Homaeipour and Razani *Fixed Point Theory and Applications* 2011, **2011**:73
http://www.fixedpointtheoryandapplications.com/content/2011/1/73

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Weak and strong convergence theorems for relatively nonexpansive multi-valued mappings in Banach spaces

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Abstract

In this paper, an iterative sequence for relatively nonexpansive multi-valued mappings by using the notion of generalized projection is introduced, and then weak and strong convergence theorems are proved.

2000 Mathematics Subject Classification: 47H09; 47H10; 47J25.

Keywords: multi-valued mapping, relatively nonexpansive, fixed point, iterative sequence

1 Introduction and preliminaries

Let D be a nonempty closed convex subset of a real Banach space X. A single-valued mapping $T : D \to D$ is called nonexpansive if $||T(x) - T(y)|| \leq ||x - y||$ for all $x, y \in D$. Let $N(D)$ and $CB(D)$ denote the family of nonempty subsets and nonempty closed bounded subsets of D, respectively. The Hausdorff metric on $CB(D)$ is defined by

$$H(A_1, A_2) = \max\left\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\right\},$$

for $A_1, A_2 \in CB(D)$, where $d(x, A_1) = \inf\{||x - y||; y \in A_1\}$. The multi-valued mapping $T : D \to CB(D)$ is called nonexpansive if $H(T(x), T(y)) \leq ||x - y||$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to N(D)$ (respectively, $T : D \to D$) if $p \in F(T)$ (respectively, $T(p) = p$). The set of fixed points of T is represented by $F(T)$.

Let X be a real Banach space with dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) := \{f^* \in X^* : \langle x, f^*\rangle = \| x\|^2 = \| f^*\|^2\},$$

where $\langle .,.\rangle$ denotes the generalized duality pairing.

The Banach space X is strictly convex if $||(x + y)/2|| < 1$ for all $x, y \in X$ with $||x|| = ||y|| = 1$ and $x \neq y$. The Banach space X is uniformly convex if $\lim_{n\to\infty} ||x_n - y_n|| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subseteq X$ with $||x_n|| = ||y_n|| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} ||(x_n + y_n)/2|| = 1$.

Lemma 1.1. [1]*Let X be a uniformly convex Banach space and $B_r = \{x \in X : ||x|| \leq r\}$, $r > 0$. Then, there exists a continuous, strictly increasing, and convex function g :*

$[0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\| \alpha x + \beta y\|^2 \leq \alpha \| x\|^2 + \beta \| y\|^2 - \alpha\beta g(\| x - y \|),$$

for all $x, y \in B_r$ *and all* $\alpha, \beta \in [0, 1]$ *with* $\alpha + \beta = 1$.

The norm of Banach space X is said to be Gâteaux differentiable if for each $x, y \in S(X) := \{x \in X : ||x|| = 1\}$ the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}, \tag{1.1}$$

exists. In this case, X is called smooth. The norm of Banach space X is said to be Fréchet differentiable if for each $x \in S(X)$, limit (1.1) is attained uniformly for $y \in S(X)$ and the norm is uniformly Fréchet differentiable if limit (1.1) is attained uniformly for $x, y \in S(X)$. In this case, X is said to be uniformly smooth. The following properties of J are well known [2]:

1. X (X^*, resp.) is uniformly convex if and only if X^* (X, resp.) is uniformly smooth;
2. If X is smooth, then J is single-valued and norm-to-weak* continuous;
3. If X is reflexive, then J is onto;
4. If X is strictly convex, then $J(x) \cap J(y) = \emptyset$ for all $x \neq y$;
5. If X has a Fréchet differentiable norm, then J is norm-to-norm continuous;
6. If X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of X.

The normalized duality mapping J of a smooth Banach space X is called weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $J(x_n) \overset{*}{\rightharpoonup} J(x)$, where $\rightharpoonup$ denotes the weak convergence and $\overset{*}{\rightharpoonup}$ denotes the weak* convergence.

