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A fixed point theorem for cyclic generalized contractions in metric spaces

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Abstract
In this paper, we extend a recent result of V. Pata (J. Fixed Point Theory Appl. 10:299-305, 2011) in the frame of a cyclic representation of a complete metric space.

1 Introduction

One of the fundamental result in fixed point theory is the Banach contraction principle. It has various non-trivial applications in many branches of pure and applied sciences (see, for instance, [2, 7, 14] and references cited therein).

Let (X,d) be a metric space and $f : X \to X$ be an operator. We say that f is a contraction if there exists $\lambda \in [0,1)$ such that, for all $x,y \in X$,

$$d\bigl(f(x),f(y)\bigr) \le \lambda d(x,y). \tag{1.1}$$

In terms of Picard operator theory (see [13]), Banach contraction principle asserts that if f is a contraction and (X,d) is complete, then f is a Picard operator. This result has been extended to other important classes of maps. Recently, Pata [8] proved that if (X,d) is a complete metric space and $f : X \to X$ is an operator such that there exists fixed constants $\gamma \ge 0$, $\alpha \ge 1$ and $\beta \in [0,\alpha]$ such that, for every $\varepsilon \in [0,1]$ and every $x,y \in X$,

$$d\bigl(f(x),f(y)\bigr) \le (1-\varepsilon)d(x,y) + \gamma\varepsilon^{\alpha}\psi(\varepsilon)\bigl[1 + \|x\| + \|y\|\bigr]^{\beta} \tag{1.2}$$

(where $\psi : [0,1] \to [0,\infty)$ is an increasing function vanishing with continuity at zero and $\|x\| := d(x,x_0)$, with arbitrary $x_0 \in X$), then f has a unique fixed point in X.

Remark 1.1 (see [8]) The condition (1.2) is weaker than the contraction condition (1.1). In fact, if

$$d\bigl(f(x),f(y)\bigr) \le \lambda d(x,y), \quad \text{for every } x,y \in X \text{ and some } \lambda \in [0,1),$$

then it can be verified that, for every $x,y \in X$, we have

$$d\bigl(f(x),f(y)\bigr) \le (1-\varepsilon)d(x,y) + \gamma\varepsilon^{1+\theta}\bigl[1 + \|x\| + \|y\|\bigr], \quad \text{for every } \theta > 0,$$

where

$$\gamma = \gamma(\theta, \lambda) = \frac{\theta^\theta}{(1+\theta)^{1+\theta}} \frac{1}{(1-\lambda)^\theta}.$$

Remark 1.2 (see [8]) The function $f : [1,\infty) \to [1,\infty)$ defined as

$$f(x) = -2 + x - 2\sqrt{x} + 4\sqrt[4]{x}$$

has a unique fixed point $x^* = 1$, but fails to be a contraction on any neighborhood both of 1 and of ∞.

Kirk, Srinivasan and Veeramani [6] obtained an extension of Banach's fixed point theorem for mappings satisfying cyclical contractive conditions. Some generalizations of the results given in [6], using the setting of so-called fixed point structures, are presented in I. A. Rus [12]. In [10], Păcurar and Rus established a fixed point theorem for cyclic φ-contractions and they further discussed fixed point theory in metric spaces. In [3], Karapinar proved a fixed point theorem for cyclic weak φ-contraction mappings. Some other recent results concerning this topic are given in [1, 4, 5, 9, 11].

In the present paper, we obtain a fixed point theorem for a generalized contraction in the sense of the assumption (1.2), defined on a cyclic representation of a complete metric space.

2 Main results

We need first to recall a known concept.

Definition 2.1 ([3]) Let X be a nonempty set, m be a positive integer and $f : X \to X$ an operator. Then, we say that $\bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f if:

(i) $X = \bigcup_{i=1}^m A_i$, where A_i are nonempty sets for each $i \in \{1, \dots, m\}$;
(ii) $f(A_1) \subset A_2, \dots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Let (X,d) be a complete metric space. Selecting an arbitrary $x_1 \in X$, we denote

$$\|x\| := d(x, x_1), \quad \text{for all } x \in X.$$

Our main result is as follows.

