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The shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-φ-nonexpansive mappings

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Abstract

In this article, we introduce a new hybrid projection iterative scheme based on the shrinking projection method for finding a common element of the set of solutions of the generalized mixed equilibrium problems and the set of common fixed points for a pair of asymptotically quasi-φ-nonexpansive mappings in Banach spaces and set of variational inequalities for an α-inverse strongly monotone mapping. The results obtained in this article improve and extend the recent ones announced by Matsushita and Takahashi (Fixed Point Theory Appl. 2004(1):37-47, 2004), Qin et al. (Appl. Math. Comput. 215:3874-3883, 2010), Chang et al. (Nonlinear Anal. 73:2260-2270, 2010), Kamraksa and Wangkeeree (J. Nonlinear Anal. Optim.: Theory Appl. 1 (1):55-69, 2010) and many others.

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1. Introduction

Let E be a Banach space with norm $||\cdot||$, C be a nonempty closed convex subset of E, and let E^* denote the dual of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction, ϕ: $C \to \mathbb{R}$ be a real-valued function, and $B : C \to E^*$ be a mapping. The *generalized mixed equilibrium problem*, is to find $x \in C$ such that

$$f(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions to (1.1) is denoted by GMEP(f, B, ϕ), i.e.,

$$\text{GMEP}(f, B, \varphi) = \{x \in C : f(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}. \tag{1.2}$$

If $B \equiv 0$, then the problem (1.1) reduces into the *mixed equilibrium problem for f*, denoted by MEP(f, ϕ), is to find $x \in C$ such that

$$f(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

If $\phi \equiv 0$, then the problem (1.1) reduces into the *generalized equilibrium problem*, denoted by GEP(f, B), is to find $x \in C$ such that

$$f(x,y) + \langle Bx,\ y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $f \equiv 0$, then the problem (1.1) reduces into the *mixed variational inequality* of Browder type, denoted by MVI(B, C), is to find $x \in C$ such that

$$\langle Bx,\ y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

If $\phi \equiv 0$, then the problem (1.5) reduces into the *classical variational inequality*, denoted by VI(B, C), which is to find $x \in C$ such that

$$\langle Bx,\ y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

If $B \equiv 0$ and $\phi \equiv 0$, then the problem (1.1) reduces into the *equilibrium problem for f*, denoted by EP(f), which is to find $x \in C$ such that

$$f(x,y) \geq 0, \quad \forall y \in C. \tag{1.7}$$

If $f \equiv 0$, then the problem (1.3) reduces into the *minimize problem*, denoted by Argmin (ϕ), which is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.8}$$

The above formulation (1.6) was shown in [1] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP(f). In other words, the EP(f) is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many articles have appeared in the literature on the existence of solutions of EP(f); see, for example [1-4] and references therein. Some solution methods have been proposed to solve the EP(f) in Hilbert spaces and Banach spaces; see, for example [5-20] and references therein.

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Let $U = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. Then, a Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - ||\frac{x+y}{2}|| : x, y \in E, ||x|| = ||y|| = 1, ||x-y|| \geq \varepsilon\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [21,22] for more details. Observe that every p-uniformly convex is uniformly convex. One should note that no Banach space is p-uniformly convex for $1 < p < 2$. It is well known that a Hilbert space is *2-uniformly convex*, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

A set valued mapping $U : E \rightrightarrows E^*$ with graph $G(U) = \{(x, x^*) : x^* \in Ux\}$, domain $D(U) = \{x \in E : Ux \neq \emptyset\}$, and rang $R(U) = \cup\{Ux : x \in D(U)\}$. U is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $x^* \in Ux$, $y^* \in Uy$. A monotone operator U is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator. We know that if U is maximal monotone, then the solution set $U^{-1} 0 = \{x \in D(U) : 0 \in Ux\}$ is closed and convex. It is knows that U is a maximal monotone if and only if $R(J + rU) = E^*$ for all $r > 0$ when E is a reflexive, strictly convex and smooth Banach space (see [23]).

Recall that let $A : C \to E^*$ be a mapping. Then, A is called

(i) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C,$$

(ii) α*-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha ||Ax - Ay||^2, \quad \forall x, y \in C.$$

The class of inverse-strongly monotone mappings has been studied by many researchers to approximating a common fixed point; see [24-29] for more details.

Recall that a mappings $T : C \to C$ is said to be *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \quad \text{for all } x, y \in C.$$

T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$, and

$$||Tx - y|| \leq ||x - y||, \quad \text{for all } x \in C, y \in F(T).$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||, \quad \text{for all } x, y \in C.$$

T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$||T^n x - y|| \leq k_n ||x - y||, \quad \text{for all } x \in C, y \in F(T).$$

T is called uniformly *L-Lipschitzian continuous* if there exists $L > 0$ such that

$$||T^n x - T^n y|| \leq L ||x - y||, \quad \text{for all } x, y \in C.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [30] in 1972. Since 1972, a host of authors have studied the weak and strong convergence of iterative processes for such a class of mappings.

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [31] recently introduced a generalized projection operator C in Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let E be a smooth, strictly convex and reflexive Banach spaces and C be a nonempty, closed convex subset of E. We consider the Lyapunov functional $\varphi : E \times E \to \mathbb{R}^+$ defined by

$$\phi(y,x) = ||y||^2 - 2\langle y, Jx\rangle + ||x||^2 \tag{1.9}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*.

Observe that, in a Hilbert space H, (1.9) reduces to $\varphi(y, x) = ||x - y||^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(y, x)$; that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y, x). \tag{1.10}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\varphi(y, x)$ and strict monotonicity of the mapping J (see, for example, [9,32-34]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function φ that

(1) $(||y|| - ||x||)^2 \le \varphi(y, x) \le (||y|| + ||x||)^2$ for all $x, y \in E$.
(2) $\varphi(x, y) = \varphi(x, z) + \varphi(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$.
(3) $\varphi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le ||x||\,||Jx - Jy|| + ||y - x||\,||y||$ for all $x, y \in E$.
(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \text{ if and only if } x = y.$$

By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$, for more details see [35,36].

Remark 1.1. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Also, it is well known that if E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one and onto (see [35]).

