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Banach operator pairs and common fixed points in modular function spaces

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Abstract

In this article, we introduce the concept of a Banach operator pair in the setting of modular function spaces. We prove some common fixed point results for such type of operators satisfying a more general condition of nonexpansiveness. We also establish a version of the well-known De Marr's theorem for an arbitrary family of symmetric Banach operator pairs in modular function spaces without Δ_2-condition.

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1. Introduction

The purpose of this article is to give an outline of fixed point theory for mappings defined on some subsets of modular function spaces which are natural generalization of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others. This article operates within the framework of convex function modulars.

The importance of applications of nonexpansive mappings in modular function spaces lies in the richness of structure of modular function spaces that besides being Banach spaces (or F-spaces in a more general settings) are equipped with modular equivalents of norm or metric notions, and also are equipped with almost everywhere convergence and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point results, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the concepts of modular function spaces. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed spaces and in metric spaces.

The theory of contractions and nonexpansive mappings defined on convex subsets of Banach spaces is rich (see, e.g., [1-4]) and has been well developed since the 1960s and generalized to other metric spaces (see, e.g., [5-7]), and modular function spaces (see, e.g., [8-11]). The corresponding fixed point results were then extended to larger classes of mappings like asymptotic mappings [12,13], pointwise contractions [14] and asymptotic pointwise contractions and nonexpansive mappings [15-17].

As noted in [16,18], questions are sometimes asked whether the theory of modular function spaces provides general methods for the consideration of fixed point properties; the situation here is the same as it is in the Banach space setting.

In this article, we introduce the concept of a Banach operator pair in modular function spaces. Then, we investigate the existence of common fixed points for such operators. Believing that the well-known De Marr's theorem [19] is not known yet in the setting of modular function spaces, we establish this classical result in this new setting.

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \mathrm{U}K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1. *Let $\rho : \mathcal{M}_\infty \to [0, \infty]$be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*:

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e., $|f(\omega)| \le |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \le \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally subadditive, i.e., $\rho(f1_{A\cup B}) \le \rho(f1_A) + \rho(f1_B)$ for any$A,B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;
(iv) ρ has the Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(v) ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$and $|g_n(\omega)| \downarrow, 0$ implies $\rho(g_n) \downarrow, 0$.

As in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind, we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \rho - \text{a.e.}\}, \tag{2.1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. When no confusion arises we will write $\mathcal{M}$. instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2. *Let ρ be a regular function pseudomodular.*

(1) We say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ-a.e.
(2) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω *will be denoted by* $\Re$.

Let us denote $\rho(f, E) = \rho(f1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Definition 2.1.1 in [20] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [20-22]; see also Musielak [23] for the basics of the general modular theory.

Remark 2.1. *We limit ourselves to convex function modulars in this article. However, omitting convexity in Definition* 2.1 *or replacing it by s-convexity would lead to the definition of nonconvex or s-convex regular function pseudomodulars, semimodulars and modulars as in* [20].

Definition 2.3. [20-22]*Let* ρ *be a convex function modular.*

(a) *A modular function space is the vector space* $L_\rho(\Omega, \Sigma)$, *or briefly* L_ρ, *defined by*

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

(b) The *following formula defines a norm in* L_ρ *(frequently called Luxemurg norm):*

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \le 1\}.$$

In the following theorem, we recall some of the properties of modular spaces that will be used later on in this article.

Theorem 2.1. [20, 21, 22] *Let* $\rho \in \Re$.

(1) L_ρ, $||f||_\rho$ *is complete and the norm* $|| \cdot ||_\rho$ *is monotone w.r.t. the natural order in* $\mathcal{M}$.
(2) $||f_n||_\rho \to 0$ *if and only if* $\rho(\alpha f_n) \to 0$ *for every* $\alpha > 0$.
(3) *If* $\rho(\alpha f_n) \to 0$ *for an* $\alpha > 0$, *then there exists a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that* $g_n \to 0$ ρ*-a.e.*
(4) *If* $\{f_n\}$ *converges uniformly to* f *on a set* $E \in \mathcal{P}$, *then* $\rho(\alpha(f_n - f), E) \to 0$ *for every* $\alpha > 0$.
(5) *Let* $f_n \to f$ ρ*-a.e. There exists a nondecreasing sequence of sets* $H_k \in \mathcal{P}$ *such that* $H_k \uparrow \Omega$ *and* $\{f_n\}$ *converges uniformly to* f *on every* H_k *(Egoroff Theorem).*
(6) $\rho(f) \le \liminf \rho(f_n)$ *whenever* $f_n \to f$ ρ*-a.e. (Note: this property is equivalent to the Fatou Property.)*
(7) *Defining* $L_\rho^0 = \{f \in L_\rho;\ \rho(f, \cdot) \text{ is order continuous}\}$ *and* $E_\rho = \{f \in L_\rho;\ \lambda f \in L_\rho^0 \text{ for every } \lambda > 0\}$, *we have:*
  (a) $L_\rho \supset L_\rho^0 \supset E_\rho$,
  (b) E_ρ *has the Lebesgue property, i.e.,* $\rho(\alpha f, D_k) \to 0$ *for* $\alpha > 0$, $f \in E_\rho$ *and* $D_k \downarrow \emptyset$.
  (c) E_ρ *is the closure of* $\mathcal{E}$ (*in the sense of* $|| \cdot ||_\rho$).