Theorem 2.2 *Let (X,d) be a complete metric space, m be a positive integer, $A_1, \dots, A_m$ be closed nonempty subsets of X, $Y := \bigcup_{i=1}^m A_i$, $\psi : [0,1] \to [0,\infty)$ be an increasing function vanishing with continuity at zero, and $f : Y \to Y$ be an operator. Assume that:*

1. *$\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;*
2. *For every $\varepsilon \in [0,1]$, $x \in A_i$, and $y \in A_{i+1}$ ($i \in \{1, \dots, m\}$, where $A_{m+1} = A_1$), we have*

$$d\bigl(f(x), f(y)\bigr) \le (1-\varepsilon) d(x,y) + \gamma \varepsilon^\alpha \psi(\varepsilon) \bigl[1 + \|x\| + \|y\|\bigr]^\beta, \tag{2.1}$$

where $\gamma \ge 0$, $\alpha \ge 1$ and $\beta \in [0,\alpha]$ are fixed constants.

Then, we have the following conclusions:

(i) f is a Picard operator, i.e., f has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$ and the Picard iteration sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to x^*, for any initial point $x \in Y$;

(ii) the following estimates hold:

$$d(x_n, x^*) \le \|x^*\|, \quad n \ge 2;$$
$$d(x_n, x_1) \le 2\|x^*\|, \quad n \ge 2.$$

Proof (i) For convenience of notation, if $j > m$, define $A_j = A_i$ where $i = j \mod m$ and $1 \le i \le m$. Let $x_1 \in A_1$. Starting from x_1, let $\{x_n\}_{n\ge1}$ be the Picard iteration defined by the sequence

$$x_n = f(x_{n-1}) = f^{n-1}(x_1), \quad n \ge 2,$$

and set $c_n = \|x_n\|$. Assume $x_n \ne x_{n+1}$ for all n. By (2.1), we have

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n) \le \cdots \le d(x_1, x_2) = c_2. \tag{2.2}$$

First, we prove that the sequence $(c_n)_{n\in\mathbb{N}^*}$ is bounded. By (2.2) we get that

$$\begin{aligned} c_n &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_2) + d(x_2, x_1) \le d(x_{n+1}, x_2) + 2c_2 \\ &= d\big(f(x_n), f(x_1)\big) + 2c_2. \end{aligned}$$

Since $x_1 \in A_1$ and $x_n \in A_n$, from (2.1), we obtain that

$$\begin{aligned} c_n &\le (1-\varepsilon)d(x_n, x_1) + \gamma\varepsilon^\alpha\psi(\varepsilon)\big[1 + \|x_n\| + \|x_1\|\big]^\beta + 2c_2 \\ &= (1-\varepsilon)c_n + \gamma\varepsilon^\alpha\psi(\varepsilon)[1 + c_n]^\beta + 2c_2 \\ &\le (1-\varepsilon)c_n + a\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha + b, \end{aligned}$$

where $c_1 = \|x_1\| = d(x_1, x_1) = 0$, $\beta \le \alpha$, and for some $a, b > 0$. Thus,

$$\varepsilon c_n \le a\varepsilon^\alpha\psi(\varepsilon)c_n^\alpha + b.$$

If there is a subsequence $(c_{n_k})_{k\in\mathbb{N}^*} \to \infty$, the choice $\varepsilon = \varepsilon_k = \frac{(1+b)}{c_{n_k}}$ leads to the contradiction

$$1 \le a(1+b)^\alpha\psi(\varepsilon_k) \to 0.$$

Therefore, the sequence (c_n) is bounded.