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [37] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n \to \infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

A point p in C is said to be a *strong asymptotic fixed point* of T [37] if C contains a sequence $\{x_n\}$ which converges strong to p such that $\lim_{n\to\infty}||x_n - Tx_n|| = 0$. The set of strong asymptotic fixed points of S will be denoted by $\widetilde{F}(T)$.

A mapping T is called *relatively nonexpansive* [38-40] if $\hat{F}(T) = F(T)$ and

$$\phi(p, Tx) \le \phi(p, x)\ \forall x \in C \text{ and } p \in F(T).$$

The asymptotic behavior of relatively nonexpansive mappings were studied in [38,39].

A mapping $T : C \to C$ is said to be *weak relatively nonexpansive* if $\widetilde{F}(T) = F(T)$ and

$$\phi(p, Tx) \le \phi(p, x)\ \forall x \in C \text{ and } p \in F(T).$$

A mapping T is called *hemi-relatively nonexpansive* if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x) \ \forall x \in C \text{ and } p \in F(T).$$

A mapping T is said to be *relatively asymptotically nonexpansive* [32,41] if $\hat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x) \ \forall x \in C,\ p \in F(T) \text{ and } n \geq 1.$$

Remark 1.2. Obviously, relatively nonexpansive implies weak relatively nonexpansive and both also imply hemi-relatively nonexpansive. Moreover, the class of relatively asymptotically nonexpansive is more general than the class of relatively nonexpansive mappings.

We note that hemi-relatively nonexpansive mappings are sometimes called quasi-φ-nonexpansive mappings.

We recall the following :

(i) $T : C \to C$ is said to be *φ-nonexpansive* [42,43] if $\varphi(Tx, Ty) \leq \varphi(x, y)$ for all x, $y \in C$.
(ii) $T : C \to C$ is said to be *quasi-φ-nonexpansive* [42,43] if $F(T) \neq \emptyset$ and $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$.
(iii) $T : C \to C$ is said to be *asymptotically φ-nonexpansive* [43] if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\varphi(T^n x, T^n y) \leq k_n \varphi(x, y)$ for all $x, y \in C$.
(iv) $T : C \to C$ is said to be *asymptotically quasi-φ-nonexpansive* [43] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\varphi(p, T^n x) \leq k_n \varphi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.3. (i) The class of (asymptotically) quasi-φ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings, which requires the strong restriction $\hat{F}(T) = F(T)$.

(ii) In real Hilbert spaces, the class of (asymptotically) quasi-φ-nonexpansive mappings is reduced to the class of (asymptotically) quasi-nonexpansive mappings.

Let T be a nonlinear mapping, T is said to be *uniformly asymptotically regular* on C if

$$\lim_{n\to\infty} \left(\sup_{x \in C} ||T^{n+1}x - T^n x|| \right) = 0.$$

$T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

We give some examples which are closed and asymptotically quasi-φ-nonexpansive.

Example 1.4. (1). Let E be a uniformly smooth and strictly convex Banach space and $U \subset E \times E^*$ be a maximal monotone mapping such that its zero set $U^{-1}0$ is nonempty. Then, $J_r = (J + rU)^{-1} J$ is a closed and asymptotically quasi-φ-nonexpansive mapping from E onto $D(U)$ and $F(J_r) = U^{-1}0$.

(2). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then Π_C is a closed and asymptotically quasi-φ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Recently, Matsushita and Takahashi [44] obtained the following results in a Banach space.

Theorem MT. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} < 1$. Suppose that $\{x_n\}$ is given by

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \ n = 0, 1, 2, \ldots, \end{cases} \tag{1.11}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_F(T)^x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$. In 2008, Iiduka and Takahashi [45] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space $E : x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \tag{1.12}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.12) converges weakly to some element of VI(A, C).

A popular method is the shrinking projection method which introduced by Takahashi et al. [46] in year 2008. Many authors developed the shrinking projection method for solving (mixed) equilibrium problems and fixed point problems in Hilbert and Banch spaces; see, [12,15,16,47-57] and references therein.

Recently, Qin et al. [58] further extended Theorem MT by considering a pair of asymptotically quasi-φ-nonexpansive mappings. To be more precise, they proved the following results.

Theorem QCK. Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-φ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ a closed and asymptotically quasi-φ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in [0, 1].

Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n J x_n + \gamma_n J(T^n x_n) + \delta_n J(S^n x_n)), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_{n+1} = \{w \in C_n : \phi(w, y_n) \le \phi(w, x_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.13}$$

where $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, J is the duality mapping on E, and $M_n = \sup\{\varphi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following restrictions :

(a) $\beta_n + \gamma_n + \delta_n = 1, \forall n \geq 1$;
(b) $\liminf_{n\to\infty} \gamma_n\delta_n$, $\lim_{n\to\infty} \beta_n = 0$;
(c) $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$.

On the other hand, Chang, Lee and Chan [59] proved a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem (1.4) and the set of common fixed points for a pair of relatively nonexpansive mappings in Banach spaces. They proved the following results.

Theorem CLC. Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a α-inverse-strongly monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$. Let $S, T : C \to C$ be two relatively nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap \text{GEP}(f, A)$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S x_n), \\ u_n \in C \textit{ such that} \\ \quad f(u_n, y) + \langle A u_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \ge 0, \ \forall y \in C, \\ H_n = \{v \in C : \phi(v, u_n) \le \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, J x_0 - J x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \ \forall n \ge 0, \end{cases} \tag{1.14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{\gamma_n\} \subset [a, 1)$ for some $a > 0$. If the following conditions are satisfied

(a) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$;
(b) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

then, $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

Very recently, Kim [60], considered the shrinking projection methods which were introduced by Takahashi et al. [46] for asymptotically quasi-φ-nonexpansive mappings in a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property.

In this article, motivated and inspired by the study of Matsushita and Takahashi [44], Qin et al. [58], Kim [60], and Chang et al. [59], we introduce a new hybrid projection iterative scheme based on the shrinking projection method for finding a common element of the set of solutions of the generalized mixed equilibrium problems, the set of the variational inequality and the set of common fixed points for a pair of asymptotically quasi-φ-nonexpansive mappings in Banach spaces. The results obtained in this article improve and extend the recent ones announced by Matsushita and Takahashi [44], Qin et al. [58], Chang et al. [59] and many others.