The following definition plays an important role in the theory of modular function spaces.

Definition 2.4. *Let $\rho \in \Re$. We say that ρ has the Δ_2-property if* $\sup_n \rho(2f_n, D_k) \to 0$ *whenever $D_k \downarrow \emptyset$and* $\sup_n \rho(f_n, D_k) \to 0$.

Theorem 2.2. *Let $\rho \in \Re$. The following conditions are equivalent:*

(a) ρ has Δ_2-property,
(b) $L^0_\rho$is a linear subspace of L_ρ,
(c) $L_\rho = L^0_\rho = E_\rho$,
(d) if $\rho(f_n) \to 0$, then $\rho(2f_n) \to 0$,
(e) if $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then $||f_n||_\rho \to 0$, i.e., the modular convergence is equivalent to the norm convergence.

The following definition is crucial throughout this article.

Definition 2.5. *Let $\rho \in \Re$.*

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f$ (ρ) if and only if $\rho(f_n - f) \to 0$.
(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n,m \to \infty$.
(c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $f_n \in B$, the convergence $f_n \to f$ (ρ) implies that f belongs to B.
(d) A set $B \subset L_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B) = \sup\{\rho(f - g); f \in B, g \in B\} < \infty$.
(e) Let $f \in L_\rho$ and $C \subset L_\rho$. The ρ-distance between f and C is defined as

$$d_\rho(f, C) = \text{int}\,\{\rho(f - g); g \in C\}.$$

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f$, $\lambda > 1$. Using Theorem 2.1, it is not difficult to prove the following.

Proposition 2.1. *Let $\rho \in \Re$.*

(i) L_ρ is ρ-complete,
(ii) ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x - y) \leq r\}$ are ρ-closed.

The following property plays in the theory of modular function spaces a role similar to the reflexivity in Banach spaces (see, e.g., [10]).

Definition 2.6. *We say that L_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection. A nonempty subset K of L_ρ is said to be ρ-compact if for any family $\{A_\alpha; A_\alpha \in 2^{L\rho}, \alpha \in \Gamma\}$ of ρ-closed subsets with $K \cap A_{\alpha 1} \cap \cdots \cap A_{\alpha n} \neq \emptyset$, for any $\alpha_1,..., \alpha_{n \in \Gamma}$, we have*

$$K \cap \left(\bigcap_{\alpha \in \Gamma} A_\alpha \right) \neq \emptyset.$$

Next, we give the modular definitions of asymptotic pointwise nonexpansive mappings. The definitions are straightforward generalizations of their norm and metric equivalents [13,15,17].

Definition 2.7. *Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called an asymptotic pointwise mapping if there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ such that*

$$\rho\left(T^n(f) - T^n(g)\right) \leq \alpha_n(f)\rho(f - g) \quad \text{for any } f, g \in L_\rho.$$

(i) If $\limsup_{n\to\infty} \alpha_n(f) \leq 1$ *for any $f \in L_\rho$, then T is called asymptotic point-wise ρ-nonexpansive.*

(ii) *If* $\sup_{n\in\mathbb{N}} \alpha_n(f) \leq 1$ *for any $f \in L_\rho$, then T is called ρ-nonexpansive. In particular, we have*

$$\rho\left(T(f) - T(g)\right) \leq \rho(f - g) \quad \text{for any } f, g \in C.$$

The fixed point set of T is defined by $Fix(T) = \{f \in C; T(f) = f\}$.

In the following definition, we introduce the concept of Banach Operator Pairs [24,25] in modular function spaces.

Definition 2.8. *Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty. The ordered pair (S, T) of two self-maps of the subset C is called a Banach operator pair, if the set $Fix(T)$ is S-invariant, namely $S(Fix(T)) \subseteq Fix(T)$.*

In [26], a result similar to Ky Fan's fixed point theorem in modular function spaces was proved. The following definition is needed:

Definition 2.9. *Let $\rho \in \Re$. Let $C \subset L_\rho$ be a nonempty ρ-closed subset. Let $T : C \to L_\rho$ be a map. T is called ρ-continuous if $\{T(f_n)\}$ ρ-converges to $T(f)$ whenever $\{f_n\}$ ρ-converges to f. Also, T will be called strongly ρ-continuous if T is ρ-continuous and*

$$\liminf_{n\to\infty} \rho(g - T(f_n)) = \rho(g - T(f))$$

for any sequence $\{f_n\} \subset C$ which ρ-converges to f and for any $g \in C$.

3. Common fixed points for Banach operator pairs

The study of a common fixed point of a pair of commuting mappings was initiated as soon as the first fixed point result was proved. This problem becomes more challenging and seems to be of vital interest in view of historically significant and negatively settled problem that a pair of commuting continuous self-mappings on the unit interval [0,1] need not have a common fixed point [27]. Since then, many fixed point theorists have attempted to find weaker forms of commutativity that may ensure the existence of a common fixed point for a pair of self-mappings on a metric space. In this context, the notions of weakly compatible mappings [28] and Banach operator pairs [24,25,29-34] have been of significant interest for generalizing results in metric fixed point theory for single valued mappings. In this section, we investigate some of these results in modular function spaces.

