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Strong convergence theorems for a common point of solution of variational inequality, solutions of equilibrium and fixed point problems

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Abstract

We introduce an iterative process which converges strongly to a common point of solution of variational inequality problem for continuous monotone mapping, solution of equilibrium problem and a common fixed point of finite family of asymptotically regular uniformly continuous relatively asymptotically nonexpansive mappings in Banach spaces. Our scheme does not involve computation of C_{n+1} from C_n for each $n \geq 1$. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

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Introduction

Let E be a real Banach space with dual E^*. A *normalized duality* mapping $J: E \to 2^{E^*}$ is defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. It is well known that E is smooth if and only if J is single-valued and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [1]).

Throughout this article, we denote by φ: $E \times E \to \mathbb{R}$ the function defined by

$$\phi(y, x) = ||y||^2 - 2\langle y, Jx\rangle + ||x||^2, \quad \text{for } x, y \in E, \tag{1.1}$$

which was studied by Alber [2], Kamimula and Takahashi [3], and Reich [4]. It is obvious from the definition of the function φ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \quad \text{for } x, y \in E. \tag{1.2}$$

where J is the normalized duality mapping. We remark that in a Hilbert space H, (1.1) reduces to $\varphi(x, y) = ||x - y||^2$, for any $x, y \in H$.

Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space E. The *generalized projection mapping*, introduced by Alber [2], is a mapping Π_C: $E \to C$ that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(y, x)$, i.e., $\prod_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x), y \in C\}. \tag{1.3}$$

A mapping A: $D(A) \subset E \to E^*$ is said to be *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x - y, Ax - Ay\rangle \geq 0. \tag{1.4}$$

A is said to be *γ-inverse strongly monotone* if there exists positive real number γ such that

$$\langle x - y, Ax - Ay\rangle \geq \gamma ||Ax - Ay||^2, \quad \text{for all } x, y \in D(A). \tag{1.5}$$

Suppose that A is monotone mapping from $C \subseteq E$ into E^*. The variational inequality problem is formulated as finding

$$\text{a point } u \in C \text{ such that } \langle v - u, Au\rangle \geq 0, \quad \text{for all } v \in C. \tag{1.6}$$

The set of solutions of the variational inequality problem is denoted by VI(C, A).

Variational inequalities were initially studied by Stampacchia [5] and ever since have been widely studied. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in C$ satisfying $0 \in Au$. If $E = H$, a Hilbert space, one method of solving a point $u \in$ VI(C, A) is the projection algorithm which starts with any point $x_0 = x \in C$ and updates iteratively as x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \alpha_n Ax_n), \qquad \text{for any } n \geq 0, \tag{1.7}$$

where P_C is the metric projection from H onto C and $\{\alpha_n\}$ is a sequence of positive real numbers. In the case that A is γ-inverse strongly monotone, Iiduka, Takahashi and Toyoda [6] proved that the sequence $\{x_n\}$ generated by (1.7) converges *weakly* to some element of VI(C, A).

When the space E is more general than a Hilbert spaces, Iduka and Takahashi [7] introduced the following iteration scheme for finding a solution of the variational inequality problem for an γ-inverse strongly monotone operator A in 2-uniformly convex and uniformly smooth spaces

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \quad \text{for any } n \geq 0, \tag{1.8}$$

where Π_C is the generalized projection from E onto C, J is the normalized duality mapping from E into E^* and $\{\alpha_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.8) *converges weakly* to some element of VI(C, A).

Our concern now is the following: *Is it possible to construct a sequence* $\{x_n\}$ *which converges strongly to some point of VI*(C, A)?

In this connection, when $E = H$, a Hilbert space and A is γ-inverse strongly monotone, Iiduka, Takahashi and Toyoda [6] studied the following iterative scheme:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ y_n = P_C(x_n - \alpha_n A x_n), \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 0, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $[0, 2\gamma]$. They proved that the sequence $\{x_n\}$ generated by (1.9) converges strongly to $P_{VI(C, A)}(x_0)$, where $P_{VI(C, A)}$ is the metric projection from H onto VI(C, A).

In the case that E is 2-uniformly convex and uniformly smooth Banach space, Iiduka and Takahashi [8] studied the following iterative scheme for a variational inequality problem for γ-inverse strongly monotone mapping A:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ y_n = \Pi_C J^{-1}(Jx_n - \alpha_n A x_n), \\ C_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \geq 0, \end{cases} \tag{1.10}$$

where $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$, J is the duality mapping from E into E^* and $\{\alpha_n\}$ is a positive real sequence satisfying certain condition. Then, they proved that the sequence $\{x_n\}$ converges strongly to an element of VI(C, A) provided that VI(C, A) ≠ ∅ and A satisfies $||A_y|| \leq ||A_y - A_u||$, for all $y \in C$, = and $u \in VI(C, A)$.

Remark **1.1**. We remark that the computation of x_{n+1} in their algorithms is not simple because of the involvement of computation of C_{n+1} from C_n and Q_n, for each $n \geq 0$.

Let C be a nonempty, closed and convex subset of a real Banach space E with dual E^*. Let T be a mapping from C into itself. An element $p \in C$ is called a *fixed point* of T if $T(p) = p$. The set of fixed points of T is denoted by $F(T)$. A point p in C is said to be an *asymptotic fixed point of* T (see [4]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is said to be *nonexpansive* if $||Tx - Ty|| \leq ||x - y||$, for each $x, y \in C$, and is called *relatively nonexpansive* if (R1) $F(T) \neq \emptyset$ (R2) $\varphi(p, Tx) \leq \varphi(p, x)$, for $x \in C$ and (R3) $F(T) = \hat{F}(T)$. T is called *relatively quasi-nonxpansive* if $F(T) = \emptyset$ and $\varphi(p, T_x) \leq \varphi(p, x)$, for all $x \in C$, and $p \in F(T)$.