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions which will be needed in proving our main results.

In the sequel, we denote the strong convergence, weak convergence and weak* convergence of a sequence $\{x_n\}$ by $x_n \to x$, $x_n \rightharpoonup^* \times$ and $x_n \rightharpoonup^* x$, respectively.

It is well known that a uniformly convex Banach space has the Kadec-Klee property, i.e. if $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $x_n \to x$.

Lemma 2.1. ([31,61]) *Let E be a smooth, strictly convex and reflexive Banach space and C be anonempty closed convex subset. Then, the following conclusion hold:*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y); \quad \forall x \in C, \quad y \in E.$$

Lemma 2.2. ([34]). *If E be a 2-uniformly convex Banach space and* $0 < c \leq 1$. *Then, for all* $x, y \in E$ *we have*

$$||x - y|| \leq \frac{2}{c^2}||Jx - Jy||,$$

where J is the normalized duality mapping of E.

The best constant $\frac{1}{c}$ in Lemma is called the *p-uniformly convex constant* of *E*.

Lemma 2.3. ([62]). *If E be a p-uniformly convex Banach space and p be a given real number with* $p \geq 2$, *then for all* $x, y \in E$, $j_x \in J_p x$ *and* $j_y \in J_p y$

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p}||x - y||^p,$$

where J_p *is the generalized duality mapping of E and* $\frac{1}{c}$*is the p-uniformly convexity constant of E.*

Lemma 2.4. ([63]) *Let E be a uniformly convex Banach space and* $B_r(0)$ *a closed ball of E. Then, there exists a continuous strictly increasing convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$||\alpha x + (1 - \alpha)y||^2 \leq \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)g(||x - y||)$$

for all $x, y \in B_r(0)$ *and* $\alpha \in [0, 1]$.

Lemma 2.5. ([58]) *Let E be a uniformly convex and smooth Banach space, C a nonempty closed convex subset of E and* $T : C \to C$ *a closed asymptotically quasi-φ-nonexpansive mapping. Then, F(T) is a closed convex subset of C.*

Lemma 2.6. ([61]) *Let E be a smooth and uniformly convex Banach space. Let* x_n *and* y_n *be sequences in E such that either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded. If* $\lim_{n\to\infty} \varphi(x_n, y_n) = 0$, *then* $\lim_{n\to\infty}||x_n - y_n|| = 0$.

Lemma 2.7. (*Alber* [31]). *Let C be a nonempty closed convex subset of a smooth Banach space E and* $x \in E$. *Then,* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Let E be a reflexive, strictly convex, smooth Banach space and J the duality mapping from E into E^*. Then, J^{-1} is also single valued, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [31]

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2, \tag{2.1}$$

for all $x \in E$ and $x^* \in E^*$; that is, $V(x, x^*) = \varphi(x, J^{-1}x^*)$.

Lemma 2.8. (*Kohsaka and Takahashi* [[64], Lemma 3.2]). *Let E be a reflexive, strictly convex smooth Banach space and let V be as in* (2.1). *Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ *and* $x^*, y^* \in E^*$.

Proof. Let $x \in E$. Define $g(x^*) = V(x, x^*)$ and $f(x^*) = ||x^*||^2$ for all $x^* \in E^*$. Since J^{-1} is the duality mapping from E^* to E, we have

$$\partial g(x^*) = \partial(-2\langle x, \cdot\rangle + f)(x^*) = -2x + 2J(-1)(x^*), \quad \forall x^* \in E^*.$$

Hence, we get

$$g(x^*) + 2\langle J^{-1}(x^*) - x,\ y^*\rangle \leq g(x^* + y^*),$$

that is,

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x,\ y^*\rangle \leq V(x, x^* + y^*),$$

for all $x^*, y^* \in E^*$.

For solving the generalized equilibrium problem, let us assume that the nonlinear mapping $A : C \to E^*$ is α-inverse strongly monotone and the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0\ \forall x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$, $\forall x, y \in C$;
(A3) $\limsup_{t\downarrow 0} f(x + t(z - x), y) \leq f(x, y)$, $\forall x, y, z \in C$;
(A4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.9. ([1]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1) - (A4). *Let* $r > 0$ *and* $x \in E$, *then there exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.10. ([65]) *Let C be a closed convex subset of a uniformly smooth and strictly convex Banach space E and let f be a bifunction from C × C to* $\mathbb{R}$ *satisfying* (A1) - (A4). *For r >0 and* × ∈ *E, define a mapping* $T_r : E \to C$ *as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C \right\},$$

for all × ∈ *C. Then, the following conclusions holds:*

(1) T_r *is single-valued;*
(2) T_r *is a firmly nonexpansive-type mapping, i.e.*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle, \quad \forall x, y \in E;$$

(A3) $F(T_r)$ = EP(*f*);
(A4) EP(*f*) *is a closed convex.*

Lemma 2.11. ([19]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from C × C to* $\mathbb{R}$ *satisfying* (A1) - (A4) *and let r* >0. *Then, for* × ∈ *E and q* ∈ $F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r(x), x) \leq \phi(q, x).$$

Lemma 2.12. ([66]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let* $B : C \to E^*$ *be a continuous and monotone mapping,* $\phi : C \to \mathbb{R}$ *be a lower semi-continuous and convex function, and f be a bifunction from C* × *C to* $\mathbb{R}$ *satisfying* (A1) - (A4). *For r* >0 *and* × ∈ *E, then there exists u* ∈ *C such that*

$$f(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle, \quad \forall y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) = \{u \in C : f(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\} \quad (2.3)$$

for all $x \in C$. Then, the following conclusions holds:

(a) K_r is single-valued ;
(b) K_r is a firmly nonexpansive-type mapping, *i.e.;*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle, \quad \forall x, y \in E;$$

(c) $F(K_r) = \hat{F}(K_r)$ = GMEP (f, B, φ);
(d) GMEP(*f*, *B*, ϕ) is a closed convex,
(e) $\varphi(q, K_r z) + \varphi(K_r z, z) \leq \varphi(q, z)$, $\forall q \in F(K_r)$, $z \in E$.

Remark 2.13. ([66]) It follows from Lemma 2.12 that the mapping $K_r : C \to C$ defined by (2.3) is a relatively nonexpansive mapping. Thus, it is quasi-φ-nonexpansive.