We first prove the following technical result.

Theorem 3.1. *Let $\rho \in \Re$. Let $K \subset L_p$ be ρ-compact convex subset. Then, any $T : K \to K$ strongly ρ-continuous has a nonempty fixed point set $Fix(T)$. Moreover, $Fix(T)$ is ρ-compact.*

Proof. The existence of a fixed point is proved in [26]. Hence, $Fix(T)$ is nonempty. Let us prove that $Fix(T)$ is ρ-compact. It is enough to show that $Fix(T)$ is ρ-closed since K is ρ-compact. Let $\{f_n\}$ be a sequence in $Fix(T)$ such that $\{f_n\}$ ρ-converges to f. Let us prove that $f \in Fix(T)$. Since T is ρ-continuous, so $\{T(f_n)\}$ ρ-converges to $T(f)$.

Since $T(f_n) = f_n$, we get $\{f_n\}$ ρ-converges to f and $T(f)$. The uniqueness of the ρ-limit implies $T(f) = f$, i.e., $f \in Fix(T)$.

Definition 3.1. *Let $K \subset L_\rho$ be nonempty subset. The mapping $T : K \to K$ is called R-map if Fix (T) is a ρ-continuous retract of K. Recall that a mapping $R : K \to Fix(T)$ is a retract if and only if $R \circ R = R$.*

Note that in general the fixed point set of ρ-continuous mappings defined on any ρ-compact convex subset of L_ρ may not be a ρ-continuous retract.

Theorem 3.2. *Let $\rho \in \Re$. Let $K \subset L_\rho$ be ρ-compact convex subset. Let $T : K \to K$ be strongly ρ-continuous R-map. Let $S : K \to K$ be strongly ρ-continuous such that (S,T) is a Banach operator pair. Then, $F(S,T) = Fix(T) \cap Fix(S)$ is a nonempty ρ-compact subset of K.*

Proof. From Theorem 3.1, we know that $Fix(T)$ is not empty and ρ-compact subset of K. Since T is an R-map, then there exists a ρ-continuous retract $R : K \to Fix(T)$. Since (S,T) is a Banach pair of operators, then $S(Fix(T)) \subset Fix(T)$. Note that $S \circ R : K \to K$ is strongly ρ-continuous. Indeed, if $\{f_n\} \subset K$ ρ-converges to f, then $\{R(f_n)\} \subset K$ ρ-converges to $R(f)$ since R is ρ-continuous. And since S is strongly ρ-continuous, then for any $g \in K$, we have

$$\liminf_{n\to\infty} \rho(g - S(R(f_n))) = \rho(g - S(R(f))),$$

which shows that $S \circ R$ is strongly ρ-continuous. Theorem 3.1 implies that $Fix(S \circ R)$ is nonempty and ρ-compact. Note that if $f \in Fix(S \circ R)$, then we have $S \circ R(f) = S(R(f)) = f \in Fix(T)$ since $S \circ R(K) \circ Fix(T)$. In particular, we have $R(f) = f$. Hence, $S(f) = f$, i.e., $f \in Fix(T) \cap Fix(S)$. It is easy to then see that $Fix(T) \cap Fix(S) = Fix(S \circ R) = F(S,T)$ which implies $F(S,T)$ is nonempty and ρ-compact subset of K.

Before we state next result which deals with ρ-nonexpansive mappings, let us recall the definition of uniform convexity in modular function spaces [18].

Definition 3.2. *Let $\rho \in \Re$. We define the following uniform convexity type properties of the function modular ρ:*

(i) Let $r > 0, \varepsilon > 0$. Define

$$D_1(r,\varepsilon) = \{(f,g); f,g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho(f-g) \ge \varepsilon r\}.$$

Let

$$\delta_1(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right); (f,g) \in D_1(r,\varepsilon)\right\} \quad \text{if } D_1(r,\varepsilon) \neq \emptyset,$$

and $\delta_1(r,\varepsilon) = 1$ if $D_1(r,\varepsilon) = \emptyset$. We say that ρ satisfies $(UC1)$ if for every $r > 0, \varepsilon > 0$, $\delta_1(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough.

(ii) We say that ρ satisfies $(UUC1)$ if for every $s \ge 0$, $\varepsilon > 0$ there exists

$$\eta_1(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0 \quad \text{for } r > s.$$

(iii) Let $r > 0$, $\varepsilon > 0$. Define

$$D_2(r,\varepsilon) = \left\{(f,g); f,g \in L_\rho, \rho(f) \le r,\ \rho(g) \le r,\ \rho\left(\frac{f-g}{2}\right) \ge \varepsilon r\right\}.$$

Let

$$\delta_2(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right); (f,g) \in D_2(r,\varepsilon)\right\} \quad \text{if } D_2(r,\varepsilon) \ne \emptyset,$$

and $\delta_2(r,\varepsilon) = 1$ if $D_2(r,\varepsilon) = \emptyset$. We say that ρ satisfies (UC2) if for every $r > 0, \varepsilon > 0$, $\delta_2(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_2(r,\varepsilon) \ne \emptyset$, for *$\varepsilon > 0$ small enough.*

(iv) We say that ρ satisfies (UUC2) if for every $s \ge 0$, $\varepsilon > 0$ there exists

$$\eta_2(s,\varepsilon) > 0$$

depending on s and ε such that

$$\delta_2(r,\varepsilon) > \eta_2(s,\varepsilon) > 0 \quad \text{for } r > s.$$

In [18], it is proved that any asymptotically pointwise ρ-nonexpansive mapping defined on a ρ-closed ρ-bounded convex subset has a fixed point. The next result improves their result by showing that the fixed point set is convex.