From (2.2) we obtain that the sequence $\{d(x_n, x_{n+1})\}$ is nonincreasing and then it is convergent to the real number

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = \inf\big\{d(x_{n-1}, x_n) : n = 2, 3, \ldots\big\}.$$

Now we show that $r = 0$. Assume that $r > 0$. Let $x_n \in A_n$ and $x_{n+1} \in A_{n+1}$. By (2.1), we have

$$\begin{aligned} r &\le d(x_n, x_{n+1}) = d\big(f(x_{n-1}), f(x_n)\big) \\ &\le (1-\varepsilon) d(x_{n-1}, x_n) + \gamma \varepsilon^\alpha \psi(\varepsilon)\big[1 + \|x_{n-1}\| + \|x_n\|\big]^\beta \\ &\le (1-\varepsilon) d(x_{n-1}, x_n) + K\varepsilon\psi(\varepsilon), \end{aligned}$$

for some $K > 0$. Letting $n \to \infty$, we obtain

$$r \le K\psi(\varepsilon), \quad \text{for every } \varepsilon \in [0,1],$$

which implies $r = 0$. This leads to a contradiction, therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

For $p \ge 1$, suppose there exists j, $0 \le j \le m-1$, such that $(n+p) - n + j = 1 \mod m$, *i.e.*, $p + j = 1 \mod m$. Now, let p be fixed, $j = 0$ and let

$$q_n = n^\alpha d(x_n, x_{n+p}).$$

So, we have

$$q_{n+1} = (n+1)^\alpha d(x_{n+1}, x_{n+1+p}) = (n+1)^\alpha d\big(f(x_n), f(x_{n+p})\big).$$

Since $p = 1 \mod m$, x_n and x_{n+p} lie in different sets A_i and A_{i+1}, for some $1 \le i \le m$. Then by (2.1) we have

$$q_{n+1} = (n+1)^\alpha (1-\varepsilon) d(x_n, x_{n+p}) + C(n+1)^\alpha \varepsilon^\alpha \psi(\varepsilon), \tag{2.3}$$

where $C = \sup \gamma(1 + 2c_n)^\beta < \infty$. Choosing for each n

$$\varepsilon = 1 - \left(\frac{n}{n+1}\right)^\alpha \le \frac{\alpha}{n+1},$$

the relation (2.3) becomes

$$q_{n+1} \le n^\alpha d(x_n, x_{n+p}) + C\alpha^\alpha \psi\left(\frac{\alpha}{n+1}\right) = q_n + C\alpha^\alpha \psi\left(\frac{\alpha}{n+1}\right).$$

Since $q_0 = 0$, it follows that

$$q_n = \sum_{k=1}^{n} (q_k - q_{k-1}) \le \sum_{k=1}^{n} C\alpha^\alpha \psi\left(\frac{\alpha}{k}\right) = C\alpha^\alpha \sum_{k=1}^{n} \psi\left(\frac{\alpha}{k}\right).$$

Consequently,

$$d(x_n, x_{n+p}) \le C\left(\frac{\alpha}{n}\right)^\alpha \sum_{k=1}^{n} \psi\left(\frac{\alpha}{k}\right).$$

This shows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (Y,d) and, thus, it is convergent to a point $y \in Y = \bigcup_{i=1}^m A_i$. The case $j \neq 0$ similar.

On the other hand, the sequence $\{x_n\}$ has an infinite number of terms in each A_i, for every $i \in \{1,\dots,m\}$. Since (Y,d) is complete, in each A_i, $i \in \{1,\dots,m\}$ we can construct a subsequence of $\{x_n\}$ which converges to y. Since each A_i is closed for $i \in \{1,\dots,m\}$, we get that $y \in \bigcap_{i=1}^m A_i$. Then $\bigcap_{i=1}^m A_i \neq \emptyset$ and we can consider the restriction

$$g := f|_{\bigcap_{i=1}^m A_i} : \bigcap_{i=1}^m A_i \to \bigcap_{i=1}^m A_i,$$

which satisfies the conditions of Theorem 1 in [8], since $\bigcap_{i=1}^m A_i$ is also closed and complete. From this result, it follows that g has a unique fixed point, say $x^* \in \bigcap_{i=1}^m A_i$.

