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Strong convergence theorem for amenable semigroups of nonexpansive mappings and variational inequalities

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Abstract

In this paper, using strongly monotone and lipschitzian operator, we introduce a general iterative process for finding a common fixed point of a semigroup of nonexpansive mappings, with respect to strongly left regular sequence of means defined on an appropriate space of bounded real-valued functions of the semigroups and the set of solutions of variational inequality for β-inverse strongly monotone mapping in a real Hilbert space. Under suitable conditions, we prove the strong convergence theorem for approximating a common element of the above two sets.

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1 Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm are denoted by $\langle . , . \rangle$ and $|| \, . \, ||$, respectively, and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called nonexpansive if $|| \, Tx - Ty \, || \leq || \, x - y \, ||$, for all $x, y \in H$. By $Fix(T)$, we denote the set of fixed points of T (i.e., $Fix(T) = \{x \in H : Tx = x\}$), it is well known that $Fix(T)$ is closed and convex. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in [0, 1)$ such that $|| \, f(x) - f(y) \, || \leq \alpha \, || \, x - y \, ||$ for all $x, y \in C$.

Let $B : C \to H$ be a mapping. The variational inequality problem, denoted by $VI(C, B)$, is to fined $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \tag{1}$$

for all $y \in C$. The variational inequality problem has been extensively studied in literature. See, for example, [1,2] and the references therein.

Definition 1.1 *Let $B : C \to H$ be a mapping. Then B*

(1) is called η-strongly monotone if there exists a positive constant η such that

$$\langle Bx - By, \; x - y \rangle \geq \eta \, \| \, x - y \, \| ^{2}, \quad \forall x, \; y \in C,$$

(2) is called k-Lipschitzian if there exist a positive constant k such that

$$\| Bx - -By \| \leq k \| x - y \|, \quad \forall x,\ y \in C,$$

(3) is called β-inverse strongly monotone if there exists a positive real number β >0 such that

$$\left\langle Bx - By, x - y \right\rangle \geq \beta \| Bx - By \|^2, \quad \forall x,\ y \in C.$$

It is obvious that any β-inverse strongly monotone mapping B is $\frac{1}{\beta}$-Lipschitzian.

Moudafi [3] introduced the viscosity approximation method for fixed point of nonexpansive mappings (see [4] for further developments in both Hilbert and Banach spaces). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{2}$$

where α_n is sequence in (0, 1). Xu [4,5] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequences $\{x_n\}$ generated by (2) strongly converges to the unique solution x^* in $Fix(T)$ of the variational inequality:

$$\langle (f - I)x^*, x - x^* \rangle \leq 0, \quad \forall x \in Fix(T).$$

Let A is strongly positive operator on C. That is, there is a constant $\bar{\gamma} > 0$ with the property that

$$\langle Ax, x \rangle \geq \bar{\gamma} \| x \|^2, \quad \forall x \in C.$$

In [5], it is proved that the sequence $\{x_n\}$ generated by the iterative method bellow with initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{3}$$

converges strongly to the unique solution of the minimization problem

$$\min_{x \in Fix(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle ,$$

where b is a given point in H.

Combining the iterative method (2) and (3), Marino and Xu [6] consider the following iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{4}$$

it is proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the following conditions:

(C_1) $\alpha_n \to 0$,

(C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

$C_3)$ either $\sum_{n=0}^{\infty} | \alpha_{n+1} - \alpha_n | < \infty$ or $\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

then, the sequence $\{x_n\}$ generated by (4) converges strongly, as $n \to \infty$, to the unique solution of the variational inequality:

$$\left\langle (\gamma f - A)x^*, x - x^* \right\rangle \leq 0, \quad \forall x \in Fix(T),$$

which is the optimality condition for minimization problem

$$\min_{x \in Fix(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for all $x \in H$). Some people also study the application of the iterative method (4) [7,8].