Let X be a smooth Banach space. The function $\varphi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \| x\|^2 - 2\langle x, J(y)\rangle + \| y\|^2, \quad \forall x, y \in X.$$

It is obvious from the definition of the function φ that

$$(\| x \| - \| y \|)^2 \leq \phi(x, y) \leq (\| x \| + \| y \|)^2, \quad \forall x, y \in X. \tag{1.2}$$

In addition, the function φ has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, J(x) - J(z)\rangle, \quad \forall x, y, z \in X. \tag{1.3}$$

Lemma 1.2. *[3, Remark 2.1] Let X be a strictly convex and smooth Banach space, then* $\varphi(x, y) = 0$ *if and only if* $x = y$.

Lemma 1.3. [4]*Let X be a uniformly convex and smooth Banach space and* $r > 0$. *Then*

$$g(\| y - z \|) \leq \phi(y, z),$$

for all $y, z \in B_r = \{x \in X; ||x|| \leq r\}$, *where* $g : [0, \infty) \to [0, \infty)$ *is a continuous, strictly increasing and convex function with* $g(0) = 0$.

Let D be a nonempty closed convex subset of a smooth Banach space X. A point $p \in D$ is called an asymptotic fixed point of $T : D \to D$ [5], if there exists a sequence $\{x_n\}$ in D which converges weakly to p and $\lim_{n\to\infty} ||x_n - T(x_n)|| = 0$. The set of asymptotic fixed points of T is represented by $\hat{F}(T)$. A mapping $T : D \to D$ is called relatively nonexpansive [3,6-8], if the following conditions are satisfied:

1. $F(T)$ is nonempty;
2. $\varphi(p, T(x)) \leq \varphi(p, x)$, $\forall x \in D$, $p \in F(T)$;
3. $\hat{F}(T) = F(T)$.

Let D be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space X. It is known that [4,9] for any $x \in X$, there exists a unique point $x_0 \in D$ such that

$$\phi(x_0, x) = \min_{y \in D} \phi(y, x).$$

Following Alber [9], we denote such an element x_0 by $\Pi_D x$. The mapping Π_D is called the generalized projection from X onto D. If X is a Hilbert space, then $\varphi(y, x) = ||y - x||^2$ and Π_D is the metric projection of X onto D.

Lemma 1.4. [4,9]*Let D be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space X. Then*

$$\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \leq \phi(x, y), \quad \forall x \in D, y \in X.$$

Lemma 1.5. [4,9]*Let D be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space X. Let $x \in X$ and $z \in D$, then*

$$z = \Pi_D x \quad \Leftrightarrow \quad \langle z - y, J(x) - J(z) \rangle \geq 0, \quad \forall y \in D.$$

In 2004, Matsushita and Takahashi [10] introduced the following iterative sequence for finding a fixed point of relatively nonexpansive mapping $T : D \to D$. Given $x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) J(T(x_n))), \tag{1.4}$$

where D is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X, Π_D is the generalized projection onto D and $\{\alpha_n\}$ is a sequence in [0, 1].

They proved weak and strong convergence theorems in uniformly convex and uniformly smooth Banach space X.

Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by some authors, see for instance [11-14].

Let D be a nonempty closed convex subset of a smooth Banach space X. We define an asymptotic fixed point for a multi-valued mapping as follows.

Definition 1.6. *A point $p \in D$ is called an asymptotic fixed point of $T : D \to N(D)$, if there exists a sequence $\{x_n\}$ in D which converges weakly to p and* $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.

Moreover, we define a relatively nonexpansive multi-valued mapping as follows.

Definition 1.7. *A multi-valued mapping $T : D \to N(D)$ is called relatively nonexpansive, if the following conditions are satisfied:*

1. *$F(T)$ is nonempty;*
2. *$\varphi(p, z) \leq \varphi(p, x)$, $\forall x \in D$, $z \in T(x)$, $p \in F(T)$;*
3. *$\hat{F}(T) = F(T)$,*

where $\hat{F}(T)$ *is the set of asymptotic fixed points of T.*

There exist relatively nonexpansive multi-valued mappings that are not nonexpansive.