A mapping T from C into itself is said to be *asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ and $||T^n x - T^n y|| \leq k_n ||x - y||$, for each $x, y \in C$, and is called *relatively asymptotically nonexpansive* if there exists $\{k_n\} \subset [1, \infty)$ such that (N1) $F(T) \neq \emptyset$ (N2) $\varphi(p, T^n x) \leq k_n \varphi(p, x)$, for $x \in C$, and (N3) $F(T) = \hat{F}(T)$, where $k_n \to 1$, as $n \to \infty$. T is called *relatively asymptotically quasinonxpansive* if there exist $\{k_n\} \subset [1, \infty)$ and $F(T) = \emptyset$ such that $\varphi(p, T^n x) \leq k_n \varphi(p, x)$, for $x \in C$, and $p \in F(T)$, where $k_n \to 1$, as $n \to \infty$. A mapping T from C into itself is said to be φ-nonexpansive (nonextensive [9]) if $\varphi(Tx, Ty) \leq \varphi(x, y)$ for all $x, y \in C$ and it is called φ-asymptotically nonexpansive if there exists $\{k_n\} \subset [1, \infty)$ such that $\varphi(T^n x, T^n y) \leq k_n \varphi(x, y)$, for all $x, y \in C$, where $k_n \to 1$, as $\to\infty$. A self-mapping on C is called *asymptotically regular* on C, if for any bounded subset $\bar{C}$ of C, there holds the following equality:

$$\limsup_{n\to\infty}\{||T^{n+1}x - T^n x|| : x \in \bar{C}\} = 0.$$

T is called *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Clearly, the class of relatively asymptotically nonexpansive mappings contains the class of relatively nonexpansive mappings.

It is well known that, in an infinite-dimensional Hilbert space, the normal *Mann's* iterative [10] algorithm has only weak convergence, in general, even for nonexpansive mappings. Consequently, to obtain strong convergence, some modifications of the normal Mann's iteration algorithm has been introduced. The so-called hybrid projection iteration algorithm (HPIA) is one of such modifications, which was introduced by Haugazeau [11] in 1968. Since then, there has been a lot of activity in this area and several modifications appeared. For details, the readers are referred to papers [12-18] and the references therein.

In 2003, Nakajo and Takahashi [17] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : || y_n - z|| \le ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \ge 0, \end{cases} \tag{1.11}$$

where C is a closed convex subset of H, Π_C is the generalized metric projection from E onto C. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.11) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denote the fixed points set of T.

In spaces more general than Hilbert spaces, Matsushita and Takahashi [16] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) < \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), n \ge 0, \end{cases} \tag{1.12}$$

They proved the following convergence theorem.

Theorem MT. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_n \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.12), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(.)$ is the generalized projection from E onto $F(T)$.

Let f: $C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for f is

$$\text{findmg } x^* \in C \text{ such that } f(x^*, y) \ge 0, \quad \forall y \in C. \tag{1.13}$$

The solution set of (1.13) is denoted by $EP(f)$.

Numerous problems in physics, optimization and economics reduce to find a solution of (1.13) (see, e.g., [19,20]). For studying the equilibrium problem (1.13), we assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$, for all $x \in C$,
(A2) f is monotone, i.e., $f(x, y)+ f(y, x) \leq 0$, for all $x, y \in C$,
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$,
(A4) for each $x \in C$, $y \to f(x, y)$ is convex and lower semicontinuous.

Recently, many authors have considered the problem of finding a common element of the fixed points set of relatively nonexpansive mapping, the solution set of equilibrium problem and solution set of variational inequality problem for γ-inverse monotone mapping (see, e.g., [21-26]). If E is uniformly convex and smooth Banach space, then Aoyama, Kohsaka and Takahashi [27] constructed a sequence which converges strongly to a common solution of variational inequality problems for two monotone mappings.

Recently, Qin et al. [22] proved the following result:

Theorem QCK. Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed and convex subset of E.

Let f: $C \times C \to R$ be a bifunction satisfying (A1)-(A4) and let T, S: $C \to C$ be two closed relatively quasi- nonexpansive mappings such that $F = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} C_1 = C, \text{ and } x_0 \in C, \text{ chosen arbitrary,} \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C: \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \ Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} \ = \{z \in C_n : \ \phi(z, \ u_n) \leq \phi(z, \ x_n)\}, \\ x_{n+1} \ = \Pi_{C_{n+1}}(x_0), n \geq 0, \end{cases} \tag{1.14}$$

where Π_C is the generalized metric projection from E onto C, J is the normalized duality mapping on E, $\{r_n\}$ is a positive sequence and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in 0[1] satisfying certain conditions. Then $\{x_n\}$ converges strongly to $\Pi_F(x_0)$.

Furthermore, Zegeye and Shahzad [28] studied the following iterative scheme for common point of solution of a variational inequality problem for γ-inverse strongly monotone mapping A and fixed point of a continuous φ-asymptotically nonexpansive mapping S in a 2-uniformly convex and uniformly smooth Banach space E

$$\begin{cases} C_0 = \ C, \text{ and } x_0 \in C, \text{ chosen arbitrary,} \\ z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = \ J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JS^n z_n), \\ u_n \in C: \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \ Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} \ = \{z \in C_n : \ \phi(z, \ u_n) \leq \phi(z, \ x_n) + \theta_n\}, \\ x_{n+1} \ = \Pi_{C_{n+1}}(x_0), n \geq 0, \end{cases} \tag{1.15}$$

where C is closed, convex and bounded subset of E, $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam}(C))^2$ and $\{\alpha_n\}$, $\{\lambda_n\}$ are sequences satisfying certain condition. Then, they proved that the sequence $\{x_n\}$ converges strongly to an element of F: $= F(S) \cap \mathrm{VI}(C, A)$ provided that

$F \neq \emptyset$ and A satisfies $||Ay|| \leq ||Ay - Ap||$, for all $y \in C$ and $p \in F$. As it is mentioned in [29], we remark that the computation of x_{n+1} in Algorithms (1.11), (1.12), (1.14) and (1.15) is not simple because of the involvement of computation of C_{n+1} from C_n, for each $n \geq 0$.

