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Semistability of iterations in cone spaces

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Abstract

The aim of this work is to prove some iteration procedures in cone metric spaces. This extends some recent results of T-stability.

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1. Introduction

Let E be a real Banach space. A subset $P \subset E$ is called a cone in E if it satisfies in the following conditions:

(i) P is closed, nonempty and $P \neq \{0\}$.

(ii) $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$, by defining; $x \leq y$ if and only if $y - x \in P$, Also, we write $x \ll y$ if $y - x \in \operatorname{int} P$, where int P denotes the interior of P. A cone P is called normal if there exists a constant $k > 1$ such that $0 \leq x \leq y$ implies $||x|| \leq k||y||$.

In the following we suppose that E is a real Banach space, P is a cone in E and $\leq$ is a partial ordering with respect to P.

Definition 1.1. ([1]) Let X be a nonempty set. Assume that the mapping $d: X \times X \to E$ satisfies in the following conditions:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

If T is a self-map of X, then by $F(T)$ we mean the set of fixed points of T. Also, $\mathbf{N_0}$ denotes the set of nonnegative integers, i.e., $\mathbf{N_0} = \mathbf{N} \cup \{\mathbf{0}\}$.

Definition 1.2. ([2]) If $0 < \alpha < 1$, $0 < \beta, \ \gamma < \frac{1}{2}$ we say that a map $T: X \to X$ is Zamfirescu with respect to (α, β, γ), if for each pair $x, y \in X$, T satisfies at least one of the following conditions:

$Z(1)$. $d(Tx, Ty) \leq \alpha d(x, y)$,

$Z(2)$. $d(Tx, Ty) \leq \beta(d(x, Tx) + d(y, Ty))$,

$Z(3)$. $d(Tx, Ty) \leq \gamma(d(x, Ty) + d(y, Tx))$.

Usually for simplicity, T is called a Zamfirescu operator if T is Zamfirescu with respect to some (α, β, γ), for some scalars α, β, γ with above restrictions. Also, T is called a f-Zamfirescu operator if the relations $Z(1)$, $Z(2)$ and $Z(3)$ hold for all $x \in X$ and all $y \in F(T)$.

Definition 1.3. ([3]) Let (X, d) be a cone metric space. A map $T: X \to X$ is called a quasi-contraction if for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists $u \in C(T; x, y) \equiv \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}$ such that $d(Tx, Ty) \leq \lambda u$. If this inequality holds for all $x \in X$ and $y \in F(T)$, we say that T is a f-quasi-contraction.

Lemma 1.4. ([4]) If T is a quasi-contraction with $0 < \lambda < \frac{1}{2}$, then T is a Zamfirescu operator.

Lemma 1.5. ([4]) Let P be a normal cone, and let $\{a_n\}$ and $\{b_n\}$ be sequences in E satisfying the inequality $a_{n+1} \leq ha_n + b_n$, where $h \in (0, 1)$ and $b_n \to 0$ as $n \to \infty$. Then $\lim_n a_n = 0$.

Definition 1.6. A self-map T of a metric space (X, d) is called nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.

Definition 1.7. A self-map T of (X, d) is called affine if $T(\alpha x + (1- \alpha)y) = \alpha Tx + (1 -\alpha)Ty$ for all $x, y \in X$, and $\alpha \in [0, 1]$.

Definition 1.8. A self-map T of (X, d) is called semi-compact if the convergence $||x_n - Tx_n|| \to 0$ implies that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^* \in X$ such that $x_{n_k} \to x^*$.

2. Main results

In this section we want to prove some iteration procedures in cone spaces. This extends some recent results of T-stability ([4]). Khamsi [5] has shown that any normal cone metric space can have a metric type defined on it. Consequently, our results are consistent for any metric spaces. Let (X, d) be a cone metric space and $\{T_n\}_n$ be a sequence of self-maps of x with $\cap_n F(T_n) \neq \emptyset$. Let x_0 be a point of X, and assume that $x_{n+1} = f(T_n, x_n)$ is an iteration procedure involving $\{T_n\}_n$, which yields a sequence $\{x_n\}$ of points from X.

Definition 2.1. The iteration $x_{n+1} = f(T_n, x_n)$ is said $\{T_n\}$-semistable (or semistable with respect to $\{T_n\}$) if $\{x_n\}$ converges to a fixed point q in $\cap_n F(T_n)$, and whenever $\{y_n\}$ is a sequence in X with $\lim_n d(y_n, f(T_n, y_n)) = 0$, and $d(y_n, f(T_n, y_n)) = o(t_n)$ for some sequence $\{t_n\} \subset \mathbf{R}^+$, then we have $y_n \to q$.

