. \tag{2.15}$$

This implies that $\{J(T^n x_n)\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $J(T^n x_n) \rightharpoonup y^* \in E^*$. In view of the reflexivity of E, we see that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned}
\phi\left(x_n, T^n x_n\right) &= \|x_n\|^2 - 2\left\langle x_n, J\left(T^n x_n\right)\right\rangle + \left\|T^n x_n\right\|^2 \\
&= \|x_n\|^2 - 2\left\langle x_n, J\left(T^n x_n\right)\right\rangle + \left\|J\left(T^n x_n\right)\right\|^2.
\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above yields that

$$\begin{aligned}
0 &\geq \|\bar{x}\|^2 - 2\left\langle \bar{x}, y^*\right\rangle + \left\|y^*\right\|^2 \\
&= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|Jy\|^2 \\
&= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|y\|^2 \\
&= \phi(\bar{x}, y).
\end{aligned}$$

That is, $\bar{x} = y$, which in turn implies that $y^* = J\bar{x}$. It follows that $J(T^n x_n) \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (2.15) that $\lim_{n\to\infty} J(T^n x_n) = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demicontinuous, we find that $T^n x_n \rightharpoonup \bar{x}$. This implies, from (2.14) and the Kadec-Klee property of E, that $\lim_{n\to\infty} T^n x_n = \bar{x}$. Notice that

$$\left\|T^{n+1} x_n - \bar{x}\right\| \leq \left\|T^{n+1} x_n - T^n x_n\right\| + \left\|T^n x_n - \bar{x}\right\|.$$

It follows from the asymptotic regularity of T that

$$\lim_{n\to\infty} \left\|T^{n+1} x_n - \bar{x}\right\| = 0,$$

that is, $TT^n x_n - \bar{x} \to 0$ as $n \to \infty$. It follows from the closedness of T that $T\bar{x} = \bar{x}$.

Finally, we show that $\bar{x} = \Pi_{F(T)} x_0$. Letting $n \to \infty$ in (2.11), we arrive at

$$\langle \bar{x} - w, Jx_0 - J\bar{x} \rangle \geq 0, \quad \forall z' \in F(T).$$

It follows from Lemma 1.1 that $\bar{x} = \Pi_{F(T)} x_0$. The proof of Theorem 2.1 is completed. □

Remark 2.2 Comparing with the results in Zhou and Gao [13], the mapping was generalized from strict quasi-ϕ-pseudocontractions to asymptotically strict quasi-ϕ-pseudocontractions.

Remark 2.3 Comparing with the results in Qin, Wang, and Cho [14], the restriction imposed on the space was relaxed from uniform convexness to strict convexness.

Since the class of asymptotically strict quasi-ϕ-pseudocontractions includes the class asymptotically quasi-ϕ-nonexpansive mappings as a special case, we find the following subresults from Theorem 2.1.

Corollary 2.4 *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that T is asymptotically regular on C, and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ C_{n+1} = \{u \in C_n : \phi(x_n, T^n x_n) \leq 2\langle x_n - u, Jx_n - JT^n x_n \rangle + \mu_n M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases}$$

where $M_n = \sup\{\phi(p, x_n) : p \in F(T)\}$. Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{F(T)} x_0$.

In Hilbert spaces, asymptotically strict quasi-ϕ-pseudocontractions are reduced to asymptotically strict quasi-pseudocontractions. The following results are not hard to derive.

Corollary 2.5 *Let C be a nonempty, closed, and convex subset of a Hilbert space E. Let $T : C \to C$ be a closed and asymptotically strict quasi-pseudocontraction with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that T is asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following* *manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ C_{n+1} = \{u \in C_n : \|x_n - T^n x_n\|^2 \leq \frac{2}{1-\kappa}\langle x_n - u, x_n - T^n x_n\rangle + \mu_n \frac{M_n}{1-\kappa}\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $M_n = \sup\{\|p - x_n\|^2 : p \in F(T)\}$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(T)}x_0$.

For strict quasi-pseudocontractions, we have the following.

Corollary 2.6 *Let C be a nonempty, closed, and convex subset of a Hilbert space E. Let $T : C \to C$ be a closed and strict quasi-pseudocontraction with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ C_{n+1} = \{u \in C_n : \|x_n - Tx_n\|^2 \leq \frac{2}{1-\kappa}\langle x_n - u, x_n - Tx_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 1. \end{cases}$$

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} = P_{F(T)}x_0$.

Competing interests
The author declares that they have no competing interests.

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