It is our purpose in this article to introduce an iterative scheme $\{x_n\}$ which converges strongly to a common point of solution of variational inequality problem for continuous monotone mapping, solution of equilibrium problem and a common fixed point of finite family of asymptotically regular uniformly continuous relatively asymptotically nonexpansive mappings in Banach spaces. Our scheme does not involve computation of C_{n+1} from C_n for each $n \geq 1$. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

Preliminaries

In the sequel, we shall use of the following lemmas.

Lemma 2.1. [2]*Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then $\forall y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x).$$

Lemma 2.2. [3]*Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$, as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Lemma 2.3. [2]*Let C be a convex subset of a real smooth Banach space E. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \ \forall z \in C.$$

We make use of the function $V: E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2, \quad \text{for all } x \in E \text{ and } x^* \in E^*,$$

studied by Alber [2], i.e., $V(x, x^*) = \varphi(x, J^{-1}x^*)$, for all $x \in E$ and $x^* \in E^*$.

We know the following lemma related to the function V.

Lemma 2.4. [2]*Let E be reflexive strictly convex and smooth Banach space with E^* as its dual. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.5. [24]*Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function g: $[0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that $||\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k||^2 \leq \sum_{i=0}^{k} \alpha_i ||x_i||^2 - \alpha_i \alpha_j g(||x_i - x_j||)$, for $0 \leq i \leq j \leq k$, and each $\alpha_i \in (0, 1)$, where $x_i \in B_R(0) := \{x \in E : ||x|| \leq R\}$, $i = 0, 1, 2, \ldots, k$ with $\sum_{i=0}^{k} \alpha_i = 1$.*

Proposition 2.6. [30]*Let E be uniformly convex and uniformly smooth Banach space, let C be closed convex subset of E, and let S be closed relatively asymptotically nonexpansive mapping from C into itself. Then F(S) is closed and convex.*

Lemma 2.7. [23]*Let C be a nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive real Banach space E. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies conditions* (A_1)-(A_4). *For* $r > 0$ *and* $x \in E$, *define the mapping* $T_r: E \to C$ *as follows:*

$$T_r x := \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then the following statements hold:

(1) T_r *is single-valued;*
(2) $F(T_r) = \mathrm{EP}(f)$;
(3) $\varphi(q, T_r x) + \varphi(T_r x, x) \leq \varphi(q, x)$, *for* $q \in F(T_r)$.
(4) $\mathrm{EP}(f)$ *is closed and convex;*

Lemma 2.8. [29]*Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E. Let* $A: C \to E^*$ *be a continuous monotone mapping. For* $r > 0$ *and* $x \in E$, *define the mapping* $F_r: E \to C$ *as follows:*

$$F_r x = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then conclusions (1)-(4) of Lemma 2.7 hold.

Lemma 2.9 [31]. *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n \delta_n, n \geq n_0, \quad \text{for some } n_0 \in \mathbb{N},$$

where $\{\beta_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset R$ *satisfying the following conditions:*

$$\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty, \text{ and } \limsup_{n\to\infty} \delta_n \leq 0. \text{ Then, } \lim_{n\to\infty} a_n = 0.$$

Lemma 2.10 [32]. *Let* $\{a_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $a_{n_i} < a_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k: a_j < a_{j+1}\}$.

Main result

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space E with dual E^*. Let $f: C \times C \to \mathbb{R}$ be a bifunction and $A: C \to E^*$ be a continuous monotone mapping. For the rest of this article, $T_{r_n}x$ and $F_{r_n}x$ are mappings defined as follows: For $x \in E$, let $F_{r_n}, T_{r_n} : E \to C$ be given by

$$F_{r_n}x := \{z \in C : f\langle y - z, Az\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\},$$

and

$$T_{r_n}x : \left\{z \in C : f(z, y) + \frac{1}{r_n}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\right\},$$

where $\{r_n\}_{n\in \mathbb{N}} \subset [c_1,\infty)$ for some $c_1 > 0$.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let f: C × C → ℝ be a bifunction which satisfies conditions (A1)-(A4). Let A: C →E* be a continuously monotone mapping. Let T_i: C → C be a asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequence $\{k_{n,\ i}\}$ for i = 1, 2, . . . , N. Assume that* $F := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap EP(f)$ *is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \text{ chosen arbitrarity,} \\ u_n = F_{r_n}x_n, \\ w_n = T_{r_n}u_n, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1\text{-}\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_{n,0}Jw_n + \sum_{i=1}^N \beta_{n,i}JT_i^n y_n), n \geq 0, \end{cases} \tag{3.1}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_{n,i}-1)}{\alpha_n} = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\{\beta_{n,\ i}\} \subset [a, b] \subset (0, 1)$, *for i = 1, 2, . . . , N, satisfying* $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,\ N} = 1$, *for each* $n \geq 0$. *Then $\{x_n\}$ converges strongly to an element of F.*