Theorem 3.3. *Assume $\rho \in \Re$is (UUC1). Let C be a ρ-closed ρ-bounded convex nonempty subset of L_ρ. Then, any $T : C \to C$ asymptotically pointwise ρ-nonexpansive has a fixed point. Moreover, the set of all fixed points Fix(T) is ρ-closed and convex.*

Proof. In [18], it is proved that $Fix(T)$ is a ρ-closed nonempty subset of C. Let us prove that $Fix(T)$ is convex. Let $f,g \in Fix(T)$, with $f \ne g$. For every $n \in \mathbb{N}$, we have

$$\rho\left(f - T^n\left(\frac{f+g}{2}\right)\right) \le \alpha_n(f)\rho\left(\frac{f-g}{2}\right)$$

and

$$\rho\left(g - T^n\left(\frac{f+g}{2}\right)\right) \le \alpha_n(g)\rho\left(\frac{f-g}{2}\right).$$

Set $R = \rho\left(\frac{f-g}{2}\right)$. Then,

$$\limsup_{n\to\infty} \rho\left(f - T^n\left(\frac{f+g}{2}\right)\right) \le R \quad \text{and} \quad \limsup_{n\to\infty} \rho\left(g - T^n\left(\frac{f+g}{2}\right)\right) \le R.$$

Since

$$\rho\left(\frac{1}{2}\left[f - T^n\left(\frac{f+g}{2}\right)\right] + \frac{1}{2}\left[T^n\left(\frac{f+g}{2}\right) - g\right]\right) = \rho\left(\frac{f-g}{2}\right) = R,$$

and ρ is (UUC2) (since (UUC1) implies (UUC2)), then we must have

$$\lim_{n\to\infty} \rho\left(\frac{1}{2}\left[f - T^n\left(\frac{f+g}{2}\right)\right] - \frac{1}{2}\left[T^n\left(\frac{f+g}{2}\right) - g\right]\right) = 0,$$

and so

$$\lim_{n\to\infty} \rho\left(\frac{f+g}{2} - T^n\left(\frac{f+g}{2}\right)\right) = 0.$$

Since ρ is convex we get

$$\rho\left(\frac{1}{2}\left[\frac{f+g}{2} - T\left(\frac{f+g}{2}\right)\right]\right) \le \frac{1}{2}\rho\left(\frac{f+g}{2} - T^n\left(\frac{f+g}{2}\right)\right) + \frac{1}{2}\rho\left(T^n\left(\frac{f+g}{2}\right) - T\left(\frac{f+g}{2}\right)\right)$$

which implies

$$\rho\left(\frac{1}{2}\left[\frac{f+g}{2} - T\left(\frac{f+g}{2}\right)\right]\right) \le \frac{1}{2}\rho\left(\frac{f+g}{2} - T^n\left(\frac{f+g}{2}\right)\right) + \frac{\alpha_1\left(\frac{f+g}{2}\right)}{2}\rho\left(\frac{f+g}{2} - T^{n-1}\left(\frac{f+g}{2}\right)\right).$$

If we let $n \to \infty$, we get $\rho\left(\frac{1}{2}\left[\frac{f+g}{2} - T\left(\frac{f+g}{2}\right)\right]\right) = 0$, i.e., $T\left(\frac{f+g}{2}\right) = \frac{f+g}{2}$ and so $\frac{f+g}{2} \in Fix(T)$. This completes the proof of our claim.

As a corollary, we obtain the following result.

Corollary 3.1. *Assume $\rho \in \Re is$ $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_p. Then, any $T : C \to C$ ρ-nonexpansive has a fixed point. Moreover, the set of all fixed points $Fix(T)$ is ρ-closed and convex.*

Next, we discuss the existence of common fixed points for Banach operator pairs of pointwise asymptotically ρ-nonexpansive mappings.

Theorem 3.4. *Assume $\rho \in \Re is$ $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_p. Let $T : C \to C$ be asymptotically pointwise ρ-nonexpansive mapping. Then, any $S : C \to C$ pointwise asymptotically ρ-nonexpansive mapping such that (S, T) is a Banach operator pair has a common fixed point with T. Moreover $F(S, T) = Fix(T) \cap Fix(S)$ is a nonempty ρ-closed convex subset of C.*

Proof. Since T is asymptotically pointwise ρ-nonexpansive, then $Fix(T)$ is nonempty ρ-closed convex subset of C. Since (S, T) is a Banach operator pair, then we must have $S(Fix(T)) \subset Fix(T)$. Theorem 3.3 implies that the restriction of S to $Fix(T)$ has a nonempty fixed point set which is ρ-closed and convex, i.e., $F(S,T) = Fix(T) \cap Fix(S)$ is a nonempty ρ-closed convex subset of C. This completes the proof of our claim.