Example 1.8. Let $I = [0,1]$, $X = L^p(I)$, $1 < p < \infty$ and $D = \{f \in X; f(x) \geq 0, \forall x \in I\}$. Let $T : D \to CB(D)$ be defined by

$$T(f) = \begin{cases} \{g \in D; f(x) - \frac{3}{4} \leq g(x) \leq f(x) - \frac{1}{4}, \forall x \in I\}, & f(x) > 1, \forall x \in I; \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is clear that $F(T) = \{0\}$. Let $h \in \hat{F}(T)$. Then, there exists a sequence $\{f_n\}$ in D which converges weakly to h, and $z_n = d(f_n, T(f_n)) \to 0$. Let $n \in \mathbb{N}$, we have

$$z_n = \begin{cases} \dfrac{1}{4}, & f_n(x) > 1, \forall x \in I; \\ \|f_n\|_p, & \text{otherwise.} \end{cases}$$

Since $z_n \to 0$, we have $||f_n||_p \to 0$. Therefore, $f_n \to 0$. Hence, $h = 0$. Therefore, $\hat{F}(T) = F(T) = \{0\}$. Let $f \in D$ such that $f(x) > 1$ for all $x \in I$, and $g \in T(f)$, then

$$\begin{aligned} \phi(0, g) &= \| g \|_p^2 \\ &\leq \| f \|_p^2 \\ &= \phi(0, f). \end{aligned}$$

Next, let $f \in D$ such that there exists $x \in I$ such that $f(x) \leq 1$, then

$$\begin{aligned} \phi(0, 0) &= 0 \\ &\leq \| f \|_p^2 \\ &= \phi(0, f). \end{aligned}$$

Hence, T is relatively nonexpansive. However, if $f(x) = 2$ and $g(x) = 1$ for all $x \in I$, we get $H(T(f), T(g)) = \frac{7}{4}$. Then, $H(T(f), T(g)) > ||f - g||_p = 1$. Hence, T is not nonexpansive.

In this article, inspired by Matsushita and Takahashi [10], we introduce the following iterative sequence for finding a fixed point of relatively nonexpansive multi-valued mapping $T : D \to N(D)$. Given $x_1 \in D$,

$$x_{n+1} = \Pi_D J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) J(z_n)), \tag{1.5}$$

where $z_n \in T(x_n)$ for all $n \in \mathbb{N}$, D is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X, Π_D is the generalized projection onto D and $\{\alpha_n\}$ is a sequence in $[0, 1]$. We prove weak and strong convergence theorems in uniformly convex and uniformly smooth Banach space X.

2 Main results

In this section, at first, concerning the fixed point set of a relatively nonexpansive multi-valued mapping, we prove the following proposition.

Proposition 2.1. *Let X be a strictly convex and smooth Banach space, and D a nonempty closed convex subset of X. Suppose* $T : D \to N(D)$ *is a relatively nonexpansive multi-valued mapping. Then, F(T) is closed and convex.*

Proof. First, we show $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x^*$. Since T is relatively nonexpansive, we have

$$\phi(x_n, z) \leq \phi(x_n, x^*),$$

for all $z \in T(x^*)$ and for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\phi(x^*, z) &= \lim_{n\to\infty}\phi(x_n, z)\\ &\leq \lim_{n\to\infty}\phi(x_n, x^*)\\ &= \phi(x^*, x^*)\\ &= 0.\end{aligned} \tag{2.1}$$

By Lemma 1.2, we obtain $x^* = z$. Hence, $T(x^*) = \{x^*\}$. So, we have $x^* \in F(T)$. Next, we show $F(T)$ is convex. Let $x, y \in F(T)$ and $t \in (0, 1)$, put $p = tx + (1 - t)y$. We show $p \in F(T)$. Let $w \in T(p)$, we have

$$\begin{aligned}\phi(p, w) &= \| p\|^2 - 2\langle p, J(w)\rangle + \| w\|^2\\ &= \| p\|^2 - 2\langle tx + (1-t)y, J(w)\rangle + \| w\|^2\\ &= \| p\|^2 - 2t\langle x, J(w)\rangle - 2(1-t)\langle y, J(w)\rangle + \| w\|^2\\ &= \| p\|^2 + t\phi(x, w) + (1-t)\phi(y, w) - t \| x\|^2 - (1-t) \| y\|^2\\ &\leq \| p\|^2 + t\phi(x, p) + (1-t)\phi(y, p) - t \| x\|^2 - (1-t) \| y\|^2\\ &= \| p\|^2 - 2\langle tx + (1-t)y, J(p)\rangle + \| p\|^2\\ &= \| p\|^2 - 2\langle p, J(p)\rangle + \| p\|^2\\ &= 0.\end{aligned} \tag{2.2}$$