Proof. Since F is nonempty closed and convex, put $x^*: = \Pi_F w$. Now, from (3.1), Lemmas 2.1, 2.7(3), 2.8(3) and property of φ we get that

$$\begin{aligned} \phi(x^*, y_n) &= \phi(x^*, \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n) \\ &\leq \phi(x^*, J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n) \\ &= ||x^*||^2 - 2\langle x^*, \alpha_n Jw + (1 - \alpha_n)Jw_n\rangle \\ &\quad + ||\alpha_n Jw + (1 - \alpha_n)Jw_n||^2 \\ &\leq ||x^*||^2 - 2\alpha_n\langle x^*, Jw\rangle - 2(1 - \alpha_n)\langle x^*, Jw_n\rangle \\ &\quad + \alpha_n||Jw||^2 + (1 - \alpha_n)||JW_n||^2, \\ &\leq \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, w_n) \\ &= \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, T_{r_n}u_n) \\ &\leq \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, u_n) \\ &= \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, F_{r_n}x_n) \\ &\leq \alpha_n\phi(x^*, w) + (1 - \alpha_n)\phi(x^*, x_n). \end{aligned} \tag{3.2}$$

Let $k_n: = \max\{k_{n,\ i}: i = 1, 2, . . . , N\}$. Then, from (3.1), Lemma 2.7(3), Lemma 2.8(3), relatively asymptotic nonexpansiveness of T_i, property of φ and (3.2) we have that

$$
\begin{aligned}
\phi(x^*,\ x_{n+1}) &= \phi\left(x^*, J^{-1}\left(\beta_{n,0}Jw_n + \sum_{i=1}^{N}\beta_{n,i}JT_i^n y_n\right)\right.\\
&\le\ \beta_{n,0}\phi(x^*, w_n) + \left.\sum_{i=1}^{N}\beta_{n,i}\phi(x^*, T_i^n y_n)\right)\\
&= \beta_{n,0}\phi(x^*, T_{r_n}u_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(x^*, T_i^n y_n)\\
&\le\ \beta_{n,0}\phi(x^*, u_n) + (1-\beta_{n,0})k_n\phi(x^*, y_n)\\
&= \beta_{n,0}\phi(x^*, F_{r_n}x_n) + (1-\beta_{n,0})k_n\phi(x^*, y_n)\\
&\le\ \beta_{n,0}\phi(x^*, x_n) + (1-\beta_{n,0})\phi(x^*, y_n)\\
&\quad + (1-\beta_{n,0})(k_n-1)\phi(x^*, y_n),\\
&\le\ \beta_{n,0}\phi(x^*,\ x_n) + (1-\beta_{n,0})[\alpha_n\phi(x^*, w) + (1-\alpha_n)\phi(x^*, x_n)]\\
&\quad + (1-\beta_{n,0})(k_n-1)[\alpha_n\phi(x^*, w) + (1-\alpha_n)\phi(x^*, x_n)]\\
&\le [\alpha_n(1-\beta_{n,0}) + (1-\beta_{n,0})(k_n-1)\alpha_n]\phi(x^*, w)\\
&\quad + [(1-\alpha_n(1-\beta_{n,0})) + (1-\beta_{n,0})(k_n-1)(1-\alpha_n)]\\
&\quad \times \phi(x^*, x_n)\\
&\le \delta_n\phi(x^*, w) + [1-(1-\varepsilon)\delta_n]\phi(x^*, x_n),
\end{aligned}
\tag{3.3}
$$

where $\delta_n = (1 - \beta_{n,0})k_n\alpha_n$, since for some $\epsilon > 0$, there exists $N_0 > 0$ such that $\frac{(k_n-1)}{\alpha_n} \le \epsilon k_n$ and $(1 - \epsilon)\delta_n \le 1$, for all $n \ge N_0$. Thus, by induction

$$
\phi(x^*, x_{n+1}) \le \max\{\phi(x^*,\ x_{N_0}), (1-\epsilon)^{-1}\phi(x^*, w)\},\quad \forall n \ge N_0.
$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$, $\{u_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n)$. Then we have that $y_n = \Pi_C z_n$. Using Lemmas 2.1, 2.4, and property of φ we obtain that

$$
\begin{aligned}
\phi(x^*, y_n) &\le \phi(x^*, z_n) = V(x^*, Jz_n)\\
&\le\ V(x^*, Jz_n - \alpha_n(Jw - Jx^*)) - 2\langle z_n - x^*, \alpha_n(Jw - Jx^*)\rangle\\
&= \phi(x^*, J^{-1}(\alpha_n Jx^* + (1-\alpha_n)Jw_n)) + 2\alpha_n\langle z_n - x^*, Jw - Jx^*\rangle\\
&\le\ \alpha_n\phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle z_n -\ x^*, Jw - Jx^*\rangle\\
&= (1-\alpha_n)\phi(x^*, w_n) + 2\alpha_n\langle z_n -\ x^*, Jw - Jx^*\rangle\\
&\le (1-\alpha_n)\phi(x^*, u_n) + 2\alpha_n\langle z_n -\ x^*, Jw - Jx^*\rangle\\
&\le (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle z_n -\ x^*, Jw - Jx^*\rangle.
\end{aligned}
\tag{3.4}
$$

Furthermore, from (3.1), Lemma 2.5, relatively asymptotic nonexpansiveness of T_i, for each $i = 1, 2, \ldots, N$, Lemmas 2.7(3), (3.4), and 2.8(3) we have that