In practice, such a sequence $\{y_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T_n, x_n)$. Let $y_0 = x_0$. Now $x_1 = f(T_0, x_0)$. Because of rounding or discretization in the function T_0, a new value y_1 approximately equal to x_1 might be obtained instead of the true value of $f(T_0, x_0)$. Then to approximate y_2, the value $f(T_1, y_1)$ is computed to yield y_2, approximation of $f(T_1, y_1)$. This computation is continued to obtain $\{y_n\}$ as an approximate sequence of $\{x_n\}$.

In the following we extend the definition of stability from a single self-map (see [6]) to a sequence of single-maps.

Definition 2.2. The iteration $x_{n+1} = f(T_n, x_n)$ is said $\{T_n\}$-stable (or stable with respect to $\{T_n\}_{n \in N_0}$) if $\{x_n\}$ converges to a fixed point q in $\cap_n F(T_n)$, and whenever $\{y_n\}$ is a sequence in X with $\lim_n d(y_{n+1}, f(T_n, y_n)) = 0$, we have $y_n \to q$.

Note that if $T_n = T$ for all n, then Definition 2.2. gives the definition of T-stability ([6]).

Definition 2.3. For a sequence of self-maps $\{T_n\}_{n\in N_0}$, the iteration $x_{n+1} = T_n x_n$ is called the Picard's S-iteration.

The stability of some iterations have been studied in metric spaces in [7,8]. Here we want to investigate the semistability and stability of Picard's S-iteration.

Theorem 2.4. Let (X, d) be a cone metric space, P a normal cone and $\{T_n\}_n = \mathbf{N_0}$ be a sequence of self-maps of X with $\cap_n F(T_n) \neq \emptyset$. Suppose that there exist nonnegative bounded sequences $\{a_n\}$, $\{b_n\}$ with $\sup_n b_n < 1$, such that

$$d(T_n x, q) \leq a_n \; d(x, T_n x) + b_n \; d(x, q) \qquad (*)$$

for each $n \in \mathbf{N_0}$, $\mathbf{x} \in \mathbf{X}$ and $q \in \cap_n F(T_n)$. Then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.

Proof. First we note that relation (*) implies that $\cap_n F(T_n)$ is a singleton. Indeed, if p and q belong to $\cap_n F(T_n)$, then by (*) we get

$$d(p, q) = d(T_n p, q) \leq a_n d(p, T_n p) + b_n d(p, q) \leq \alpha d(p, q),$$

where $\alpha = \sup_n b_n$. This implies that $p = q$. So let $\cap_n F(T_n) = \{q_0\}$ and $\{y_n\} \subset X$ be such that $\lim_n d(y_{n+1}, T_n y_n) = \lim_n d(T_n y_n, y_n) = 0$. Now we show that $y_n \to q_0$. For this by using the relation (*) we have:

$$\begin{aligned} d(y_{n+1}, q_0) &\leq d(y_{n+1}, T_n y_n) + d(T_n y_n, q_0) \\ &\leq d(y_{n+1}, T_n y_n) + a_n d(T_n y_n, y_n) + b_n d(y_n, q_0) \\ &= c_n + \alpha d(y_n, q_0), \end{aligned}$$

where $c_n = d(y_{n+1}, T_n y_n) + a_n \; d(T_n y_n, y_n)$ tends to 0 as $n \to \infty$, and $0 \leq \alpha < 1$. Now by Lemma 1.5, $y_n \to q_0$ and so the Picard's S-iteration is $\{T_n\}_n$-semistable. This completes the proof.□

Corollary 2.5. Let (X, d) be a cone metric space, P a normal cone and $\{T_n\}_{n\in N_0}$ be a sequence of self-maps of X with $\cap_n F(T_n) \neq \emptyset$. If there exists a nonnegative sequence $\{\lambda_n\}$ with $\sup_n \lambda_n < 1$ such that $d(T_n x, T_n y) \leq \lambda_n \; d(x, y)$ for each $x, y \in X$ and $n \in \mathbf{N_0}$, then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.