Let C be a nonempty closed convex subset of a Banach space E and let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of [0, 1] into E^*, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \quad \forall y \in C\}. \tag{2.4}$$

Lemma 2.14. (*Rockafellar* [23]). *Let C be a nonempty, closed convex subset of a Banach space E, and A a monotone, hemicontinuous operator of C into* E^*. *Let* $U : E \rightrightarrows E^*$ *be an operator defined as follows:*

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{2.5}$$

Then, U is maximal monotone and $U^{-1}0 = \mathrm{VI}(A, C)$.

3. Main results

In this section, we shall prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem (1.2), set of variational inequalities for an α-inverse strongly monotone mapping and the set of common fixed points for a pair of asymptotically quasi-φ-nonexpansive mappings in Banach spaces.

Theorem 3.1. *Let E be a uniformly smooth and 2-uniformly convex Banach space, C be a nonempty closed convex subset of E. Let A be an* α*-inverse-strongly monotone mapping of C into* E^* *satisfying* $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ *and* $u \in \mathrm{VI}(A, C) \neq \emptyset$. *Let* $B : C \to E^*$ *be a continuous and monotone mapping and* $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1) - (A4), *and* $\phi : C \to \mathbb{R}$ *be a lower semi-continuous and convex function. Let* $T : C \to C$ *be a closed and asymptoticallyquasi-*φ*-nonexpansive mapping with the sequence* $\{k_n^{(t)}\} \subset [1, \infty)$ *such that* $k_n^{(t)} \to 1$ *as* $n \to \infty$ *and* $S : C \to C$ *be a closed and asymptotically quasi-*φ*-nonexpansive mapping with the sequence* $\{k_n^{(s)}\} \subset [1, \infty)$ *such that* $k_n^{(s)} \to 1$ *as* $n \to \infty$. *Assume that T and S are uniformly asymptotically regular on C and* $\Omega := F(T) \cap F(S) \cap \mathrm{VI}(A, C) \cap \mathrm{GMEP}(f, B, \phi.) \neq \emptyset$.

Let $\{x_n\}$ *be the sequence defined by* $x_0 \in E$ *and*

$$\begin{cases} x_1 = \Pi_{C_1} x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n w_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JS^n z_n), \\ u_n \in C, \text{such that} \\ f(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\theta_n = (1 - \beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\varphi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied: $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω.

Proof. We have several steps to prove this theorem as follows:

Step 1. We first show that C_{n+1} is closed and convex for each $n \geq 1$. Indeed, it is obvious that $C_1 = C$ is closed and convex. Suppose that C_i is closed and convex for each $i \in \mathbb{N}$. Next, we prove that C_{i+1} is closed and convex. For any $z \in C_{i+1}$, we know that $\varphi(z, u_i) \leq \varphi(z, x_i) + \theta_i$ is equivalent to

$$2\langle z, Jx_i - Ju_i\rangle \leq ||x_i||^2 - ||u_i||^2 + \theta_i,$$

where $\theta_i = (1-\beta_i)(k_i^2 - 1)M_i$ and $M_i = \sup\{\varphi(z, x_i) : z \in \Omega\}$ for each $i \geq 1$. Hence, C_{i+1} is closed and convex. Then, for each $n \geq 1$, we see that C_n is closed and convex. Hence, Π_{C_n} is well defined.

By the same argument as in the proof of [[43], Lemma 2.4], one can show that $F(T) \cap F(S)$ is closed and convex. We also know that $VI(A, C) = U^{-1}0$ is closed and convex, and hence from Lemma 2.12(d), $\Omega := F(S) \cap F(T) \cap VI(A, C) \cap GMEP(f, B, \phi)$ is a nonempty, closed and convex subset of C. Consequently, Π_Ω is well defined.

Step 2. We show that the sequence $\{x_n\}$ is well defined. Next, we prove that $\Omega \subset C_n$ for each $n \geq 1$. If $n = 1$, $\Omega \subset C_1 = C$ is obvious. Suppose that $\Omega \subset C_i$ for some positive integer i. For every $q \in \Omega$, we obtain from the assumption that $q \in C_i$. It follows, from Lemma 2.1 and Lemma 2.8, that

$$\begin{aligned}
\phi(q, w_i) &= \phi(q, \Pi_C J^{-1}(Jx_i - \lambda_i Ax_i)) \\
&\leq \phi(q, J^{-1}(Jx_i - \lambda_i Ax_i)) \\
&= V(q, Jx_i - \lambda_i Ax_i) \\
&\leq V(q, (Jx_i - \lambda_i Ax_i) + \lambda_i Ax_i) - 2\langle J^{-1}(Jx_i - \lambda_i Ax_i) - q, \lambda_i Ax_i\rangle \\
&= V(q, Jx_i) - 2\lambda_i\langle J^{-1}(Jx_i - \lambda_i Ax_i) - q, Ax_i\rangle \\
&= \phi(q, x_i) - 2\lambda_i\langle x_i - q, Ax_i\rangle + 2\langle J^{-1}(Jx_i - \lambda_i Ax_i) - x_i, -\lambda_i Ax_i\rangle.
\end{aligned} \tag{3.2}$$

Thus, $q \in VI(A, C)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned}
-2\lambda_i\langle x_i - q, Ax_i\rangle &= -2\lambda_i\langle x_i - q, Ax_i - Aq\rangle - 2\lambda_i\langle x_i - q, Aq\rangle \\
&\leq -2\lambda_i\langle x_i - q, Ax_i - Aq\rangle \\
&= -2\alpha\lambda_i||Ax_i - Aq||^2.
\end{aligned} \tag{3.3}$$