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J^{-1}\left(\beta_{n,0}Jw_n + \sum_{k=1}^{N}\beta_{n,i}JT_i^n y_n\right)\right)\\
&\le \beta_{n,0}\phi(x^*, w_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(x^*, JT_i^n y_n)\\
&\quad - \beta_{n,0}\beta_{n,i}g\ (||Jw_n\ JT_i^n y_n||),
\end{aligned}
$$

for each $i = 1, 2, \ldots, N$. This implies that

$$\begin{aligned}
&\phi(x^*, x_{n+1}) \\
&\leq \beta_{n,0}\phi(x^*, w_n) + (1-\beta_{n,0})k_n\phi(x^*, y_n) \\
&\quad - \beta_{n,0}\beta_{n,i}g(||Jw_n - JT_i^n y_n||) \\
&\leq \beta_{n,0}\left(\phi(x^*, u_n) - \phi(u_n, w_n)\right) + (1+\beta_{n,0})\phi(x^*, y_n) \\
&\quad + (1-\beta_{n,0})(k_n - 1)\phi(x^*, y_n) - \beta_{n,0}\beta_{n,i}g(||Jw_n - JT_i^n y_n||) \\
&\leq \beta_{n,0}(\phi(x^*, x_n) - \phi(u_n, x_n)) - \beta_{n,0}\phi(u_n, w_n) + (1-\beta_{n,0})\phi(x^*, y_n) \\
&\quad + (1-\beta_{n,0})(k_n - 1)\phi(x^*, y_n) - \beta_{n,0}\beta_{n,i}g(||Jw_n - JT_i^n y_n||), \\
&\leq \beta_{n,0}\phi(x^*, x_n) - \beta_{n,0}\left(\phi(u_n, x_n) + \phi(u_n, w_n)\right) \\
&\quad + (1-\beta_{n,0})\left[(1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle z_n - x^*, Jw - Jx^*\rangle\right] \\
&\quad + (1-\beta_{n,0})(k_n - 1)\phi(x^*, y_n) - \beta_{n,0}\beta_{n,i}g(||Jw_n - JT_i^n y_n||),
\end{aligned}$$

and hence

$$\begin{aligned}
&\phi(x^*, x_{n+1}) \\
&\leq (1-\theta_n)\phi(x^*, x_n) + 2\theta_n\langle z_n - x^*, Jw - Jx^*\rangle + (k_n - 1)M \\
&-\beta_{n,0}(\phi(u_n, x_n) + \phi(u_n, w_n)) - \beta_{n,0}\beta_{n,i}g(||Jw_n - JT_i^n y_n||)
\end{aligned} \tag{3.5}$$

$$\leq (1-\theta_n)\phi(x^*, x_n) + 2\theta_n\langle z_n - x^*, Jw - Jx^*\rangle + (k_n - 1)M, \tag{3.6}$$

for some $M > 0$, where $\theta_n := \alpha_n(1 - \beta_{n,0})$, for all $n \in \mathbb{N}$. Note that θ_n satisfies $\lim_n \theta_n = 0$ and $\sum_{n=1}^{\infty}\theta_n = \infty$.

Now, the rest of the proof is divided into two parts:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(x^*, x_n)\}$ is nonincreasing for all $n \geq n_0$. In this situation, $\{\{\varphi(x^*, x_n)\}$ is then convergent. Then from (3.5) we have that $\varphi(u_n, x_n)$, $\varphi(w_n, u_n) \to 0$ and hence Lemma 2.2 implies that

$$u_n - x_n \to 0,\ u_n - w_n \to 0, \quad \text{as } n \to \infty. \tag{3.7}$$

Moreover, from (3.5) we have that $\beta_{n,0}\beta_{n,i}g(||Jw_n - JT_i^n y_n|| \to 0$, for $i = 1, 2, \ldots, N$, which implies by the property of g that $Jw_n - JT_i^n y_n \to 0$, as $n \to \infty$, and hence, since J^{-1} uniformly continuous on bounded sets, we obtain that

$$w_n - T_i^n y_n \to 0,\ \text{as } n \to \infty, \quad \text{for each } i \in \{1, 2, \ldots, N\}. \tag{3.8}$$

Furthermore, Lemma 2.1, property of φ and the fact that $\alpha_n \to 0$, as $n \to \infty$, imply that

$$\begin{aligned}
\phi(w_n, y_n) = \phi(w_n, \Pi_C z_n) &\leq \phi(w_n, z_n) \\
&= \phi(w_n, J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n) \\
&\leq \alpha_n\phi(w_n, w) + (1-\alpha_n)\phi(w_n, w_n) \\
&= \alpha_n\phi(w_n, w) \to 0,\ \text{as } n \to \infty,
\end{aligned} \tag{3.9}$$

and that

$$w_n - y_n \to 0 \quad \text{and} \quad w_n - z_n \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

Therefore, from (3.7), (3.8), and (3.10) we obtain that

$$x_n - z_n \to 0, y_n - x_n \to 0 \quad \text{and} \quad y_n - T_i^n y_n \to 0, \quad \text{as } n \to \infty, \tag{3.11}$$

for each $i \in \{1, 2, \ldots, N\}$. Therefore, since

$$\begin{aligned} \|y_n - T_i y_n\| &\leq \|y_n - T_i^n y_n\| + \|T_i^n y_n - T_i^{n+1} y_n\| + \|T_i^{n+1} y_n - T_i y_n\|, \\ &= \|y_n - T_i^n y_n\| + \|T_i^n y_n - T_i^{n+1} y_n\| + \|T_i(T_i^n y_n) - T_i y_n\|, \end{aligned} \tag{3.12}$$

we have from (3.11), asymptotic regularity and uniform continuity of T_i that

$$\|y_n - T_i y_n\| \to 0, \quad \text{as } n \to \infty, \quad \text{for each } i = 1, 2, \ldots, N. \tag{3.13}$$

Since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty} \langle z_n - x^*, Jw - Jx^*\rangle = \lim_{i\to\infty} \langle z_{n_i} - x^*, Jw - Jx^*\rangle$. Then, from (3.7), (3.11) and the uniform continuity of J we get that

$$u_{n_i} w_{n_i} \rightharpoonup z \text{ and } Ju_n - Jx_n, Ju_n - Jw_n \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