Corollary 2.6. Let (X, d) be a cone metric space, P a normal cone and $\{T_n\}_{n\in N_0}$ be a sequence of self-maps of X with $\cap_n F(T_n) \neq \emptyset$. If for all $n \in \mathbf{N_0}$, T_n is a f-Zamfirescu operator with respect to $(\alpha_n, \beta_n, \gamma_n)$ with $\sup_n \gamma_n < 1/2$, then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.

Proof. It is sufficient to show that condition (*) in Theorem 2.4 is consistent. Clearly the conditions $Z(1)$ and $Z(2)$ imply that (*) holds. Also, note that by using condition Z (3) for T_n we have:

$$d(T_n x, q) \leq \gamma_n (d(q, T_n x) + d(x, q)),$$

where $q \in \cap_n F(T_n)$. Thus we get

$$d(T_n x, q) \leq \gamma_n d(x, T_n x) + 2\gamma_n d(x, q).$$

Since $\sup_n \gamma_n < 1/2$, so clearly (*) holds.□

Corollary 2.7. Under the conditions of Corollary 2.6 if T_n is a Zamfirescu operator for all n, then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.

Corollary 2.8. Let (X, d) be a cone metric space, P a normal cone and $\{T_n\}_{n\in N_0}$ be a sequence of self-maps of X with $\cap_n F(T_n) \neq \emptyset$. If for all $n \in \mathbf{N_0}$, T_n is a f-quasi-contraction with λ_n such that $\sup_n \lambda_n < 1$, then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.

Proof. It is sufficient to show that condition (*) holds. For every $x \in X$ and $q \in \cap_n F(T_n)$ we have $d(T_n x, q) \leq \gamma_n u_n$ for some $u_n \in C(T_n; x, q)$. Hence

$$d(T_n x, q) \leq t_n d(x, T_n x) + s_n d(x, q),$$

where $s_n, t_n \in \{0, \lambda_n\}$. This completes the proof. □

Theorem 2.9. Under the conditions of Theorem 2.4, suppose that there exists a sequence of nonnegative scalars $\{\lambda_n\}_{n \in \mathbf{N_0}}$ with $\sup_n \lambda_n < 1/2$, such that for all $x, y \in X$, $n \geq 1$ we have $d(T_n x, T_{n-1} y) \leq \lambda_n u_n$ where $u_n = d(T_n x, y)$ or $u_n = d(T_{n-1} y, y)$. Then the Picard's S-iteration is semistable with respect to $\{T_n\}_n$.

Proof. It is sufficient to show that $d(y_n, T_n y_n) \to 0$ whenever $d(y_{n+1}, T_n y_n) \to 0$. Put $b_n = d(y_n, T_n y_n)$ and $c_n = d(y_n, T_{n-1} y_{n-1})$. We have

$$b_n \leq d(y_n, T_{n-1} y_{n-1}) + d(T_n y_n, T_{n-1} y_{n-1}) \leq c_n + s_n b_{n-1},$$

where $s_n = \lambda_n$ or $s_n = \frac{\lambda_n}{1-\lambda_n}$. Hence by Lemma 1.5, $b_n \to 0$, and so by the proof of Theorem 2.4, the proof is complete.□

Now we want to investigate the semistability in the cone normed spaces.

Definition 2.10. Let X be a vector space over the field F. Assume that the function $p: X \to E$ having the properties:

(a) $p(x) \geq 0$ for all x in X.

(b) $p(x + y) \leq p(x) + p(y)$ for all x, y in X.

(c) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in F$ and $x \in X$.

Then p is called a cone seminorm on X. A cone norm is a cone seminorm p such that

(d) $x = 0$ if $p(x) = 0$.

We will denote a cone norm by $||\cdot||_c$ and $(X, ||\cdot||_c)$ is called a cone normed space. Also, $d_c(x, y) = ||x - y||_c$ defines a cone metric on X.

Lemma 2.11. Let P be a normal cone, and the sequences $\{t_n\}$ and $(s_n\}$ be such that $0 \leq t_{n+1} \leq t_n + s_n$ for all $n \geq 1$. If $\sum_{n \in N} s_n$ converges, then $\lim_n ||t_n||$ exists.