From Lemma 2.2 and $||Ay|| \leq ||Ay - Au||$ for all $y \in C$ and $q \in \Omega$, we obtain

$$\begin{aligned}
2\langle J^{-1}(Jx_i - \lambda_i Ax_i) - x_i, -\lambda_i Ax_i\rangle &= 2\langle J^{-1}(Jx_i - \lambda_i Ax_i) - J^{-1}(Jx_i), -\lambda_i Ax_i\rangle \\
&\leq 2||J^{-1}(Jx_i - \lambda_i Ax_i) - J^{-1}(Jx_i)||\ ||\lambda_i Ax_i|| \\
&\leq \frac{4}{c^2}||JJ^{-1}(Jx_i - \lambda_i Ax_i) - JJ^{-1}(Jx_i)||\ ||\lambda_i Ax_i|| \\
&= \frac{4}{c^2}||Jx_i - \lambda_i Ax_i - Jx_i||\ ||\lambda_i Ax_i|| \\
&= \frac{4}{c^2}||\lambda_i Ax_i||^2 \\
&= \frac{4}{c^2}\lambda_i^2||Ax_i||^2 \\
&\leq \frac{4}{c^2}\lambda_i^2||Ax_i - Aq||^2.
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we have

$$\begin{aligned}\phi(q, w_i) &\leq \phi(q, x_i) - 2\alpha\lambda_i||Ax_i - Aq||^2 + \frac{4}{c^2}\lambda_i^2||Ax_i - Aq||^2 \\ &= \phi(q, x_i) + 2\lambda_i(\frac{2}{c^2}\lambda_i - \alpha)||Ax_i - Aq||^2 \\ &\leq \phi(q, x_i).\end{aligned} \tag{3.5}$$

As T^i is asymptotically quasi-φ-nonexpansive mapping, we also have

$$\begin{aligned}\phi(q, z_i) &= \phi(q, J^{-1}(\alpha_i Jx_i + (1 - \alpha_i)JT^i w_i)) \\ &= ||q||^2 - 2\langle q, \alpha_i Jx_i + (1 - \alpha_i)JT^i w_i\rangle + ||\alpha_i Jx_i + (1 - \alpha_i)JT^i w_i||^2 \\ &\leq ||q||^2 - 2\alpha_i\langle q, Jx_i\rangle - 2(1 - \alpha_i)\langle q, JT^i w_i\rangle \\ &\quad + \alpha_i||x_i||^2 + (1 - \alpha_i)||T^i w_i||^2 \\ &= \alpha_i\phi(q, x_i) + (1 - \alpha_i)\phi(q, T^i w_i) \\ &\leq \alpha_i\phi(q, x_i) + (1 - \alpha_i)k_i^{(t)}\phi(q, w_i) \\ &\leq \alpha_i\phi(q, x_i) + (1 - \alpha_i)k_i\phi(q, w_i) \\ &\leq \phi(q, x_i) + (k_i - 1)\phi(q, w_i).\end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}\phi(q, u_i) &= \phi(q, K_{r_i}y_i) \leq \phi(q, y_i) \\ &\leq \phi(q, J^{-1}(\beta_i Jx_i + (1 - \beta_i)JS^i z_i)) \\ &= ||q||^2 - 2\langle q,\ \beta_i Jx_i + (1 - \beta_i)JS^i z_i\rangle + ||\beta_i Jx_i + (1 - \beta_i)JS^i z_i||^2 \\ &\leq ||q||^2 - 2\beta_i\langle q,\ Jx_i\rangle - 2(1 - \beta_i)\langle q,\ JS^i z_i\rangle + \beta_i||x_i||^2 + (1 - \beta_i)||S^i z_i||^2 \\ &= \beta_i\phi(q, x_i) + (1 - \beta_i)\phi(q, S^i z_i) \\ &\leq \beta_i\phi(q, x_i) + (1 - \beta_i)k_i^{(s)}\phi(q, z_i) \\ &\leq \beta_i\phi(q, x_i) + (1 - \beta_i)k_i\phi(q, z_i) \\ &= (1 - (1 - \beta_n))\phi(q, x_i) + (1 - \beta_i)k_i\phi(q, z_i) \\ &= \phi(q, x_i) + (1 - \beta_i)[k_i\phi(q, z_i) - \phi(q, x_i)] \\ &\leq \phi(q, x_i) + (1 - \beta_i)[k_i(\phi(q, x_i) + (k_i - 1)\phi(q, w_i)) - \phi(q, x_i)] \\ &\leq \phi(q, x_i) + (1 - \beta_i)[k_i(\phi(q, x_i) + (k_i - 1)\phi(q, x_i)) - \phi(q, x_i)] \\ &= \phi(q, x_i) + (1 - \beta_i)[k_i\phi(q, x_i) + (k_i^2 - k_i)\phi(q, x_i) - \phi(q, x_i))] \\ &= \phi(q, x_i) + (1 - \beta_i)(k_i^2 - 1)\phi(q, x_i) \\ &\leq \phi(q, x_i) + (1 - \beta_i)(k_i^2 - 1)M_i \\ &= \phi(q, x_i) + \theta_i.\end{aligned} \tag{3.7}$$

This shows that $q \in C_{i+1}$. This implies that $\Omega \subset C_n$ for each $n \geq 1$.
From $x_n = \prod_{C_n} x_0$, we see that

$$\langle x_n - q, Jx_0 - Jx_n\rangle \geq 0, \quad \forall q \in C_n.$$

Since $\Omega \subset C_n$ for each $n \geq 1$, we arrive at

$$\langle x_n - q, Jx_0 - Jx_n\rangle \geq 0, \quad \forall q \in \Omega. \tag{3.8}$$

Hence, the sequence $\{x_n\}$ is well defined.
Step 3. Now, we prove that $\{x_n\}$ is bounded.

In view of Lemma 2.1, we see that

$$\phi(x_n, x_0) = \phi(\prod_{C_n} x_0, x_0) \leq \phi(q, x_0) - \phi(q, x_n) \leq \phi(q, x_0),$$

for each $q \in C_n$. Therefore, we obtain that the sequence $\varphi(x_n, x_0)$ is bounded, and so are $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{T^n w_n\}$ and $\{S^n x_n\}$.

Step 4. We show that $\{x_n\}$ is a Cauchy sequence.

Since $x_n = \prod_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1.$$

This implies that $\{\varphi(x_n, x_0)\}$ is nondecreasing, and $\lim_{n \to \infty} \varphi(x_n, x_0)$ exists.

For $m > n$ and from Lemma 2.1, we have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{3.9}$$

Letting $m, n \to \infty$ in (3.9), we see that $\varphi(x_m, x_n) \to 0$. It follows from Lemma 2.6 that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we can assume that $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Step 5. We will show that $p \in \Omega := F(T) \cap F(S) \cap VI(A, C) \cap GMEP(f, B, \phi)$.