Yamada [9] introduce the following hybrid iterative method for solving the variational inequality:

$$x_{n+1} = Tx_n - \mu\alpha_n F(Tx_n), \quad n \in \mathbb{N}, \tag{5}$$

where F is k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$, $0 < \mu < \frac{2\eta}{k^2}$, then he proved that if $\{\alpha_n\}$ satisfying appropriate conditions, then $\{x_n\}$ generated by (5) converges strongly to the unique solution of the variational inequality:

$$\left\langle Fx^*, x - x^* \right\rangle \geq 0, \quad \forall x \in Fix(T).$$

In 2010, Tian [10] combined the iterative (4) with the iterative method (5) and considered the iterative methods:

$$x_{n+1} = (I - \mu\alpha_n F)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{6}$$

and he prove that if the sequence $\{\alpha_n\}$ of parameters satisfies the conditions (C_1), (C_2), and (C_3), then the sequences $\{x_n\}$ generated by (6) converges strongly to the unique solution $x^* \in Fix(T)$ of the variational inequality:

$$\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T).$$

In this paper motivated and inspired by Atsushiba and Takahashi [11], Ceng and Yao [12], Kim [13], Lau et al. [14], Lau et al [15], Marino and Xu [6], Piri and Vaezi [16], Tian [10], Xu [5] and Yamada [9], we introduce the following general iterative algorithm: Let $x_0 \in C$ and

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) P_C(x_n - \delta_n B x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n, \quad n \geq 0. \end{cases} \tag{7}$$

where P_C is a metric projection of H onto C, B is β-inverse strongly monotone, $\phi = \{T_t : t \in S\}$ is a nonexpansive semigroup on H such that the set $\mathcal{F} = Fix(\varphi) \cap VI(C, B) \neq \emptyset$,, X is a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t x, y \rangle$ is an element of X for each $x, y \in H$, and $\{\mu_n\}$ is a sequence of means on X. Our purpose in this paper is to introduce this general iterative algorithm for approximating a common element of the set of common fixed point of a semigroup of nonexpansive mappings and the set of solutions of variational inequality for β-inverse strongly monotone mapping which solves some variational inequality. We will prove that if $\{\mu_n\}$ is left regular and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ of parameters satisfies appropriate conditions, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (7) converges strongly to the unique solution $x^* \in \mathcal{F}$ of the variational inequalities:

$$\begin{cases} \left\langle (\mu F - \gamma f)x^*, x - x^* \right\rangle \geq 0, & \forall x \in \mathcal{F}, \\ \left\langle Bx^*, y - x^* \right\rangle \geq 0 & \forall y \in C. \end{cases}$$

2 Preliminaries

Let S be a semigroup and let $B(S)$ be the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by

$$(l_s f)(t) = f(st), \quad (r_s f)(t) = f(ts), \quad \forall t \in S.$$

Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $|| \mu || = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup, see [15]. A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_{\alpha} \| l_s{}^* \mu_\alpha - \mu_\alpha \| = 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let S be a semigroup and let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\phi = \{T_t : t \in S\}$ of mapping from C into itself is said to be a nonexpansive semigroup on C if T_t is nonexpansive and $T_{ts} = T_t T_s$ for each $t, s \in S$. By $Fix(\phi)$, we denote the set of common fixed points of ϕ, i.e.,

$$Fix(\varphi) = \bigcap_{t \in S} \{x \in C : T_t x = x\}.$$

Lemma 2.1 [15]*Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let* $\phi = \{T_t : t \in S\}$ *be a nonexpansive semigroup on H such that* $\{T_t x : t \in S\}$ *is bounded for some* $x \in C$, *let X be a subspace of* $B(S)$ *such that* $1 \in X$ *and the mapping* $t \to \langle T_t x, y^* \rangle$ *is an element of X for each* $x \in C$ *and* $y^* \in E^*$, *and* μ *is a mean on X. If we write* $T_\mu x$ *instead of* $\int T_t x d\mu(t)$, *then the followings hold.*

(i) T_μ *is non-expansive mapping from C into C.*

(ii) $T_\mu x = x$ *for each* $x \in Fix(\phi)$.

(iii) $T_\mu x \in \overline{co}\{T_t x : t \in S\}$ *for each* $x \in C$.