Proof. Let $t_1 = 0$ and P be normal with constant k. Since $t_{n+1} - t_n \leq s_n$, thus $\sum_n (t_{n+1} - t_n) \leq \sum_n s_n$. Hence $||\sum_n (t_{n+1} - t_n)|| \leq k \, ||\sum_n s_n|| < \infty$. So $\lim_k \| \sum_{n=1}^{k} (t_{n+1} - t_n) \|$ exists. But $\sum_{n=1}^{k} (t_{n+1} - t_n) = t_{k+1} - t_1$. Thus indeed $\lim_n ||t_n||$ exists.□

Theorem 2.12. Let $(X, ||\cdot||_c)$ be a cone normed space with respect to a normal cone P in the real Banach space E, and $\{T_n\}_{n \in N_0}$ be a sequence of self-maps of X with $\cap_n F(T_n) \neq \emptyset$, $T_0 = I$ and $d_c(T_n x, q) \leq (1 + \alpha_n)\, d_c(x, q)$ for all $n \in \mathbf{N_0}$, $x \in X$ and $q \in \cap_n F(T_n)$ where $\sum_{n \in \mathbf{N}_0} \alpha_n < \infty$. Suppose that there exists a sequence $\{\beta_n\} \subset (0, 1]$ such that $\sum_n \frac{1-\beta_n}{n} < \infty$ and the sequence $\{x_n\}_n$ obtained by the iteration procedure $x_{n+1} = \beta_n x_n + (1 - \beta_n) S_n x_n$ be bounded where $S_n = \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$. Then $\lim d_c(x_n, q)$ exists for all $q \in \cap_n F(T_n)$. Moreover, if for all m, T_m is a continuous semi-compact mapping and $d_c(T_m x_n, x_n) \to 0$ as $n \to \infty$, then $\{x_n\}$ converges to a point in $\cap_n F(T_n)$.

Proof. Let $q \in \cap_n F(T_n)$ and put $\alpha = \sum_n \alpha_n$, $\gamma_0 = \sup d_c(x_n, q)$ and $b_n = d_c(x_n, q)$ for each n. By taking $\alpha_0 = 0$, we get

$$\begin{aligned} b_{n+1} &= d_c(x_{n+1}, q) \\ &= d_c(\beta_n x_n + (1-\beta_n)S_n x_n, q) \\ &\leq \beta_n d_c(x_n, q) + (1-\beta_n) d_c(S_n x_n, q) \\ &= \beta_n b_n + (1-\beta_n) d_c(S_n x_n, q). \end{aligned}$$

But,

$$\begin{aligned} d_c(S_n x_n, q) &= d_c\left(\frac{1}{n}(x_n + T_1 x_n + \cdots + T_{n-1}x_n), q\right) \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1} d_c(T_i x_n, q) \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1}(1+\alpha_i) d_c(x_n, q) \\ &= \frac{1}{n} b_n \sum_{i=0}^{n-1}(1+\alpha_i) \\ &= b_n + \frac{1}{n}\sum_{i=1}^{n-1} b_n \alpha_i. \end{aligned}$$

Hence we get

$$\begin{aligned} b_{n+1} &\leq \beta_n b_n + (1-\beta_n)\left(b_n + \frac{1}{n} b_n \sum_{i=1}^{n-1}\alpha_i\right) \\ &= b_n + \frac{1}{n}(1-\beta_n)\sum_{i=1}^{n-1}\alpha_i b_n \\ &\leq b_n + \frac{1}{n}(1-\beta_n)\alpha b_n \\ &\leq b_n + \frac{1}{n}(1-\beta_n)\alpha\gamma_0. \end{aligned}$$

But $\sum_n \frac{1-\beta_n}{n} < \infty$, so by lemma 2.11 we conclude that $\lim_n b_n$ exits and so the proof of the first part is complete. Now let T_m 's be continuous semi-compact and for all m, $d_c(T_m x_n, x_n) \to 0$ as $n \to \infty$. Since T_m is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in X$ such that $d_c(x_{n_k}, q) \to 0$. But T_m is continuous, thus for all $m, d_c(T_m x_{n_k}, T_m q) \to 0$ as $k \to \infty$.