(a) First, we show that $p \in F(T) \cap F(S)$.

By taking $m = n + 1$ in (3.9), we obtain that

$$\lim_{n-\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.10}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, from definition of C_{n+1}, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n, \quad \forall n \geq 1, \tag{3.11}$$

and from (3.5) and (3.6), we also have

$$k_n \phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + (k_n^2 - 1)M_n, \quad \forall n \geq 1. \tag{3.12}$$

Since E is uniformly smooth and uniformly convex, from (3.10)-(3.12), $\theta_n \to 0$ as $n \to \infty$ and

Lemma 2.6, it follows that

$$\lim_{n-\infty} ||x_{n+1} - x_n|| = \lim_{n-\infty} ||x_{n+1} - u_n|| = \lim_{n-\infty} ||x_{n+1} - z_n|| = 0, \tag{3.13}$$

and by using triangle inequality, we have

$$\lim_{n-\infty} ||x_n - u_n|| = \lim_{n-\infty} ||x_n - z_n|| = \lim_{n-\infty} ||x_n - z_n|| = 0. \tag{3.14}$$

Since J is uniformly norm-to-norm continuous, we also have

$$\lim_{n-\infty} ||Jx_n - Ju_n|| = 0. \tag{3.15}$$

and

$$\lim_{n-\infty} ||Jx_n - Jz_n|| = 0. \tag{3.16}$$

Since $u_n = K_{r_n} y_n$, and from (3.7), we have

$$\phi(u, y_n) \leq \phi(u, x_n) + \theta_n, \quad \forall u \in \Omega. \tag{3.17}$$

Since $||x_n - u_n|| \to 0$ and J is uniformly continuous, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, K_{r_n} y_n) + \theta_n \\ &= \phi(u, x_n) - \phi(u, u_n) + \theta_n \\ &= ||x_n||^2 - ||u_n||^2 - 2\langle u, Jx_n - Ju_n\rangle + \theta_n \\ &\leq ||x_n - u_n||(||x_n|| + ||u_n||) - 2\langle u, Jx_n - Ju_n\rangle + \theta_n \to 0. \end{aligned} \tag{3.18}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded, it follows from (3.14) and (3.15) that $\varphi(y_n, u_n) \to 0$ as $n \to \infty$. Since E is smooth and uniformly convex, from Lemma 2.6, we have

$$||y_n - u_n|| \to 0, \text{ and so } ||y_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous, we also have

$$||Jy_n - Ju_n|| \to 0, \text{ and } ||Jy_n - Jx_n|| \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Again from (3.1) and (3.16), we have

$$||Jz_n - Jx_n|| = (1 - \alpha_n)||JT^n w_n - Jx_n|| \to 0 \text{ as } n \to \infty. \tag{3.21}$$

This implies that $||JT^n w_n - Jx_n|| \to 0$. Again since J^{-1} is uniformly norm-to-norm continuous, we also have

$$||T^n w_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.22}$$

For $p \in \Omega$, we note that

$$||T^n w_n - p|| \leq ||T^n w_n - x_n|| + ||x_n - p||. \tag{3.23}$$

It follows from (3.22) and $x_n \to p$ as $n \to \infty$, that

$$\lim_{n - \infty} ||T^n w_n - p|| = 0. \tag{3.24}$$

On other hand, we have

$$||T^{n+1} w_n - p|| \leq ||T^{n+1} w_n - T^n w_n|| + ||T^n w_n - p||.$$

Since T is uniformly asymptotically regular and from (3.24), we obtain that

$$||T^{n+1} w_n - p|| = 0. \tag{3.25}$$

Thai is, $TT^{nw}n \to p$ as $n \to \infty$. From the closedness of T, we see that $p \in F(T)$. Furthermore, For $q \in \Omega$, from (3.7) and (3.18) that

$$
\begin{aligned}
\phi(q,u_n) &\le \phi(q,y_n)\\
&\le \phi(q,x_n) + (1-\beta_n)[k_n(\phi(q,x_n) + (k_n-1)\phi(q,w_n)) - \phi(q,x_n)]\\
&\le \phi(q,x_n) + (1-\beta_n)[k_n\phi(q,x_n) + k_n^2\phi(q,w_n) - \phi(q,x_n)]\\
&\le \phi(q,x_n) + (1-\beta_n)[k_n\phi(q,x_n) + k_n^2(\phi(q,x_n) - 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)||Ax_n - Aq||^2) - \phi(q,x_n)]\\
&\le \phi(q,x_n) + (1-\beta_n)k_n\phi(q,x_n) + (1-\beta_n)k_n^2\phi(q,x_n)\\
&\quad - (1-\beta_n)k_n^2 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)||Ax_n - Aq||^2 - (1-\beta_n)\phi(q,x_n)\\
&\le \phi(q,x_n) + (1-\beta_n)k_n^2\phi(q,x_n) - (1-\beta_n)k_n^2 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)||Ax_n - Aq||^2\\
&= \phi(q,x_n) + \theta_n - (1-\beta_n)k_n^2 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)||Ax_n - Aq||^2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
2a(\alpha - \frac{2b}{c^2})||Ax_n - Aq||^2 &\le 2\lambda_n(\alpha - \frac{2}{c^2}\lambda_n)||Ax_n - Aq||^2\\
&\le \frac{1}{(1-\beta_n)k_n^2}(\phi(q,x_n) - \phi(q,u_n) + \theta_n).
\end{aligned}
\tag{3.26}
$$

From (3.18) and $\liminf_{n\to\infty}(1 - \beta_n) > 0$, obtain that

$$
\lim_{n\to\infty} ||Ax_n - Aq|| = 0 \tag{3.27}
$$

From Lemma 2.1, Lemma 2.8 and (3.4), we compute

$$
\begin{aligned}
\phi(x_n,w_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n))\\
&\le \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n))\\
&= V(x_n, Jx_n - \lambda_n Ax_n)\\
&\le V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n,\ \lambda_n Ax_n\rangle\\
&= \phi(x_n,x_n) + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n,\ -\lambda_n Ax_n\rangle\\
&= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n,\ -\lambda_n Ax_n\rangle\\
&\le \frac{4\lambda_n^2}{c^2}||Ax_n - Aq||^2\\
&\le \frac{4b^2}{c^2}||Ax_n - Aq||^2.
\end{aligned}
$$