As a corollary, we get the following result.

Corollary 3.2. *Assume $\rho \in \Re is$ $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_p. Let $T : C \to C$ be ρ-nonexpansive mapping. Then, any $S : C \to C$ ρ-nonexpansive mapping such that (S,T) is a Banach operator pair has a common fixed point with T. Moreover, $F(S,T) = Fix(T) \cap Fix(S)$ is a nonempty ρ-closed convex subset of C.*

4. Common fixed point of Banach operator family

The aim of this section is to extend the common fixed point results found in the previous section to a family of Banach operator mappings. In particular, we prove an analogue of De Marr's result in modular function spaces. In order to obtain such extension we need to introduce the concept of symmetric Banach operator pairs.

Definition 4.1. *Let T and S be two self-maps of a set C. The pair* (S,T) *is called symmetric Banach operator pair if both* (S, T) *and* (T, S) *are Banach operator pairs, i.e.,* $T(Fix(S)) \subseteq Fix(S)$ *and* $S(Fix(T)) \subseteq Fix(T)$.

Let $\rho \in \Re$ and C be a ρ-closed nonempty subset of L_ρ. Let $\mathcal{T}$ be a family of self-maps defined on C. Then, the family $\mathcal{T}$ has a common fixed point if it is the fixed point of each member of $\mathcal{T}$. The set of common fixed points is denoted by $Fix(\mathcal{T})$. We have by definition $Fix(\mathcal{T}) = \bigcap_{T\in\mathcal{T}} Fix(T)$.

Next, we state an analogue of De Marr's result in modular function spaces.

Theorem 4.1. *Let* $\rho \in \Re$. *Let* $K \subset L_\rho$ *be nonempty* ρ*-compact convex subset. Let* $\mathcal{T}$ *be a family of self-maps defined on K such that any map in* $\mathcal{T}$ *is strongly* ρ*-continuous R-map. Assume that any two mappings in* $\mathcal{T}$ *form a symmetric Banach operator pair. Then, the family* $\mathcal{T}$ *has a common fixed point. Moreover,* $Fix(\mathcal{T})$ *is a* ρ*-compact subset of K.*

Proof. Using Theorem 3.2, we deduce that for any $T_1,T_2,\ldots,T_n$ in $\mathcal{T}$, we have $Fix(T_1) \cap Fix(T_2) \cap \ldots \cap Fix(T_n)$ is a nonempty ρ-compact subset of K. Therefore, any finite family of the subsets $\{Fix(T); T \in \mathcal{T}\}$ has a nonempty intersection. Since these sets are all ρ-closed and K is ρ-compact, we conclude that $Fix(\mathcal{T}) = \bigcap_{T\in\mathcal{T}} Fix(T)$ is not empty and is ρ-closed. Therefore, $Fix(\mathcal{T})$ is a ρ-compact subset of K which finishes the proof of our theorem.

As commuting operators are symmetric Banach operators, so we obtain:

Corollary 4.1. *Let* $\rho \in \Re$. *Let* $K \subset L_\rho$ *be nonempty* ρ*-compact convex subset. Let* $\mathcal{T}$ *be a family of commuting self-maps defined on K such that any map in* $\mathcal{T}$ *is strongly* ρ*-continuous R-map. Then, the family* $\mathcal{T}$ *has a common fixed point. Moreover,* $Fix(\mathcal{T})$ *is a* ρ*-compact subset of K.*

Next, we discuss a similar conclusion in modular function spaces L_ρ when ρ is $(UUC1)$. Prior to obtain such result we will need an intersection property which seems to be new. Indeed, it is well known [18] that if $\rho \in \Re$ is $(UUC2)$, then any countable family $\{C_n\}$ of ρ-bounded ρ-closed convex subsets of L_ρ has a nonempty intersection provided that the intersection of any finite subfamily has a nonempty intersection. Such intersection property is known as property (R). This intersection property is parallel to the well-known fact that uniformly convex Banach spaces are reflexive. The property (R) is essential for the proof of many fixed point theorems in metric and modular function spaces. But since it is not clear that this intersection property is related to any topology, we did not know if such intersection property is in fact valid for any family. Therefore, the next result seems to be new.

Theorem 4.2. *Assume* $\rho \in \Re$ *is* $(UUC1)$. *Let* $\{C_a\}_{a\in\Gamma}$ *be a nonincreasing family of nonempty, convex,* ρ*-closed* ρ*-bounded subsets of* L_ρ, *where* Γ *is a directed index set. then,* $\bigcap_{\alpha\in\Gamma} C_\alpha \neq \emptyset$.