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ a mapping. Then T is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C, \quad Tx_n \to v \quad imply \quad Tu = v,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

Lemma 2.2 [17]*Let C be a nonempty closed convex subset of a Hilbert space H and suppose that* $T : C \to H$ *is nonexpansive. Then, the mapping I - T is demiclosed at zero.*

Lemma 2.3 [18]*For a given* $x \in H$, $y \in C$,

$$y = P_C x \Leftrightarrow \langle y - x, z - y \rangle \geq 0, \quad \forall z \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\| P_C x - P_C y \|^2 \le \langle P_C x - P_C y,\ x - y \rangle, \quad \forall x,\ y \in H \tag{8}$$

Moreover, P_C is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

$$\langle x - P_C x, y - P_C x \rangle \le 0. \tag{9}$$

It is easy to see that (9) is equivalent to the following inequality

$$\| x - y \|^2 \ge \| x - P_C x \|^2 + \| y - P_C x \|^2. \tag{10}$$

Using Lemma 2.3, one can see that the variational inequality (24) is equivalent to a fixed point problem.

It is easy to see that the following is true:

$$u \in VI(C, B) \Leftrightarrow u = P_C(u - \lambda B u), \quad \lambda > 0. \tag{11}$$

A set-valued mapping $U : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Ux$ and $g \in Uy$ imply $\langle x - y, f - g \rangle \ge 0$. A monotone mapping $U : H \to 2^H$ is maximal if the graph of $G(U)$ of U is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping U is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \ge 0$ for every $(y, g) \in G(U)$ implies that $f \in Ux$. Let B be a monotone mapping of C into H and let $N_C x$ be the normal cone to C at $x \in C$, that is, $N_C x = \{y \in H : \langle z - x, y \rangle \le 0, \forall z \in C\}$ and define

$$Ux = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases} \tag{12}$$

Then U is the maximal monotone and $0 \in Ux$ if and only if $x \in VI(C, B)$; see [19].

The following lemma is well known.

Lemma 2.4 *Let H be a real Hilbert space. Then, for all $x, y \in H$*

$$\| x - y \|^2 \le \| x \|^2 + 2\langle y, x + y \rangle, \ .$$

Lemma 2.5 [5]*Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - b_n)a_n + b_n c_n, \quad n \ge 0,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) either $\limsup_{n\to\infty} c_n \le 0$ *or* $\sum_{n=0}^{\infty} | b_n c_n | < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

As far as we know, the following lemma has been used implicitly in some papers; for the sake of completeness, we include its proof.

Lemma 2.6 *Let H be a real Hilbert space and F be a k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $\tau = \mu(\eta - \frac{\mu k^2}{2})$. Then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, $I - t_\mu F$ is contraction with constant $1 - t\tau$.*

Proof. Notice that

$$
\begin{aligned}
&\| (I - t\mu F)x - (I - t\mu F)y \|^2 \\
&\quad = \langle (I - t\mu F)x - (I - t\mu F)y, (I - t\mu F)x - (I - t\mu F)y \rangle \\
&\quad = \| x - y \|^2 + t^2\mu^2 \| Fx - Fy \|^2 - 2t\mu \langle x - y, Fx - Fy \rangle \\
&\quad \le \| x - y \|^2 + t^2\mu^2 k^2 \| x - y \|^2 - 2t\mu\eta \| x - y \|^2 \\
&\quad \le \| x - y \|^2 + t\mu^2 k^2 \| x - y \|^2 - 2t\mu\eta \| x - y \|^2 \\
&\quad = \left(1 - 2t\mu\left(\eta - \frac{\mu k^2}{2}\right)\right) \| x - y \|^2 \\
&\quad = (1 - 2t\tau) \| x - y \|^2 \\
&\quad \le (1 - t\tau)^2 \| x - y \|^2.
\end{aligned}
$$