Now for all m we have

$$d_c(T_m q, q) \leq d_c(T_m q, T_m x_{n_k}) + d_c(T_m x_{n_k}, q) + d_c(q, x_{n_k})$$

which tends to 0 as $k \to \infty$. Hence $T_m q = q$ for all m. So $q \in \cap_m F(T_m)$ and $d_c(x_{n_k}, q) \to 0$. Also, we saw by the first part of the proof, $\lim_n d_c(x_{n_k}, q)$ exists. This implies that $d_c(x_{n_k}, q) \to 0$ and so the proof is complete.□

Theorem 2.13. Let $(X, ||\cdot||_c)$ be a cone normed space with respect to a normal cone P in the real Banach space E, and $\{T_n\}_{n \in N_0}$ be a sequence of self-maps of X with $T_0 = I$, $\cap_n F(T_n) \neq \emptyset$, and $||T_m x - T_{m-1}x|| \leq ||T_{m-1}x - T_{m-2}x||$ for all $x \in X$, $m \geq 2$. Consider the iteration procedure $x_{n+1} = f(T_n, x_n) = \alpha_n x_n + (1 - \alpha_n)S_n x_n$ where $S_n = \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$ and $\alpha_n \in [0, 1)$. If there exist $a \geq 0$ and $b \in (0, 1)$ such that

$$d_c(f(T_n, y_n), q) \leq a\ d_c(f(T_n, x_n), y_n) + b\ d_c(y_n, q) \qquad (*)$$

for all sequences $\{y_n\}$ with $d_c(T_1y_n, y_n) = o(\frac{1}{(1-\alpha_n)(n-1)})$, and all $q \in \cap_n F(T_n)$, then the given iteration is $\{T_n\}$-semistable.

Proof. First note that the relation (*) implies that $\cap_n F(T_n)$ is a singleton. Indeed, if p and q belong to $F(T)$, then by setting $y_n = p$ in (*) for all n, we get $d_c(p, q) \leq bd_c(p, q)$. This implies that $p = q$. Now let $F(T) = \{q_0\}$ and $\{y_n\} \subset X$ be such that $\lim_n d_c(y_{n+1}, f(T_n, y_n) = \lim_n((1- \alpha_n)(n- 1))\ d_c(T_1y_n, y_n) = 0$. Now we show that $y_n \to q_0$. To see this note that by using the relation (*) we have:

$$\begin{aligned} d_c(y_{n+1}, q_0) &\leq d_c(y_{n+1}, f(T_n, y_n)) + d_c(f(T_n, y_n), q_0) \\ &\leq d_c(y_{n+1}, f(T_n, y_n)) + ad_c(f(T_n, y_n), y_n) + bd_c(y_n, q_0) \\ &= c_n + bd_c(y_n, q_0), \end{aligned}$$

where $c_n = d_c(y_{n+1}, f(T_n, y_n)) + a\ d_c(f(T_n, y_n), y_n)$. By Lemma 1.5, it suffices to show that $c_n \to 0$. For this we show that $d_c(f(T_n, y_n), y_n) \to 0$ as $n \to \infty$. We have

$$\begin{aligned} d_c(f(T_n, y_n), y_n) &= \| f(T_n, y_n) - y_n\|_c \\ &= \| \alpha_n y_n + (1-\alpha_n)S_n y_n - y_n\|_c \\ &= (1-\alpha_n) \| y_n - S_n y_n\|_c \\ &\leq \frac{1-\alpha_n}{n} \sum_{i=1}^{n-1} \| (T_i y_n - y_n)\|_c. \end{aligned}$$

But for $i \geq 1$, we have

$$\begin{aligned} \| T_i y_n - y_n\|_c &\leq d_c(T_i y_n - T_{i-1}y_n) + \cdots + d_c(T_1 y_n - y_n) \\ &\leq id_c({}_1Ty_n, y_n). \end{aligned}$$

Therefore,

$$d_c(f(T_n, y_n), y_n) \leq \frac{1-\alpha_n}{n} \sum_{i=1}^{n-1} id_c(T_1y_n, y_n) = \frac{(1-\alpha_n)(n-1)}{2} d_c(T_1y_n, y_n),$$

which tends to 0 since $d_c(T_1y_ny_n) = o(\frac{1}{(1-\alpha_n)(n-1)})$. Thus $y_n \to q_0$ and so the iteration $x_{n+1} = f(T_n, x_n)$ is $\{T_n\}$-semistable. This completes the proof.□

Corollary 2.14. Let $(X, ||\cdot||_c)$ be a cone normed space with respect to a normal cone P in the real Banach space E, and $\{T_n\}_{n\in N_0}$ be a sequence of self-maps of X with $T_0 = I$, $\cap_n F(T_n) \neq \emptyset$, and $||T_m x - T_{m-1}x|| \leq ||T_{m-1}x- T_{m-2}x||$ for all $x \in X$, $m \geq 2$. Consider the iteration procedure $x_{n+1} = S_n x_n$ where $S_n = \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$. If there exist non-negative bounded sequences $\{a_n\}$ and $\{b_n\}$ with $\sup_n b_n < 1$, such that

$$d_c(S_n y_n, q) \leq a_n d_c(S_n y_n, y_n) + b_n d_c(y_n, q)$$

for all sequences $\{y_n\}$ with $d_c(T_1y_n, y_n) = o(\frac{1}{n-1})$, and for all $q \in \cap_n F(T_n)$, then the given iteration is $\{T_n\}$-semistable.