Proof. Recall that Γ is directed if there exists an order $\preccurlyeq$ defined on Γ such that for any $\alpha,\beta \in \Gamma$, there exists $\gamma \in \Gamma$ such that $\alpha \preccurlyeq \gamma$ and $\beta \preccurlyeq \gamma$. And $\{C_a\}_{a\in\Gamma}$ is nonincreasing if and only if for any $\alpha, \beta \in \Gamma$ such that $\alpha \preccurlyeq \beta$, then $C_\beta \subset C_\alpha$. Note that for any $\alpha_0 \in \Gamma$, we have

$$\bigcap_{\alpha\in\Gamma} C_\alpha = \bigcap_{\alpha_0\preccurlyeq\alpha} C_\alpha.$$

Therefore, without of any generality, we may assume that there exists $C \subset L_p$ ρ-closed ρ-bounded convex subset such that $C_\alpha \subset C$ for any $\alpha \in \Gamma$. If $\delta_P(C) = 0$, then all subsets C_α are reduced to a single point. In this case, we have nothing to prove. Hence, let us assume $\delta_P(C) > 0$. Let $f \in C$. Then, the proximinality of ρ-closed convex subsets of L_p when ρ is $(UUC2)$ (see [18]) implies the existence of $f_\alpha \in C_\alpha$ such that

$$\rho(f - f_\alpha) = d_\rho(f, C_\alpha) = \inf\{\rho(f - g); g \in C_\alpha\}.$$

Set $A_\alpha = \{f_\beta;\ \alpha \preccurlyeq \beta\}$, for any $\alpha \in \Gamma$. Then, $A_\alpha \subset C_\alpha$, for any $\alpha \in \Gamma$. Notice that

$$\delta_\rho(A_\alpha) = \delta_\rho\left(\overline{conv}^\rho(A_\alpha)\right) \quad \text{for any } \alpha \in \Gamma.$$

Indeed, let $g \in A_\alpha$, then $A_\alpha \subset B(g,\delta_\rho(A_\alpha))$. Since $B(g,\delta_\rho(A_\alpha))$ is ρ-closed and convex, then we must have $\overline{conv}^\rho(A_\alpha) \subset B(g, \delta_\rho(A_\alpha))$. Hence, for any $h \in \overline{conv}^\rho(A_\alpha)$, we have $\rho(g - h) \le \delta_\rho(A_\alpha)$. Since g was arbitrary in A_α we conclude that $A_\alpha \subset B(h, \delta_\rho(A_\alpha))$. Again for the same reason we get $\overline{conv}^\rho(A_\alpha) \subset B(h, \delta_\rho(A_\alpha))$. Hence, for any $g, h \in \overline{conv}^\rho(A_\alpha)$ we have $\rho(g - h) \le \delta_\rho(A_\alpha)$, which implies $\delta_\rho\left(\overline{conv}^\rho(A_\alpha)\right) \le \delta_\rho(A_\alpha)$. This is enough to have $\delta_\rho\left(\overline{conv}^\rho(A_\alpha)\right) = \delta_\rho(A_\alpha)$. Set $R = \sup_{\alpha\in\Gamma} \rho(f - f_\alpha)$. Without loss of any generality, we may assume $R > 0$. Let us prove that $\inf_{\alpha\in\Gamma} \delta_\rho(A_\alpha) = 0$. Assume not. Then, $\inf_{\alpha\in\Gamma} \delta_\rho(A_\alpha) > 0$. Set $\delta = \frac{1}{2}\inf_{\alpha\in\Gamma} \delta_\rho(A_\alpha)$. Then, for any $\alpha \in \Gamma$, there exist $\beta,\gamma \in \Gamma$ such that $\alpha \preccurlyeq \beta$ and $\alpha \preccurlyeq \gamma$ and

$$\rho(f_\beta - f_\gamma) > \delta.$$

Since $\rho(f - f_\gamma) \le R$ and $\rho(f - f_\beta) \le R$, then we have

$$\rho\left(f - \frac{f_\beta + f_\gamma}{2}\right) \le R\left(1 - \delta_1\left(R, \frac{\delta}{R}\right)\right).$$

Since $f_\beta, f_\gamma \in C_\alpha$ and C_α is convex, we get

$$\rho(f - f_\alpha) \le R\left(1 - \delta_1\left(R, \frac{\delta}{R}\right)\right),$$

using the definition of f_α. Since α was chosen arbitrarily in Γ we get

$$R = \sup_{\alpha\in\Gamma} \rho(f - f_\alpha) \le R\left(1 - \delta_1\left(R, \frac{\delta}{R}\right)\right).$$