Now, we show that $z \in VI(C, A)$. But from the definition of u_n we have that

$$\langle y - u_n, Au_n\rangle + \left\langle y - u_n, \frac{Ju_n - Jx_n}{r_n}\right\rangle \geq 0, \quad \forall y \in C. \tag{3.15}$$

and hence

$$\langle y - u_{n_i}, Au_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{Ju_{n_i} - Jx_{n_i}}{r_{n_i}}\right\rangle \geq 0, \quad \forall y \in C. \tag{3.16}$$

Set $v_t = ty + (1 - t)z$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we get that $v_t \in C$. Now, from (3.16) it follows that

$$\begin{aligned} \langle v_t - u_{n_i}, Av_t\rangle &\geq \langle v_t - u_{n_i}, Av_t\rangle - \langle v_t - u_{n_i}, Au_{n_i}\rangle \\ &\quad - \left\langle v_t - u_{n_i}, \frac{Ju_{n_i} - Jx_{n_i}}{r_{n_i}}\right\rangle \\ &= \langle v_t - u_{n_i}, Av_t - Au_{n_i}\rangle - \left\langle v_t - u_{n_i}, \frac{Ju_{n_i} - Jx_{n_i}}{r_{n_i}}\right\rangle. \end{aligned}$$

But, from (3.14) we have that $\frac{Ju_{n_i} - Jx_{n_i}}{r_{n_i}} \to 0$, as $i \to \infty$ and the monotonicity of A implies that $\langle v_t - u_{n_i}, Av_t - Au_{n_i}\rangle \geq 0$. Thus, it follows that

$$0 \leq \lim_{i\to\infty} \langle v_t - u_{n_i}, Av_t\rangle = \langle v_t - z, Av_t\rangle,$$

and hence

$$\langle y - z, Av_t\rangle \geq 0, \quad \forall y \in C.$$

If $t \to 0$, the continuity of A implies that

$$\langle y - z, Az\rangle \geq 0, \quad \forall y \in C.$$

This implies that $z \in VI(C, A)$.

Next, we show that $z \in$ EP(*f*). From the definition of w_n and (A2) we note that

$$\frac{1}{r_{n_i}}\langle y - w_{n_i}, Jw_{n_i} - Ju_{n_i}\rangle \geq -f(w_{n_i}, y) \geq f(y, w_{n_i}), \quad \forall y \in C. \tag{3.17}$$

Letting $i \to \infty$, we have from (3.14) and (A4) that $f(y, z) \leq 0$, for all $y \in C$. Now, for $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $f(y_t, z) \leq 0$. So, from the convexity of the equilibrium bifunction $f(x, y)$ on the second variable y, we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, z) \leq tf(y_t, y),$$

and hence $f(y_t, y) \geq 0$. Now, letting $t \to 0$, and condition (A3), we obtain that $f(z, y) \geq 0$, for all $y \in C$, and hence $z \in EP(f)$.

Finally, we show that $z \in \cap_{i=1}^N F(T_i)$. But, since each T_i satisfies condition (N3) we obtain from (3.13) that $z \in F(T_i)$ for each $i = 1, 2, \ldots, N$ and hence $z \in \cap_{i=1}^N F(T_i)$. Thus, from the above discussions we obtain that $z \in F := \cap_{i=1}^N F(T_i) \cap \mathrm{VI}(C, A) \cap \mathrm{EP}(f)$. Therefore, by Lemma 2.3, we immediately obtain that $\limsup_{n\to\infty}\langle z_n - x^*, Jw - Jx^*\rangle = \lim_{i\to\infty}\langle z_{n_i} - x^*, Jw - Jx^*\rangle = \langle z - x^*, Jw - Jx^*\rangle \leq 0$. It follows from Lemma 2.9 and (3.6) that $\varphi(x^*, x_n) \to 0$, as $n \to \infty$. Consequently, $x_n \to x^*$ by Lemma 2.2.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\phi(x^*, x_{n_i}) < \phi(x^*, x_{n_i+1})$, for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1})$ and $\phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$, for all $k \in \mathbb{N}$. Then from (3.5) and the fact that $\theta_n \to 0$ we have that

$$\begin{aligned}
&\beta_{m_k,0}\left(\phi(u_{m_k}, x_{m_k}) + \phi(u_{m_k}, w_{m_k})\right) + \beta_{m_k,0}\beta_{m_k,i}g(||Jw_{m_k} - JT_i^{m_k}y_{m_k}||)\\
&\quad \leq \left(\phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1})\right) - \theta_{m_k}\phi(x^*, x_{m_k})\\
&\qquad +2\theta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle + (k_{m_k} - 1)M \to 0, \text{ as } k \to \infty.
\end{aligned}$$

Thus, using the same proof of Case 1, we obtain that $u_{m_k} - x_{m_k} \to 0$, $u_{m_k} - w_{m_k} \to 0$ and $y_{m_k} - T_i y_{m_k} \to 0$, as $k \to \infty$, for each $i = 1, 2, \ldots, N$ and hence

$$\limsup_{n\to\infty}\langle z_{m_k} - x^*, Jw - Jx^*\rangle \leq 0. \tag{3.18}$$

Then from (3.6) we have that

$$\begin{aligned}
\phi(x^* x_{m_k+1}) &\leq (1 - \theta_{m_k})\phi(x^*, x_{m_k}) + 2\theta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle\\
&\quad + (k_{m_k} - 1)M.
\end{aligned} \tag{3.19}$$

Since $\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k+1})$, (3.19) implies that

$$\begin{aligned}
\theta_{m_k}\phi(x^*, x_{m_k}) &\leq \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_k+1})\\
&\quad + 2\theta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle + (k_{m_k} - 1)M\\
&\leq 2\theta_{m_k}\langle z_{m_k} - x^*, Jw - Jx^*\rangle + (k_{m_k} - 1)M.
\end{aligned}$$