Corollary 2.15. Let $(X, ||\cdot||_c)$ be a cone normed space with respect to a normal cone P in the real Banach space E, and $\{T_n\}_{n\in N_0}$ be a sequence of self-maps of X with $T_0 = I$,

$\cap_n F(T_n) \neq \emptyset$, and $||T_m x - T_{m-1}x|| \leq ||T_{m-1}x - T_{m-2}x||$ for all $x \in X$, $m \geq 2$. Consider the iteration procedure $x_{n+1} = S_n x_n$ where $S_n = \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$. If there exist $a \geq 0$ and $b \in (0, 1)$ such that

$$d_c(S_n y_n, q) \leq a d_c(S_n y_n, y_n) + b d_c(y_n, q)$$

for all sequences $\{y_n\}$ with $d_c(T_1 y_n, y_n) = o(\frac{1}{n-1})$, and for all $q \in \cap_n F(T_n)$, then the given iteration is $\{T_n\}$-semistable.

Theorem 2.16. Let $(X, ||\cdot||_c)$ be a cone normed space with respect to a normal cone P in the real Banach space E, and $\{T_n\}_{n\in N_0}$ be a sequence of affine self-maps of X with $T_0 = I$, $\cap_n F(T_n) \neq \emptyset$, and $d_c(T_m x - T_{m-1}y) \leq d_c(T_{m-1}x - T_{m-2}y)$ for all $x \in X$, $m \geq 2$. Consider the iteration procedure $x_{n+1} = f(T_n, x_n) = (1 - \alpha_n)x_n + \alpha_n T_n z_n$ where $z_n = (1 - \beta_n)x_n + \beta_n T_n x_n$ and $\alpha_n, \beta_n \in [0, 1]$. Suppose that there exist $a \geq 0$ and $b \in (0, 1)$ such that

$$d_c(f(T_n, y_n), q) \leq a\ d_c(f(T_n, y_n), y_n) + b\ d_c(y_n, q) \qquad (*)$$

for all sequences $\{y_n\}$ with $d_c(T_1 y_n, y_n) = o(\frac{1}{n\alpha_n})$, and all $q \in \cap_n F(T_n)$. Then the given iteration is $\{T_n\}$-semistable.

Proof. If p and q belong to $\cap_n F(T_n)$, then by setting $y_n = p$ in (*) for all n, we get $d_c(p, q) \leq b d_c(p, q)$. This implies that $p = q$. Now let $\cap_n F(T_n) = \{q_0\}$ and $\{y_n\} \subseteq X$ be such that

$$\lim_n d_c(y_{n+1}, f(T_n, y_n)) = \lim_n n\alpha_n d_c(T_1 y_n, y_n) = 0.$$

Now we show that $y_n \to q_0$. To see this note that by using the notation (*) we have:

$$\begin{aligned} d_c(y_{n+1}, q_0) &\leq d_c(y_{n+1}, f(T_n, y_n)) + d_c(f(T_n, y_n), q_0) \\ &\leq d_c(y_{n+1}, f(T_n, y_n)) + a d_c(f(T_n, y_n), y_n) + b d_c(y_n, q_0) \\ &= c_n + b d_c(y_n, q_0), \end{aligned}$$

where $c_n = d_c(y_n+_1, f(T_n, y_n)) + a\ d_c(f(T_n, y_n), y_n)$. By Lemma 1.5, it is sufficient to show that $c_n \to 0$. For this we show that $d_c(f(T_n, y_n), y_n) \to 0$ as $n \to \infty$. Note that