This is a contradiction. Therefore, we have $\inf_{\alpha\in\Gamma} \delta_\rho(A_\alpha) = 0$. Since Γ is directed, there exists $\{\alpha_n\} \subset \Gamma$ such that $\alpha_n \preccurlyeq \alpha_{n+1}$ and $\inf_{n\ge1} \delta_\rho(A_{\alpha n}) = 0$. In particular, we have $A_{\alpha n+1} \subset A_{\alpha n}$ which implies $\overline{conv}^\rho\left(A_{\alpha_{n+1}}\right) \subset \overline{conv}^\rho\left(A_{\alpha_n}\right)$. Using the property (R) satisfied by L_ρ, we conclude $A = \bigcap_{n\ge1} \overline{conv}^\rho\left(A_{\alpha_n}\right) \ne \emptyset$. Since $\inf_{\alpha\in\Gamma}\delta_\rho\left(\overline{conv}^\rho\left(A_{\alpha_n}\right)\right) = \inf_{\alpha\in\Gamma}\delta_\rho(A_\alpha) = 0$, we conclude that $A = \{h\}$ for some $h \in C$. Let us prove that for any $\alpha \in \Gamma$ we have $h \in \overline{conv}^\rho(A_\alpha)$. Indeed, let $\alpha \in \Gamma$. If there exists $n \ge 1$ such that $\alpha \preccurlyeq \alpha_n$, then we have $A_{\alpha n} \subset A_\alpha$. Hence, $\overline{conv}^\rho\left(A_{\alpha_n}\right) \subset \overline{conv}^\rho(A_\alpha)$. This clearly implies $h \in \overline{conv}^\rho(A_\alpha)$. Otherwise, assume that for any $n \ge 1$ such that $\alpha_n \preccurlyeq \alpha$, so $A_\alpha \preccurlyeq A_{\alpha n}$. Hence, $\overline{conv}^\rho(A_\alpha) \subset \overline{conv}^\rho\left(A_{\alpha_n}\right)$. In particular, we have $\overline{conv}^{\rho}(A_{\alpha}) \subset \bigcap_{n\geq 1} \overline{conv}^{\rho}\left(A_{\alpha_n}\right) = \{h\}$. Which forces $h \in \overline{conv}^{\rho}(A_{\alpha})$. Therefore, $h \in \bigcap_{\alpha\in\Gamma} \overline{conv}^{\rho}(A_{\alpha})$. Since $\bigcap_{\alpha\in\Gamma} \overline{conv}^{\rho}(A_{\alpha}) \subset \bigcap_{\alpha\in\Gamma} C_{\alpha}$, we conclude that $h \in \bigcap_{\alpha\in\Gamma} C_{\alpha}$. Hence, $\bigcap_{\alpha\in\Gamma} C_{\alpha} \neq \emptyset$.

Using Theorem 4.2, we get the following common fixed point result.

Theorem 4.3. *Assume $\rho \in \Re$. is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_p. Let $\mathcal{T}$ be a family of self-maps defined on C such that any map in $\mathcal{T}$ is asymptotically pointwise ρ-nonexpansive. Assume that any two mappings in $\mathcal{T}$ form a symmetric Banach operator pair. Then, the family $\mathcal{T}$ has a common fixed point. Moreover, $Fix(\mathcal{T})$ is a ρ-closed convex subset of C.*

Proof. Using Theorem 3.4, we deduce that for any $T_1, T_2, ..., T_n$ in $\mathcal{T}$, we have $Fix(T_1) \cap Fix(T_2) \cap ... \cap Fix(T_n)$ is a nonempty ρ-closed convex subset of C. Therefore, any finite family of the subsets $\{Fix(T); T \in \mathcal{T}\}$ has a nonempty intersection. Since these sets are all ρ-closed and convex subsets of C, then Theorem 4.2 implies that $Fix(\mathcal{T}) = \bigcap_{T\in\mathcal{T}} Fix(T)$ is not empty and is ρ-closed and convex. Therefore, $Fix(\mathcal{T})$ is a ρ-closed convex subset of C which finishes the proof of our theorem.

As corollaries we get the following common fixed point results which seem to be new.

Corollary 4.2. *Assume $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_{ρ}. Let $\mathcal{T}$ be a family of self-maps defined on C such that any map in $\mathcal{T}$ is ρ-nonexpansive. Assume that any two mappings in $\mathcal{T}$ form a symmetric Banach operator pair. Then, the family $\mathcal{T}$ has a common fixed point. Moreover, $Fix(\mathcal{T})$ is a ρ-closed convex subset of C.*

Corollary 4.3. *Assume $\rho \in \Re$ is $(UUC1)$. Let C be a ρ-closed ρ-bounded convex nonempty subset of L_p. Let $\mathcal{T}$ be a family of commuting self-maps defined on C such that any map in $\mathcal{T}$ is asymptotically pointwise ρ-nonexpansive. Then, the family $\mathcal{T}$ has a common fixed point. Moreover, $Fix(\mathcal{T})$ is a ρ-closed convex subset of C.*

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References

1. Browder, FE: Nonexpansive nonlinear operators in a Banach space. Proc Natl Acad Sci USA. **54**, 1041–1044 (1965). doi:10.1073/pnas.54.4.1041
2. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762–772 (2006). doi:10.1016/j.na.2005.09.044
3. Goebel, K, Reich, S: Uniform convexity, hyperbolic geometry, and nonexpansive mappings. In Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 83,Dekker, New York (1984)
4. Gohde, D: Zum Prinzip der kontraktiven Abbildung. Math Nachr. **30**, 251–258 (1965). doi:10.1002/mana.19650300312
5. Baillon, JB: Nonexpansive mappings and hyperconvex spaces. Contemp Math. **72**, 11–19 (1988)