In particular, since $\theta_{m_k} > 0$, we get

$$\phi(x^*, x_{m_k}) \leq 2\langle z_{m_k} - x^*, Jw - Jx^*\rangle + \frac{(k_{m_k} - 1)}{\theta_{m_k}} M.$$

Then, from (3.18) and the fact that $\frac{(k_{m_k}-1)}{\theta_{m_k}} \to 0$, we obtain that $\phi(x^*, x_{m_k}) \to 0$, as $k \to \infty$. This together with (3.19) gives $\phi(x^*, x_{m_k+1}) \to 0$, as $k \to \infty$. But $\phi(x^*, x_k) \leq \phi(x^*, x_{m_k+1})$, for all $k \in \mathbb{N}$, thus we obtain that $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to x^* and the proof is complete. □

Now, we give an example of asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping which is not uniformly Lipschitzian.

Example **3.2**. Let $C := [\frac{-1}{\pi}, \frac{1}{\pi}]$ and define $T: C \to C$ by

$$T(x) = \begin{cases} \frac{x}{2}\sin(\frac{1}{x}), & x \neq 0, \\ x, & x = 0. \end{cases}$$

Then clearly, T is continuous and $F(T) = \{0\}$. Moreover, following the method in [33] we obtain that $T^n x \to 0$, uniformly, for each $x \in C$, but T is not a Lipschitz function. We now show that it is relatively asymptotically nonexpansive, asymptotically regular and uniformly continuous mapping. But for any $x \in C$ we have that $|T^n x - T^n 0| \leq |(\frac{1}{2})^n x| = |(\frac{1}{2})^n x - 0| \leq k_n |x - 0|$, for $k_n := \max\{(\frac{1}{2})^n, 1\} = 1$, for each $n \geq 1$ and $|T^{n+1}x - T^n x| \leq |T^{n+1}x| + |T^n x| \to 0$, as $n \to \infty$. Moreover, since $T: C \to C$ is continuous, it follows that it is uniformly continuous. Therefore, T is relatively asymptotically nonexpansive, asymptotically regular and uniformly continuous mapping.

Next, we give an example of uniformly Lipschitzian relatively asymptotically nonexpansive mapping which is not relatively nonexpansive.

Example **3.3** [34]. Let $X = l^p$, where $1 < p < \infty$, and $C = \{x = (x_1, x_2, \ldots) \in X; x_n \geq 0\}$. Then C is closed and convex subset of X. Note that C is not bounded. Obviously, X is uniformly convex and uniformly smooth. Let $\{\lambda_n\}$ and $\{\bar{\lambda}_n\}$ be sequences of real numbers satisfying the following properties:

(i) $0 < \lambda_n < 1$, $\bar{\lambda}_n > 1$, $\lambda_n \uparrow 1$ and $\bar{\lambda}_n \downarrow 1$,

(ii) $\lambda_{n+1}\bar{\lambda}_n = 1$ and $\lambda_{j+1}\bar{\lambda}_{n+j} < 1$, for all n and j (for example: $\lambda_n = 1 - \frac{1}{n+1}$, $\bar{\lambda}_n = 1 - \frac{1}{n+1}$).

Then the map $T: C \to C$ defined by

$$Tx := (0, \bar{\lambda}_1|\sin x_1|, \lambda_2 x_2, \bar{\lambda}_2 x_3, \lambda_3 x_4, \bar{\lambda}_3 x_5, \ldots),$$

for all $x = (x_1, x_2, \ldots) \in C$ is uniformly Lipschitzian which is relatively asymptotically nonexpansive but not relatively nonexpansive (see [34] for the details). Note also that $F(T) = \{0\}$.

Remark **3.4**. *We note that the asymptotic regularity assumption on T_i in Theorem 3.1 can be weakened to the assumption that $T_i^{n+1}y_n - T_i^n y_n \to 0$, as $n \to \infty$, for $i = 1, 2, \ldots, N$.*

Recall that T is *uniformly L-Lipschitzian* if there exists some $L > 0$ such that

$$||T^n x - T^n y|| \le L||x - y||, \quad \text{for all } n \ge 1 \text{ and } x, y \in C. \tag{3.20}$$

We also note that the assumption $T_i^{n+1}y_n - T_i^n y_n \to 0$, as $n \to \infty$ and uniform continuity of T_i can be replaced by the unform Lipschitz continuity of T_i.

With the above observation we have the following convergence result.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E.* Let *f: $C \times C \to \mathbb{R}$, be a bifunction which satisfies conditions (A1)-(A4). Let A: $C \to E^*$ be a continuously monotone mapping. Let T_i: $C \to C$ be uniformly Li-Lipschtzian relatively asymptotically nonexpansive mapping with sequence $\{k_{n,\ i}\}$, for $i = 1, 2, \ldots, N$. Assume that $F := \cap_{i=1}^N F(T_i) \cap VI(C, A) \cap EP(f)$ is nonempty. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to an element of F.*