We claim now that for any initial value $x \in Y$, we get the same limit point $x^* \in \bigcap_{i=1}^m A_i$. Indeed, for $x \in Y = \bigcup_{i=1}^m A_i$, by repeating the above process, the corresponding iterative sequence yields that g has a unique fixed point, say $z \in \bigcap_{i=1}^m A_i$. Since x^*, $z \in \bigcap_{i=1}^m A_i$, we have x^*, $z \in A_i$ for all $i \in \{1,\dots,m\}$ and, hence, $d(x^*,z)$ and $d(f(x^*),f(z))$ are well defined. We can write (2.1) in the form

$$d(x^*,z) = d(f(x^*),f(z)) \leq (1-\varepsilon)d(x^*,z) + K\varepsilon\psi(\varepsilon),$$

for some $K > 0$. Suppose that $\varepsilon = 0$. Then we have

$$d(f(x^*),f(z)) \leq d(x^*,z).$$

If equality occurs, the relation

$$d(x^*,z) \leq K\psi(\varepsilon)$$

is valid for every $\varepsilon \in [0,1]$, which implies $d(x^*,z) = 0$. Thus, x^* is the unique fixed point of f for any initial value $x \in Y$.

To prove that the Picard iteration converges to x^*, let us consider $x_1 \in Y = \bigcup_{i=1}^m A_i$. Then there exists $i_0 \in \{1,\dots,m\}$ such that $x_n \in A_{i_0}$. As $x^* \in \bigcap_{i=1}^m A_i$ it follows that $x^* \in A_{i_0+1}$ as well. By the continuity of f, we obtain

$$d(f^{n-1}(x_1),x^*) = d(f(x_{n-1}),x^*) = d(x_n,x^*) = \lim_{p\to\infty} d(x_n,x_{n+p}) \leq C\left(\frac{\alpha}{n}\right)^{\alpha} \sum_{k=1} n\psi\left(\frac{\alpha}{k}\right).$$

Letting $n \to \infty$, it follows that $(x_n) \to x^*$, *i.e.*, the Picard iteration converges to the unique fixed point of f for any initial point $x_1 \in Y$.

(ii) Since x^* is a fixed point and $x^* \in \bigcap_{i=1}^m A_i$, we obtain that

$$d(x_n,x^*) = d(f(x_{n-1}),f(x^*)) \leq d(x_{n-1},x^*) \leq \cdots \leq d(x_1,x^*) = \|x^*\|. \tag{2.4}$$

By (2.4), it follows that

$$d(x_n,x_1) \leq d(x_n,x^*) + d(x^*,x_1) \leq \|x^*\| + d(x^*,x_1) \leq 2\|x^*\|. \qquad \square$$

In view of Remark 1.1, we immediately obtain the following corollary.

Corollary 2.3 (Kirk, Srinivasan, Veeramani [2, Theorem 1.3]) *Let (X,d) be a complete metric space, m be a positive integer, $A_1,\dots,A_m$ be closed nonempty subsets of X, $Y := \bigcup_{i=1}^m A_i$ and $f : Y \to Y$ be an operator. Assume that:*

(i) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;

(ii) there exists $\lambda \in [0,1)$ such that, for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$, we have

$$d\bigl(f(x),f(y)\bigr) \le \lambda d(x,y).$$

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$.