6. Goebel, K, Sekowski, T, Stachura, A: Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball. Nonlinear Anal. **4**, 1011–1021 (1980). doi:10.1016/0362-546X(80)90012-7
7. Khamsi, MA, Kirk, WA: An Introduction to Metric Spaces and Fixed Point Theory. Wiley, New York (2001)
8. Khamsi, MA: A convexity property in modular function spaces. Math Japonica. **44**(2), 269–279 (1996)
9. Khamsi, MA, Kozlowski, WK, Reich, S: Fixed point theory in modular function spaces. Nonlinear Anal. **14**, 935–953 (1990). doi:10.1016/0362-546X(90)90111-S
10. Khamsi, MA, Kozlowski, WM, Shutao, C: Some geometrical properties and fixed point theorems in Orlicz spaces. J Math Anal Appl. **155**(2), 393–412 (1991). doi:10.1016/0022-247X(91)90009-O
11. Kutbi, MA, Latif, A: Fixed point of multivalued maps in modular function spaces. Fixed Point Theory Appl (2009). Article ID 786357, 12 pages (2009)
12. Khamsi, MA: On asymptotically nonexpansive mappings in hyperconvex metric spaces. Proc Am Math Soc. **132**, 365–373 (2004). doi:10.1090/S0002-9939-03-07172-7
13. Kirk, WA: Fixed points of asymptotic contractions. J Math Anal Appl. **277**, 645–650 (2003). doi:10.1016/S0022-247X(02)00612-1
14. Kirk, WA: Mappings of generalized contractive type. J Math Anal Appl. **32**, 567–572 (1970). doi:10.1016/0022-247X(70)90278-7
15. Hussain, N, Khamsi, MA: On asymptotic pointwise contractions in metric spaces. Nonlinear Anal. **71**(10), 4423–4429 (2009). doi:10.1016/j.na.2009.02.126
16. Khamsi, MA, Kozlowski, WK: On asymptotic pointwise contractions in modular function spaces. Nonlinear Anal. **73**, 2957–2967 (2010). doi:10.1016/j.na.2010.06.061
17. Kirk, WA, Xu, HK: Asymptotic pointwise contractions. Nonlinear Anal. **69**, 4706–4712 (2008). doi:10.1016/j.na.2007.11.023
18. Khamsi, MA, Kozlowski, WK: On asymptotic pointwise nonexpansive mappings in modular function spaces. J Math Anal Appl. **380**, 697–708 (2011). doi:10.1016/j.jmaa.2011.03.031
19. De Marr, R: Common fixed points for commuting contraction mappings. Pacific J Math. **13**, 1139–1141 (1963)
20. Kozlowski, WM: Modular function spaces. In Series of Monographs and Textbooks in Pure and Applied Mathematics, vol, vol. 122,Dekker, New York, Basel (1988)
21. Kozlowski, WM: Notes on modular function spaces I. Comment Math. **28**, 91–104 (1988)
22. Kozlowski, WM: Notes on modular function spaces II. Comment Math. **28**, 105–120 (1988)
23. Musielak, J: Orlicz spaces and modular spaces. In Lecture Notes in Mathematics, vol. 1034,Springer, Berlin (1983)
24. Chen, J, Li, Z: Banach operator pair and common fixed points for nonexpansive maps. Nonlinear Anal. **74**, 3086–3090 (2011). doi:10.1016/j.na.2010.12.010
25. Pathak, HK, Hussain, N: Common fixed points for Banach operator pairs with applications. Nonlinear Anal. **69**, 2788–2802 (2008). doi:10.1016/j.na.2007.08.051
26. Khamsi, MA, Latif, A, Al-Sulami, H: KKM and Ky Fan theorems in modular function spaces. Fixed Point Theory Appl. **2011**, 57 (2011). doi:10.1186/1687-1812-2011-57
27. Huneke, JP: On common fixed points of commuting continuous functions on an interval. Trans Am Math Soc. **139**, 371–381 (1969)
28. Jungck, G: Common fixed point theorems for compatible self maps of Hausdorff topolog-ical spaces. Fixed Point Theory Appl. 355–363 (2005)
29. Ciric, LjB, Hussain, N, Cakic, N: Common fixed points for Ciric type f-weak contraction with applications. Publ Math Debrecen. **76**(1-2), 31–49 (2010)
30. Ciric, LjB, Hussain, N, Akbar, F, Ume, JS: Common fixed points for Banach operator pairs from the set of best approximations. Bull Belgian Math Soc Simon Stevin. **16**, 319–336 (2009)
31. Hussain, N: Common fixed points in best approximation for Banach operator pairs with Ciric type I-contractions. J Math Anal Appl. **338**, 1351–1362 (2008). doi:10.1016/j.jmaa.2007.06.008
32. Hussain, N, Khamsi, MA, Latif, A: Banach operator pairs and common fixed points in hyperconvex metric spaces. Nonlinear Anal. **743**, 5956–5961 (2011)
33. Khan, AR, Akbar, F: Best simultaneous approximations, asymptotically nonexpansive mappings and variational inequalities in Banach spaces. J Math Anal Appl. **354**, 469–477 (2009). doi:10.1016/j.jmaa.2009.01.007
34. Khan, AR, Akbar, F: Common fixed points from best simultaneous approximations. Taiwan J Math. **13**, 1379–1386 (2009)

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