It follows that

$$\| (I - t\mu F)x - (I - t\mu F)y \| \le (1 - t\tau) \| x - y \|,$$

and hence $I - t\mu F$ is contractive due to $1 - t\tau \in (0, 1)$. □

Notation Throughout the rest of this paper, F will denote a k-Lipschitzian and η-strongly monotone operator of C into H with $k > 0$, $\eta > 0$, f is a contraction on C with coefficient $0 < \alpha < 1$. We will also always use γ to mean a number in $(0, \frac{\tau}{\alpha})$, where $\tau = \mu(\eta - \frac{\mu k^2}{2})$ and $0 < \mu < \frac{2\eta}{k^2}$. The open ball of radius r centered at 0 is denoted by B_r and for a subset D of H, by $\overline{co}D$, we denote the closed convex hull of D. For $\varepsilon > 0$ and a mapping $T : D \to H$, we let $F_\varepsilon(T; D)$ be the set of ε-approximate fixed points of T, i.e., $F_\varepsilon(T; D) = \{x \in D : ||x - T_x|| \le \varepsilon\}$. Weak convergence is denoted by $\rightharpoonup$ and strong convergence is denoted by $\to$.

3 Main results

Theorem 3.1 *Let S be a semigroup, C a nonempty closed convex subset of real Hilbert space H and $B : C \to H$ be a β-inverse strongly monotone. Let $\phi = \{T_t : t \in S\}$ be a nonexpansive semigroup of C into itself such that $\mathcal{F} = Fix(\varphi) \cap VI(C, B) \neq \emptyset$,, X a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$, $\{\mu_n\}$ a left regular sequence of means on X such that $\sum_{n=1}^{\infty} \| \mu_{n+1} - \mu_n \| < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ and $\{\delta_n\}$ be a sequence in $[a, b]$, where $0 < a < b < 2\beta$. Suppose the following conditions are satisfied.*

(B_1) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$,
(B_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(B_3) $\sum_{n=1}^{\infty} | \alpha_{n+1} - \alpha_n | < \infty$, $\sum_{n=1}^{\infty} | \beta_{n+1} - \beta_n | < \infty$, $\sum_{n=1}^{\infty} | \delta_{n+1} - \delta_n | < \infty$.

If $\{x_n\}$ and $\{y_n\}$ be generated by $x_0 \in C$ and

$$
\begin{cases}
y_n = \beta_n x_n + (1 - \beta_n) P_C(x_n - \delta_n B x_n), \\
x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n, \quad n \ge 0.
\end{cases}
$$

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly, as $n \to \infty$, to $x^* \in \mathcal{F}$, which is a unique solution of the variational inequalities:

$$\begin{cases} \langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, & \forall x \in \mathcal{F}, \\ \langle Bx^*, y - x^* \rangle \geq 0 & \forall y \in C. \end{cases}$$

Proof. Since $\{\alpha_n\}$ satisfies in condition (B_1), we may assume, with no loss of generality, that $\alpha_n < \min\{1, \frac{1}{\tau}\}$. Since B is β-inverse strongly monotone and $\delta_n < 2\beta$, for any $x, y \in C$, we have

$$\begin{aligned} &\| (I - \delta_n B)x - (I - \delta_n B)y \|^2 \\ &\quad = \| (x - y) - \delta_n (Bx - By) \|^2 \\ &\quad = \| x - y \|^2 - 2\delta_n \langle x - y, Bx - By \rangle + \delta_n^2 \| Bx - By \|^2 \\ &\quad \leq \| x - y \|^2 - 2\delta_n \beta \| Bx - By \|^2 + \delta_n^2 \| Bx - By \|^2 \\ &\quad = \| x - y \|^2 + \delta_n (\delta_n - 2\beta) \| Bx - By \|^2 \\ &\quad \leq \| x - y \|^2. \end{aligned}$$

It follows that

$$\| (I - \delta_n B)x - (I - \delta_n B)y \| \leq \| x - y \| . \tag{13}$$

Let $p \in \mathcal{F}$, in the context of the variational inequality problem, the characterization of projection (11) implies that $p = P_C(p - \delta_n B_p)$. Using (13), we get

$$\begin{aligned} \| y_n - p \| &= \| \beta_n x_n + (1 - \beta_n) P_C(x_n - \delta_n B x_n) - p \| \\ &= \| \beta_n [x_n - p] + (1 - \beta_n)[P_C(x_n - \delta_n B x_n) - P_C(p - \delta_n B p)] \| \\ &\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| P_C(x_n - \delta_n B x_n) - P_C(p - \delta_n B p) \| \\ &\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| x_n - p \| = \| x_n - p \| . \end{aligned} \tag{14}$$