By Lemma 1.2, we obtain $p = w$. Hence, $T(p) = \{p\}$. So, we have $p \in F(T)$. Therefore, $F(T)$ is convex. □

Remark 2.2. Let X be a strictly convex and smooth Banach space, and D a nonempty closed convex subset of X. Suppose $T : D \to N(D)$ is a relatively nonexpansive multivalued mapping. If $p \in F(T)$, then $T(p) = \{p\}$.

Proposition 2.3. *Let X be a uniformly convex and smooth Banach space, and D a nonempty closed convex subset of X. Suppose $T : D \to N(D)$ is a relatively nonexpansive multivalued mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ for all $n \in \mathbb{N}$. For a given $x_1 \in D$, let $\{x_n\}$ be the iterative sequence defined by (1.5). Then, $\{\Pi_{F(T)}x_n\}$ converges strongly to a fixed point of T, where $\Pi_{F(T)}$ is the generalized projection from D onto $F(T)$.*

Proof. By Proposition 2.1, $F(T)$ is closed and convex. So, we can define the generalized projection $\Pi_{F(T)}$ onto $F(T)$. Let $p \in F(T)$. From Lemma 1.4, we have

$$\begin{aligned}\phi(p, x_{n+1}) &= \phi(p, \Pi_D J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(z_n)))\\ &\leq \phi(p, J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(z_n)))\\ &= \| p\|^2 - 2\langle p, \alpha_n J(x_n) + (1-\alpha_n)J(z_n)\rangle\\ &\quad + \| \alpha_n J(x_n) + (1-\alpha_n)J(z_n)\|^2\\ &\leq \| p\|^2 - 2\alpha_n\langle p, J(x_n)\rangle - 2(1-\alpha_n)\langle p, J(z_n)\rangle + \alpha_n \| x_n\|^2\\ &\quad + (1-\alpha_n) \| z_n\|^2\\ &= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n)\\ &\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n)\\ &= \phi(p, x_n).\end{aligned} \tag{2.3}$$

Hence, $\lim_{n\to\infty} \varphi(p, x_n)$ exists. So, $\{\varphi(p, x_n)\}$ is bounded. Then, by (1.2) we have $\{x_n\}$ is bounded, and hence, $\{z_n\}$ is bounded. Let $u_n = \Pi_{F(T)}x_n$, for all $n \in \mathbb{N}$. Then, we have

$$\phi(u_n, x_{n+1}) \le \phi(u_n, x_n). \tag{2.4}$$

Therefore

$$\phi(u_n, x_{n+m}) \le \phi(u_n, x_n), \tag{2.5}$$

for all $m \in \mathbb{N}$. From Lemma 1.4, we obtain

$$\begin{aligned} \phi(u_{n+1}, x_{n+1}) &= \phi(\Pi_{F(T)}x_{n+1}, x_{n+1}) \\ &\le \phi(u_n, x_{n+1}) - \phi(u_n, \Pi_{F(T)}x_{n+1}). \end{aligned} \tag{2.6}$$

By (2.4) and (2.6) we have

$$\phi(u_{n+1}, x_{n+1}) \le \phi(u_n, x_n). \tag{2.7}$$

It follows that $\{\varphi(u_n, x_n)\}$ converges. From $u_{n+m} = \Pi_{F(T)}x_{n+m}$ and Lemma 1.4, we have