Applying Lemma 2.6 and (3.27) that

$$
\lim_{n\to\infty} ||x_n - w_n|| = 0. \tag{3.28}
$$

Since J is uniformly norm-to-norm continuous on bounded sets, by (3.28), we have

$$
\lim_{n\to\infty} ||Jx_n - Jw_n|| = 0. \tag{3.29}
$$

From(3.1), (3.20) and (ii), we have

$$
||Jy_n - Jx_n|| = (1-\beta_n)||JS^n z_n - Jx_n|| \to 0 \text{ as } n \to \infty. \tag{3.30}
$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets

$$
||S^n z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.31}
$$

We observe that

$$
||S^n z_n - p|| \le ||S^n z_n - x_n|| + ||x_n - p||. \tag{3.32}
$$

It follows from (3.31) and $x_n \to p$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} ||S^n z_n - p|| = 0. \tag{3.33}$$

On other hand, we have

$$||S^{n+1}z_n - p|| \leq ||S^{n+1}z_n - S^n z_n|| + ||S^n z_n - p||.$$

Since S is uniformly asymptotically regular and (3.33), we obtain that

$$||S^{n+1}z_n - p|| = 0. \tag{3.34}$$

that is, $SS^n z_n \to p$ as $n \to \infty$. From the closedness of S, we see that $p \in F(S)$. Hence, $p \in F(T) \cap F(S)$.

(b) We show that $p \in \mathrm{GMEP}(f, B, \phi)$. From (A2), we have

$$\langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq f(y, u_n), \quad \forall y \in C,$$

and hence

$$\langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \langle y - u_n, \frac{(Ju_n - Jy_n)}{r_n}\rangle \geq f(y, u_n), \quad \forall y \in C. \tag{3.35}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = t_y + (1 - t)p$. Then, we get $y_t \in C$. From (3.35), it follows that

$$\begin{aligned}\langle By_t, y_t - u_n\rangle &\geq \langle By_t, y_t - u_n\rangle - \langle Bu_n, y_t - u_n\rangle - \varphi(y_t) + \varphi(u_n) - \langle y_t - u_n, \frac{(Ju_n - Jy_n)}{r_n}\rangle + f(y_t, u_n)\\ &\geq \langle By_t - Bu_n, y_t - u_n\rangle - \varphi(y_t) + \varphi(u_n) - \langle y_t - u_n, \frac{(Ju_n - Jy_n)}{r_n}\rangle + f(y_t, u_n), \quad \forall y_t \in C.\end{aligned}$$

we know that $y_n, u_n \to p$ as $n \to \infty$, and $\frac{||Ju_n - Jy_n||}{r_n} \to 0$ as $n \to \infty$. Since B is monotone, we know that $\langle By_t - Bu_n, y_t - u_n\rangle \geq 0$. Thus, it follows from (A4) that

$$\begin{aligned}f(y_t, p) - \varphi(y_t) + \varphi(p) &\leq \liminf_{n\to\infty} f(y_t, u_n) - \varphi(y_t) + \varphi(u_n) \leq \lim_{n\to\infty}\langle By_t, y_t - u_n\rangle\\ &= \langle By_t, y_t - p\rangle.\end{aligned}$$

Based on the conditions (A1), (A4) and convexity of ϕ, we have

$$\begin{aligned}0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t)\\ &\leq tf(y_t, y) + (1 - t)f(y_t, p) + t\varphi(y) + (1 - t)\varphi(p) - \varphi(y_t)\\ &= t[f(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)[f(y_t, p) + \varphi(p) - \varphi(y_t)]\\ &\leq t[f(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)[\langle By_t, y_t - p\rangle]\\ &= t[f(y_t, y) + \varphi(y) - \varphi(y_t)] + (1 - t)t[\langle By_t, y - p\rangle]\end{aligned}$$

and hence

$$0 \leq f(y_t, y) + \varphi(y) - \varphi(y_t) + (1 - t)\langle By_t, y - p\rangle.$$

From (A3) and the weakly lower semicontinuity of ϕ, and letting $t \to 0$, we also have

$$f(p, y) + \langle Bp, y - p\rangle + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C.$$

This implies that $p \in \mathrm{GMEP}(f, B, \phi)$.

(c) We show that $p \in \mathrm{VI}(A, C)$. Indeed, define a set-valued $U : E \rightrightarrows E^*$ by Lemma 2.14, U is maximal monotone and $U^{-1}0 = \mathrm{VI}(A, C)$. Let $(v, w) \in G(U)$. Since $w \in Uv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.

From $w_n \in C$, we have

$$\langle v - w_n, w - Av\rangle \geq 0. \tag{3.36}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then from Lemma 2.7, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0,$$

and thus

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\right\rangle \leq 0. \tag{3.37}$$

It follows from (3.36) and (3.37) that

$$\begin{aligned}
\langle v - w_n, w\rangle &\geq \langle v - w_n, Av\rangle \\
&\geq \langle v - w_n, Av\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\rangle \\
&= \langle v - w_n, Av - Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\rangle \\
&= \langle v - w_n, Av - Aw_n\rangle + \langle v - w_n, Aw_n - Ax_n\rangle + \langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\rangle \\
&\geq -||v - w_n||\frac{||w_n - x_n||}{\alpha} - ||v - w_n||\frac{||Jx_n - Jw_n||}{a} \\
&\geq -M(\frac{||w_n - x_n||}{\alpha} + \frac{||Jx_n - Jw_n||}{a}),
\end{aligned}$$

where $M = \sup_{n\geq 1} ||v - w_n||$. Takeing the limit as $n \to \infty$, (3.28) and (3.29), we obtain $\langle v - p, w\rangle \geq 0$. Based on the maximality of U, we have $p \in U^{-1}0$ and hence $p \in \mathrm{VI}(A, C)$. Hence, by (a), (b) and (c), we obtain $p \in \Omega$.