We claim that $\{x_n\}$ is bounded. Let $p \in \mathcal{F}$, using Lemma 2.6 and (14), we have

$$\begin{aligned} \| x_{n+1} - p \| &= \| \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n - p \| \\ &= \| \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n - (I - \alpha_n \mu F)p - \alpha_n \mu F p \| \\ &\leq \alpha_n \| \gamma f(x_n) - \mu F p \| + \| (I - \alpha_n \mu F) T_{\mu_n} y_n - (I - \alpha_n \mu F) p \| \\ &\leq \alpha_n \| \gamma f(x_n) - \gamma f(p) \| \\ &\quad + \alpha_n \| \gamma f(p) - \mu F p \| + (1 - \alpha_n \tau) \| T_{\mu_n} y_n - p \| \\ &\leq \alpha_n \gamma \alpha \| x_n - p \| + \alpha_n \| \gamma f(p) - \mu F p \| + (1 - \alpha_n \tau) \| y_n - p \| \\ &\leq \alpha_n \gamma \alpha \| x_n - p \| + \alpha_n \| \gamma f(p) - \mu F p \| + (1 - \alpha_n \tau) \| x_n - p \| \\ &= (1 - \alpha_n (\tau - \gamma \alpha)) \| x_n - p \| + \alpha_n \| \gamma f(p) - \mu F p \| \\ &\leq \max\{\| x_n - p \|, (\tau - \gamma \alpha)^{-1} \| \gamma f(p) - \mu F p \|\}. \end{aligned}$$

By induction we have,

$$\| x_n - p \| \leq \max\{(\tau - \gamma \alpha)^{-1} \| \gamma f(p) - \mu F p \|, \| x_0 - p \|\} = M_0.$$

Hence, $\{x_n\}$ is bounded and also $\{y_n\}$ and $\{f(x_n)\}$ are bounded. Set $D = \{y \in H : ||y - p|| \leq = M_0\}$. We remark that D is ϕ-invariant bounded closed convex set and $\{x_n\}$, $\{y_n\} \subset D$. Now we claim that

$$\limsup_{n \to \infty} \sup_{y \in D} \| T_{\mu_n} y - T_t T_{\mu_n} y \| = 0, \quad \forall t \in S. \tag{15}$$

Let $\varepsilon > 0$. By [[20], Theorem 1.2], there exists $\delta > 0$ such that

$$\overline{co} F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D), \quad \forall t \in S. \tag{16}$$

Also by [[20], Corollary 1.1], there exists a natural number N such that

$$\left\| \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y - T_t\left(\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\right)\right\| \le \delta, \tag{17}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\| \mu_n - l^*_{t^i}\mu_n \| \le \frac{\delta}{(M_0 + \|p\|)}$ for $n \ge N_0$ and $i = 0, 1, 2,..., N$. Then we have

$$\begin{aligned}
&\sup_{y\in D}\left\| T_{\mu_n}y - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y \mathrm{d}\mu_n s\right\| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \langle T_{\mu_n}y, z\rangle - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y \mathrm{d}\mu_n s, z\right\rangle\right| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left| \frac{1}{N+1}\sum_{i=0}^{N} (\mu_n)_s\langle T_s y, z\rangle - \frac{1}{N+1}\sum_{i=0}^{N} (\mu_n)_s\langle T_{t^i s}y, z\rangle\right| \\
&\quad \le \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1} | (\mu_n)_s\langle T_s y, z\rangle - (l^*_{t^i}\mu_n)_s\langle T_s y, z\rangle | \\
&\quad \le \max_{i=1,2,\ldots N} \| \mu_n - l^*_{t^i}\mu_n \| (M_0 + \| p \|) \le \delta, \ \forall n \ge N_0.
\end{aligned} \tag{18}$$

By Lemma 2.1, we have

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y \mathrm{d}\mu_n s \in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s y) : s \in s\right\}. \tag{19}$$