$$\phi(u_n, u_{n+m}) + \phi(u_{n+m}, x_{n+m}) \le \phi(u_n, x_{n+m}).$$

Hence, by (2.5) we obtain

$$\phi(u_n, u_{n+m}) \le \phi(u_n, x_n) - \phi(u_{n+m}, x_{n+m}), \tag{2.8}$$

for all $m, n \in \mathbb{N}$. Let $r = \sup_{n\in\mathbb{N}} ||u_n||$. From Lemma 1.3, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} g(\| u_m - u_n \|) &\le \phi(u_m, u_n) \\ &\le \phi(u_m, x_m) - \phi(u_n, x_n), \end{aligned} \tag{2.9}$$

for all $m, n \in \mathbb{N}$, $n > m$. Therefore, $\{u_n\}$ is a Cauchy sequence. Since X is complete and $F(T)$ is closed, there exists $q \in F(T)$ such that $\{u_n\}$ converges strongly to q. □

If the duality mapping J is weakly sequentially continuous, we have the following weak convergence theorem.

Theorem 2.4. *Let X be a uniformly convex and uniformly smooth Banach space, and D a nonempty closed convex subset of X. Suppose $T : D \to N(D)$ is a relatively nonexpansive multi-valued mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that* $0 \le \alpha_n \le 1$ *for all $n \in \mathbb{N}$ and* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. *For a given $x_1 \in D$, let $\{x_n\}$ be the iterative sequence defined by (1.5). If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. As in the proof of Proposition 2.3, $\{x_n\}$ and $\{z_n\}$ are bounded. So, there exists $r > 0$ such that $x_n, z_n \in B_r$ for all $n \in \mathbb{N}$. Since X is a uniformly smooth Banach space, X^* is a uniformly convex Banach space. Let $p \in F(T)$. By Lemma 1.1, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, \Pi_D J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(z_n))) \\ &\le \phi(p, J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(z_n))) \\ &= \| p\|^2 - 2\langle p, \alpha_n J(x_n) + (1-\alpha_n)J(z_n)\rangle \\ &\quad + \| \alpha_n J(x_n) + (1-\alpha_n)J(z_n)\|^2 \\ &\le \| p\|^2 - 2\alpha_n\langle p, J(x_n)\rangle - 2(1-\alpha_n)\langle p, J(z_n)\rangle + \alpha_n \| x_n\|^2 \\ &\quad + (1-\alpha_n) \| z_n\|^2 - \alpha_n(1-\alpha_n)g(\| J(x_n) - J(z_n) \|) \\ &= \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n) - \alpha_n(1-\alpha_n)g(\| J(x_n) - J(z_n) \|) \\ &\le \phi(p, x_n) - \alpha_n(1-\alpha_n)g(\| J(x_n) - J(z_n) \|). \end{aligned} \tag{2.10}$$

Hence

$$\alpha_n(1-\alpha_n)g(\| J(x_n)-J(z_n)\|)\le \phi(p,x_n)-\phi(p,x_{n+1}).$$

Since $\lim_{n\to\infty} \varphi(p, x_n)$ exists and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, we obtain

$$\lim_{n\to\infty} g(\| J(x_n)-J(z_n)\|)=0.$$

Therefore,

$$\lim_{n\to\infty} \| J(x_n)-J(z_n)\|=0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \| x_n-z_n\|=0.$$

Since $d(x_n, T(x_n)) \le ||x_n - z_n||$, we obtain

$$\lim_{n\to\infty} d(x_n,T(x_n))=0. \tag{2.11}$$

Let $u_n = \Pi_{F(T)} x_n$. By Lemma 1.5, we have

$$\langle u_n-w,J(x_n)-J(u_n)\rangle\ge 0, \tag{2.12}$$

for each $w \in F(T)$. From Proposition 2.3, there exists $p \in F(T)$ such that $\{u_n\}$ converges strongly to p. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to q. Then, by (2.11) we have $q \in F(T)$. It follows from (2.12) that

$$\langle u_{n_j}-q,J(x_{n_j})-J(u_{n_j})\rangle\ge 0. \tag{2.13}$$

Let $j \to \infty$ in inequality (2.13), since J is weakly sequentially continuous we have

$$\langle p-q,J(q)-J(p)\rangle\ge 0. \tag{2.14}$$

Since J is monotone, we have

$$\langle q-p,J(q)-J(p)\rangle\ge 0. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\langle q-p,J(q)-J(p)\rangle= 0. \tag{2.16}$$