$$\begin{aligned} d_c(f(T_n, y_n), y_n) &= \| f(T_n, y_n) - y_n\|_c \\ &= \| (1-\alpha_n)y_n + \alpha_n T_n(z_n) - y_n\|_c \\ &= \alpha_n \| T_n z_n - y_n\|_c \\ &= \alpha_n \| T_n((1-\beta_n)y_n + \beta_n T_n y_n) - y_n\|_c \\ &= \alpha_n \| ((1-\beta_n)T_n y_n + \beta_n T_n^2 y_n) - y_n\|_c \\ &\leq \alpha_n(1-\beta_n)d_c(T_n y_n, y_n) + \alpha_n\beta_n d_c(T_n^2 y_n, y_n) \\ &\leq \alpha_n(1-\beta_n)[d_c(T_n y_n, T_{n-1}y_n) + \cdots + d_c(T_1 y_n, y_n)] \\ &+ \alpha_n\beta_n[d_c(T_n^2 y_n, T_{n-1}T_n y_n) + \cdots + d_c(T_1 T_n y_n, T_n y_n)] \\ &\leq \alpha_n(1-\beta_n)d_c(T_1 y_n, y_n) + n\alpha_n\beta_n d_c(T_1 T_n y_n, T_n y_n) \\ &\leq \alpha_n(1-\beta_n)d_c(T_1 y_n, y_n) + n\alpha_n\beta_n d_c(T_n T_1 y_n, T_n y_n) \\ &\leq \alpha_n(1-\beta_n)d_c(T_1 y_n, y_n) + n\alpha_n\beta_n d_c(T_1 y_n, y_n) \\ &= [n\alpha_n(1-\beta_n) + n\alpha_n\beta_n]d_c(T_1 y_n, y_n) \\ &= n\alpha_n d_c(T_1 y_n, y_n) \end{aligned}$$

which tends to 0 since $d_c(T_1 y_n, y_n) = o(\frac{1}{n\alpha_n}),$. Thus $y_n \to q_0$ and so the iteration $x_{n+1} = f(T_n, x_n)$ is $\{T_n\}$-semistable. This completes the proof.□

Corollary 2.17. Let $(X, ||\cdot||_c)$ be a cone normed space with respect to a normal cone P in the real Banach space E, and $\{T_n\}_{n\in N_0}$ be a sequence of self-maps of X with $T_0 = I$, $\cap_n F(T_n) \neq \emptyset$, and $||T_m x - T_{m-1}x|| \leq ||T_{m-1}x - T_{m-2}x||$ for all $\in X$, $m \geq 2$. Consider the iteration procedure $x_{n+1} = f(T_n, x_n) = \alpha_n x_n + (1 - \alpha_n) T_n x_n$ where $S_n = \frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$ and $\alpha_n \in [0, 1)$. If there exist $a \geq 0$ and $b \in (0, 1)$ such that

$$d_c(f(T_n, y_n), q) \leq a d_c(f(T_n, x_n), y_n) + b d_c(y_n, q)$$

for all sequences $\{y_n\}$ with $d_c(T y_n, y_n) = o(\frac{n+n^2}{1-\alpha_n})$, and all $q \in \cap_n F(T_n)$, then the given iteration is $\{T_n\}$-semistable.

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Authors' contributions
All authors concieved of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

1. Huang, LG, Zheng, X: Cone metric space and fixed point theorems of contractive mapping. J Math Anal Appl. **332**(2), 1468–1476 (2007). doi:10.1016/j.jmaa.2005.03.087
2. Zamfirescu, T: Fixed point theorem in metric spaces. Arch Math (Basel). **23**, 91–101 (1972)
3. Ilic, D, Rakocevic, V: Quasi-contraction on a cone metric space. Appl Math Lett. **22**(5), 728–7310 (2009). doi:10.1016/j.aml.2008.08.011
4. Asadi, M, Soleimani, H, Vaezpour, SM, Rhoades, BE: On T-stability of picard iteration in cone metric spaces. Fixed Point Theory Appl **2009**, 6 (2009). Article ID 751090
5. Khamsi, MA: Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl **2010**, 7 (2010). Article ID 315398
6. Harder, AM, Hicks, TL: Stability results for fixed point iteration procedures. Math Japanica. **33**(5), 693–706 (1988)
7. Qing, Y, Rhoades, BE: T-stability of picard iteration in metric spaces. Fixed Point Theory Appl **2008**, 4 (2008). Article ID 418971
8. Rhoades, BE, Soltus, SM: The equivalence between the T-stabilities of Mann and Ishikawa iterations. J Math Anal. **318**, 472–475 (2006). doi:10.1016/j.jmaa.2005.05.066

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