Finally, we will prove a periodic point theorem. For this purpose, notice first that if f satisfies (1.2) with constants α, β, γ and function ψ, and if $\|f(x)\| \le \|x\|$ for each $x \in X$, then its m-iterate f^m also satisfies the condition (1.2) with constants α, β, $m\gamma$ and function ψ. Indeed, let us suppose that f satisfies (1.2) with constants α, β, γ. Then, for every $\varepsilon \in [0,1]$, we have

$$\begin{aligned}
&d\bigl(f^2(x),f^2(y)\bigr)\\
&\quad\le (1-\varepsilon)d\bigl(f(x),f(y)\bigr) + \gamma\varepsilon^\alpha\psi(\varepsilon)\bigl[1+\bigl\|f(x)\bigr\|+\bigl\|f(y)\bigr\|\bigr]^\beta\\
&\quad\le (1-\varepsilon)\bigl[(1-\varepsilon)d(x,y) + \gamma\varepsilon^\alpha\psi(\varepsilon)\bigl(1+\|x\|+\|y\|\bigr)^\beta\bigr]\\
&\qquad + \gamma\varepsilon^\alpha\psi(\varepsilon)\bigl[1+\bigl\|f(x)\bigr\|+\bigl\|f(y)\bigr\|\bigr]^\beta\\
&\quad\le (1-\varepsilon)\bigl[(1-\varepsilon)d(x,y) + \gamma\varepsilon^\alpha\psi(\varepsilon)\bigl(1+\|x\|+\|y\|\bigr)^\beta\bigr]\\
&\qquad + \gamma\varepsilon^\alpha\psi(\varepsilon)\bigl[1+\|x\|+\|y\|\bigr]^\beta\\
&\quad= (1-\varepsilon)^2 d(x,y) + (1-\varepsilon)\gamma\varepsilon^\alpha\psi(\varepsilon)\bigl(1+\|x\|+\|y\|\bigr)^\beta\\
&\qquad + \gamma\varepsilon^\alpha\psi(\varepsilon)\bigl[1+\|x\|+\|y\|\bigr]^\beta\\
&\quad= (1-\varepsilon)^2 d(x,y) + (2-\varepsilon)\gamma\varepsilon^\alpha\psi(\varepsilon)\bigl(1+\|x\|+\|y\|\bigr)^\beta\\
&\quad\le (1-\varepsilon)d(x,y) + 2\gamma\varepsilon^\alpha\psi(\varepsilon)\bigl(1+\|x\|+\|y\|\bigr)^\beta.
\end{aligned}$$

Thus, we immediately get that, for $m \in \mathbb{N}$ with $m \ge 2$, we have

$$d\bigl(f^m(x),f^m(y)\bigr) \le (1-\varepsilon)d(x,y) + m\gamma\varepsilon^\alpha\psi(\varepsilon)\bigl(1+\|x\|+\|y\|\bigr)^\beta.$$

Notice also that if $\bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f, then each A_i ($i \in \{1,2,\dots,m\}$) is an invariant set with respect to f^m. Using these two remarks, we get the following periodic point theorem.

Theorem 2.4 *Let (X,d) be a complete metric space, m be a positive integer, $A_1,\dots,A_m$ be nonempty subsets of X, $Y := \bigcup_{i=1}^m A_i$, $\psi : [0,1] \to [0,\infty)$ be an increasing function vanishing with continuity at zero and $f : Y \to Y$ be an operator such that $\|f(x)\| \le \|x\|$ for each $x \in Y$. Assume that:*

1. *$\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f.*
2. *There exists $i_0 \in \{1,\dots,m\}$ such that A_{i_0} is closed.*

3. *For every* $\varepsilon \in [0,1]$ *and each* $x, y \in A_{i_0}$, *we have*

$$d\big(f(x), f(y)\big) \leq (1-\varepsilon)d(x,y) + \gamma\varepsilon^{\alpha}\psi(\varepsilon)\big[1 + \|x\| + \|y\|\big]^{\beta}, \tag{2.1}$$

where $\gamma \geq 0$, $\alpha \geq 1$ *and* $\beta \in [0, \alpha]$ *are fixed constants.*

Then, f^m *has a fixed point.*

Proof Notice that, by the above considerations, f^m is a self mapping on A_{i_0} and it satisfies the condition (1.2) with constants α, β, $m\gamma$ and function ψ. Thus, by Theorem 1 in [8] we get the conclusion. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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