Since X is strictly convex, we have $p = q$. Therefore, $\{x_n\}$ converges weakly to p. The proof is complete. □

Theorem 2.5. *Let X be a uniformly convex and uniformly smooth Banach space, and D a nonempty closed convex subset of X. Suppose $T : D \to N(D)$ is a relatively nonexpansive multi-valued mapping. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ for all $n \in \mathbb{N}$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. For a given $x_1 \in D$, let $\{x_n\}$ be the iterative sequence defined by (1.5). If the interior of $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since the interior of $F(T)$ is nonempty, there exists $p \in F(T)$ and $r > 0$ such that $p + rh \in F(T)$, whenever $||h|| \le 1$. By (1.3) for any $q \in F(T)$ we have

$$\phi(q,x_n)=\phi(x_{n+1},x_n)+\phi(q,x_{n+1})+2\langle x_{n+1}-q,J(x_n)-J(x_{n+1})\rangle. \tag{2.17}$$

Therefore,

$$\frac{1}{2}(\phi(q,x_n)-\phi(q,x_{n+1}))=\frac{1}{2}\phi(x_{n+1},x_n)+\langle x_{n+1}-q,J(x_n)-J(x_{n+1})\rangle. \tag{2.18}$$

Since $p + rh \in F(T)$, as in the proof of Proposition 2.3, we have

$$\phi(p+rh,x_{n+1})\leq\phi(p+rh,x_n). \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$\frac{1}{2}\phi(x_{n+1},x_n)+\langle x_{n+1}-(p+rh),J(x_n)-J(x_{n+1})\rangle\geq 0. \tag{2.20}$$

Then, by (2.18) and (2.20) we have

$$\begin{aligned}\langle h,J(x_n)-J(x_{n+1})\rangle &\leq \tfrac{1}{r}(\langle x_{n+1}-p,J(x_n)-J(x_{n+1})\rangle+\tfrac{1}{2}\phi(x_{n+1},x_n))\\ &= \tfrac{1}{2r}(\phi(p,x_n)-\phi(p,x_{n+1})),\end{aligned} \tag{2.21}$$

whenever $||h|| \leq 1$. Therefore, we obtain

$$\| J(x_n)-J(x_{n+1})\| \leq \frac{1}{2r}(\phi(p,x_n)-\phi(p,x_{n+1})).$$

It follows that

$$\begin{aligned}\| J(x_m)-J(x_n)\| &\leq \Sigma_{i=m}^{n-1}\| J(x_i)-J(x_{i+1})\|\\ &\leq \Sigma_{i=m}^{n-1}\tfrac{1}{2r}(\phi(p,x_i)-\phi(p,x_{i+1}))\\ &= \tfrac{1}{2r}(\phi(p,x_m)-\phi(p,x_n)),\end{aligned} \tag{2.22}$$

for all $m, n \in \mathbb{N}$, $n > m$. As in the proof of Proposition 2.3, $\{\varphi(p, x_n)\}$ converges. Hence, $\{J(x_n)\}$ is a Cauchy sequence. Since X^* is complete, $\{J(x_n)\}$ converges strongly to a point in X^*. Since X^* has a Fréchet differentiable norm, then J^{-1} is norm-to-norm continuous on X^*. Hence, $\{x_n\}$ converges strongly to some point u in D. As in the proof of Theorem 2.4, $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Hence, we have $u \in F(T)$, where $u = \lim_{n\to\infty} \Pi_{F(T)}x_n$. □

Acknowledgements
AR would like to thank the School of Mathematics of the Institute for Research in Fundamental Sciences, Tehran, Iran, for supporting this research (Grant No. 90470122).

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Authors' contributions
Both authors contributed to this work equally. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 2 June 2011 Accepted: 31 October 2011 Published: 31 October 2011

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doi:10.1186/1687-1812-2011-73
Cite this article as: Homaeipour and Razani: **Weak and strong convergence theorems for relatively nonexpansive multi-valued mappings in Banach spaces.** *Fixed Point Theory and Applications* 2011 **2011**:73.