Proof. Clearly, T_i for each $i = 1, 2, \ldots, N$ is uniformly continuous. Now we show that $T_i^{n+1}y_n - T_i^n y_n \to 0$, as $n \to \infty$. But observe that from (3.1) and (3.8) we have

$$\begin{aligned} ||Jx_{n+1} - Jw_n|| &\le \beta_{n,1}||T_1^n y_n - w_n|| + \beta_{n,2}||T_2^n y_n - w_n|| \\ &\quad + \cdots + \beta_{n,N}||T_2^n y_n - w_n|| \to 0, \end{aligned} \tag{3.21}$$

as $n \to \infty$. Thus, as J^{-1} is uniformly continuous on bounded sets we have that x_{n+1} - $w_n \to 0$ which implies from (3.10) that x_{n+1} - $y_n \to 0$, as $n \to \infty$. Thus, this with (3.11) implies that

$$||y_{n+1} - y_n|| \le ||y_{n+1} - x_{n+1}|| + ||x_{n+1} - y_n|| \to 0, \quad \text{as } n \to \infty. \tag{3.22}$$

and hence (3.22) and (3.11) imply that

$$\begin{aligned} ||T_i^n y_n - T_i^{n+1} y_n|| &\le ||T_i^{n+1} y_n - T_i^{n+1} y_{n+1}|| + ||T_i^{n+1} y_{n+1} - y_{n+1}|| \\ &\quad + ||y_{n+1} - y_n|| + ||y_n - T_i^n y_n|| \\ &\le (1 + L)||y_{n+1} - y_n|| + ||T_i^{n+1} y_{n+1} - y_{n+1}|| \\ &\quad + ||T_i^n y_n - y_n|| \to 0, \text{ as } n \to \infty, \end{aligned} \tag{3.23}$$

for each $i = 1, 2, \ldots, N$, where $L := \max_{1 \le i \le N}\{L_i\}$. Therefore, Remark 3.4 with Theorem 3.1 imply the desired conclusion. □

If in Theorem 3.1 we have $N = 1$ we get the following corollary.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let f: $C \times C \to \mathbb{R}$, be a bifunction which satisfies conditions (A1)-(A4). Let A: $C \to E^*$ be a continuously monotone mapping. Let T: $C \to C$ be a asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequence $\{k_n\}$. Assume that F: = F(T) ∩ VI(C, A) ∩ EP(f) is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \text{ chosen arbitrary}, \\ u_n = F_{r_n} x_n, \\ w_n = T_{r_n} u_n, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_n Jw_n + (1 - \beta_n)JT^n y_n), \ n \ge 0, \end{cases} \tag{3.24}$$

where $\alpha_n \in (0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_n-1)}{\alpha_n} = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$, *for each* $n \geq 0$. *Then* $\{x_n\}$ *converges strongly to an element of F.*

If in Theorem 3.1 we assume that each T_i is relatively nonexpansive we get the following theorem.

Theorem 3.7. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let f*: $C \times C \to \mathbb{R}$, *be a bifunction which satisfies conditions (A1)-(A4). Let A*: $C \to E^*$ *be a continuously monotone mapping. Let* T_i: $C \to C$ *be a relatively nonexpansive mapping for each* $i = 1, 2, \ldots, N$. *Assume that* $F := \cap_{i=1}^{N} F(T_i) \cap VI(C, A) \cap EP(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \text{ chosen arbitrarily,} \\ u_n = F_{r_n} x_n, \\ w_n = T_{r_n} u_n, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_{n,0} Jw_n + \sum_{i=1}^{N} \beta_{n,i} JT_i y_n), \ n \geq 0, \end{cases} \tag{3.25}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, *for* $i = 1, 2, \ldots, N$ *satisfying* $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, *for each* $n \geq 0$. *Then* $\{x_n\}$ *converges strongly to an element of F.*

Proof. Following the methods of proof of Theorem 3.1 we obtain the required assertion. □

If in Theorem 3.1 we assume that $f \equiv 0$ and $A \equiv 0$ we get the following corollary.

Corollary 3.8. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let* T_i: $C \to C$ *be a asymptotically regular uniformly continuous relatively asymptotically nonexpansive mapping with sequence* $\{k_{n,i}\}$, *for* $i = 1, 2, \ldots, N$. *Assume that* $F := \cap_{i=1}^{N} F(T_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \text{ chosen arbitrarily,} \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_{n,0} Jx_n + \sum_{i=1}^{N} \beta_{n,i} JT_i^n y_n), \ n \geq 0, \end{cases} \tag{3.26}$$

where $\alpha_n \in (0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_{n,i}-1)}{\alpha_n} = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, *for* $i = 1, 2, \ldots, N$ *satisfying* $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, *for each* $n \geq 0$. *Then* $\{x_n\}$ *converges strongly to an element of F.*

If in Theorem 3.1 we assume that $f \equiv 0$ and $T \equiv I$ we get the following corollary.

Corollary 3.9. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let A*: $C \to E^*$ *be a continuously monotone mapping. Assume that F*: = *VI(C, A) is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = w \in C, \text{ chosen arbitrarily,} \\ w_n = F_{r_n} x_n, \\ y_n = \Pi_C J^{-1}(\alpha_n Jw + (1-\alpha_n)Jw_n), \\ x_{n+1} = J^{-1}(\beta_n Jw_n + (1-\beta_n)Jy_n), \ n \geq 0, \end{cases} \tag{3.27}$$

where $\alpha_n \in (0, 1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$, *for each* $n \geq 0$. *Then* $\{x_n\}$ *converges strongly to an element of* F.

Remark **3.10**. Theorem 3.1 improves and extends the corresponding results of Nakajo and Takahashi [17], Kim and Xu [35] in the sense that the space is extended from Hilbert spaces to uniformly smooth and uniformly convex Banach spaces. Moreover, Corollary 3.8 improves and extends Theorem MT of Matsushita and Takahashi [16] and Theorem 3.4 of Nilsrakoo and Saejung [36] from a relatively nonexpansive mappings to a finite family of asymptotically regular uniformly continuous relatively asymptotically nonexpansive mappings. Corollary 3.9 extends the corresponding results of Iiduka, Takahashi and Toyoda [6] and Iiduka and Takahashi [8] in the sense that either the space is extended from Hilbert space to uniformly smooth and uniformly convex Banach space or our scheme is used for approximating solutions of variational problems for a more general class of monotone mappings. Moreover, our scheme does not involve computation of C_{n+1} from sets C_n and Q_n for each $n \geq 1$.

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