Step 5. Finally, we prove that $p = \Pi_\Omega x_0$. Taking the limit as $n \to \infty$ in (3.8), we obtain that

$$\langle p - q, Jx_0 - Jp\rangle \geq 0, \quad \forall q \in \Omega$$

and hence, $p = \Pi_\Omega x_0$ by Lemma 2.1. This completes the proof.

The following Theorems can readily be derived from Theorem 3.1.

Corollary 3.2. *Let E be a uniformly smooth and 2-uniformly convex Banach space, and C be a nonempty closed convex subset of E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$,. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1) - (A4), and $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-φ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$such that $k_n^{(t)} \to 1$as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-φ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$such that $k_n^{(s)} \to 1$as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S) \cap \mathrm{VI}(A, C) \cap \mathrm{MEP}(f, \phi) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by $x_0 \in E$ and*

$$\begin{cases} x_1 = \Pi_{C_1}x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n w_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ u_n \in C, \text{ such that} \\ f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n\phi(z, x_n) + (1-\beta_n)k_n\phi(z, z_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases} \tag{3.38}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\varphi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} (1 - \alpha_n) > 0$,
(ii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω.

Proof. Putting $B \equiv 0$ in Theorem 3.1, the conclusion of Theorem 3.2 can be obtained.

Corollary 3.3. *Let E be a uniformly smooth and 2-uniformly convex Banach space, C be a nonempty closed convex subset of E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in$ VI(A, C) ≠ ∅. Let $B : C \to E^*$ be a continuous and monotone mapping and $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-φ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-φ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S) \cap$ VI(A, C) ∩ MVI(B, C) ≠ ∅. Let $\{x_n\}$ be the sequence defined by $x_0 \in E$ and*

$$\begin{cases} x_1 = \Pi_{C_1}x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT^n w_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS^n z_n), \\ u_n \in C, \text{ such that} \\ \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n\phi(z, x_n) + (1-\beta_n)k_n\phi(z, z_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases} \tag{3.39}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\varphi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} (1 - \alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω.

Proof. Putting $f \equiv 0$ in Theorem 3.1, the conclusion of Theorem 3.2 can be obtained.

Since every closed relatively asymptotically nonexpansive mapping is asymptotically quasi-φ-nonexpansive, we obtain the following corollary.

Corollary 3.4. *Let E be a uniformly smooth and 2-uniformly convex Banach space, C be a nonempty closed convex subset of E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $B : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1) - (A4), and ϕ. $C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let T. $C \to C$ be a closed and relatively asymptotically nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and S. $C \to C$ be a closed and relatively asymptotically nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega := F(T) \cap F(S) \cap VI(A, C) \cap GMEP(f, B, \phi) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by $x_0 \in E$ and*

$$\begin{cases} x_1 = \Pi_{C_1} x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n w_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JS^n z_n), \\ u_n \in C, \text{ such that} \\ f(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n \phi(z, x_n) + (1-\beta_n) k_n \phi(z, z_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.40}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\varphi(z, x_n). z \in \Omega\}$ for each $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} (1 - \alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω.

Since every closed relatively nonexpansive mapping is asymptotically quasi-φ-nonexpansive, we obtain the following corollary.

Corollary 3.5. *Let E be a uniformly smooth and 2-uniformly convex Banach space, C be a nonempty closed convex subset of E. Let A be an α-inverse-strongly monotone* *mapping of C into E^* satisfying $||Ay|| \leq ||Ay\ Au||$, $\forall y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$. Let $B : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1) - (A4), and $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T, S : C \to C$ be closed relatively nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap \mathrm{VI}(A, C) \cap \mathrm{GMEP}(f, B, \phi) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by $x_0 \in E$ and*

$$\begin{cases} x_1 = \Pi_{C_1} x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T w_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S z_n), \\ u_n \in C, \text{ such that} \\ f(u_n, y) + \langle B u_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n \phi(z, x_n) + (1-\beta_n) k_n \phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.41}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} (1 - \alpha_n) > 0$,
(ii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω.

Corollary 3.6. *Let E be a uniformly smooth and 2-uniformly convex Banach space, C be a nonempty closed convex subset of E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in \mathrm{VI}(A, C) \neq \emptyset$. Let $B : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1) - (A4), and $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T, S : C \to C$ be a closed quasi-φ-nonexpansive mappings $\Omega := F(T) \cap F(S) \cap \mathrm{VI}(A, C) \cap \mathrm{GMEP}(f, B, \phi) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by $x_0 \in E$ and*

$$\begin{cases} x_1 = \Pi_{C_1} x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T w_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S z_n), \\ u_n \in C, \text{ such that} \\ f(u_n, y) + \langle B u_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n \phi(z, x_n) + (1-\beta_n) k_n \phi(z, z_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{3.42}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} (1 - \alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω

Proof. Since every closed quasi-φ-nonexpansive mapping is asymptotically quasi-φ-nonexpansive, the result is implied by Theorem 3.1.

Corollary 3.7. *Let E be a uniformly smooth and 2-uniformly convex Banach space, C be a nonempty closed convex subset of E. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $||Ay|| \leq ||Ay - Au||$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $B : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1) - (A4), and $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T, S : C \to C$ be closed relatively nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap VI(A, C) \cap GMEP(f, B, \phi) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by $x_0 \in E$ and*

$$\begin{cases} x_1 = \Pi_{C_1}x_0 \text{ and } C_1 = C, \\ w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTw_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSz_n), \\ u_n \in C, \text{ such that} \\ f(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \beta_n\phi(z, x_n) + (1-\beta_n)k_n\phi(z, z_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1], $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} (1 - \alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} (1 - \beta_n) > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is generalized projection of E onto Ω.

Proof. Since every closed relatively nonexpansive mapping is quasi-φ-nonexpansive, the result is implied by Theorem 3.1.

Remark 3.8. Corollaries 3.7, 3.6 and 3.7 improve and extend the corresponding results of Saewan et al. [[51], Theorem 3.1] in the sense of changing the closed relatively quasi-nonexpansive mappings to be the more general than the closed and asymptotically quasi-φ-nonexpansive mappings and adjusting a problem from the classical equilibrium problem to be the generalized equilibrium problem.

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Authors' contributions

SS designed and performed all the steps of proof in this research and also wrote the paper. PK participated in the design of the study and suggest many good ideas that made this paper possible and helped to draft the first manuscript. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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