It follows from (16), (17), (18), and (19) that

$$\begin{aligned}
T_{\mu_n}(y) &\in \overline{co}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y) : s \in S\right\} + B_\delta \\
&\subset \overline{co}F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D),
\end{aligned}$$

for all $y \in D$ and $n \ge N_0$. Therefore,

$$\limsup_{n\to\infty}\sup_{y\in D} \| T_t(T_{\mu_n}y) - T_{\mu_n}y \| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get (15). In this stage, we will show

$$\lim_{n\to\infty} \| x_n - T_t x_n \| = 0, \quad \forall t \in S. \tag{20}$$

Let $t \in S$ and $\varepsilon > 0$. Then, there exists $\delta > 0$, which satisiies (16). From $\lim_{n\to\infty} \alpha_n = 0$ and (15) there exists $N_1 \in \mathbb{N}$ such that $\alpha_n \le \frac{\delta}{(\tau+\mu k)M_0}$ and $T_{\mu_n}y_n \in F_\delta(T_t; D)$, for all $n \ge N_1$. By Lemma 2.6 and (14), we have

$$\begin{aligned}
&\alpha_n \|\gamma f(x_n) - \mu F T_{\mu_n}y_n \| \\
&\quad \le \alpha_n(\gamma \| f(x_n) - f(p) \| + \| \gamma f(p) - \mu F p \| + \| \mu F p - \mu F T_{\mu_n}y_n \|) \\
&\quad \le \alpha_n(\gamma\alpha \| x_n - p \| + \| \gamma f(p) - \mu F p \| + \mu k \| y_n - p \|) \\
&\quad \le \alpha_n(\gamma\alpha M_0 + (\tau - \gamma\alpha)M_0 + \mu k M_0) \\
&\quad \le \alpha_n(\tau + \mu k)M_0 \le \delta,
\end{aligned}$$

for all $n \geq N_1$. Therefore, we have

$$\begin{aligned} x_{n+1} &= T_{\mu_n} y_n + \alpha_n [\gamma f(x_n) + \mu F(T_{\mu_n} y_n)] \\ &\in F_\delta(T_t; D) + B_\delta \subset F_\varepsilon(T_t; D), \end{aligned}$$

for all $n \geq N_1$. This shows that

$$\| x_n - T_t x_n \| \leq \varepsilon, \ \forall n \geq N_1.$$

Since $\varepsilon > 0$ is arbitrary, we get (20).

Let

$Q = P_{\mathcal{F}}$. Then $Q(I - \mu F + \gamma f)$ is a contraction of H into itself. In fact, we see that

$$\begin{aligned} &\| Q(I - \mu F + \gamma f)x - Q(I - \mu F + \gamma f)y \| \\ &\quad \leq \| (I - \mu F + \gamma f)x - (I - \mu F + \gamma f)y \| \\ &\quad \leq \| (I - \mu F)x - (I - \mu F)y \| + \gamma \| f(x) - f(y) \| \\ &\quad = \lim_{n\to\infty} \left\| \left(I - \left(1 - \frac{1}{n} \right) \mu F \right) x - \left(I - \left(1 - \frac{1}{n} \right) \mu F \right) y \right\| + \gamma \| f(x) - f(y) \| \\ &\quad \leq \lim_{n\to\infty} (1 - (1 - \frac{1}{n})\tau) \| x - y \| + \gamma\alpha \| x - y \| \\ &\quad = (1 - \tau) \| x - y \| + \gamma\alpha \| x - y \|, \end{aligned}$$

and hence $Q(I - \mu F + \gamma f)$ is a contraction due to $(1 - (\tau - \gamma\alpha)) \in (0, 1)$.

Therefore, by Banachs contraction principal, $P_{\mathcal{F}}(I - \mu F + \gamma f)$ has a unique fixed point x^*. Then using (9), x^* is the unique solution of the variational inequality:

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}. \tag{21}$$

We show that

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - \mu F x^*, \ x_n - x^* \rangle \leq 0. \tag{22}$$

Indeed, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - \mu F x^*, \ x_n - x^* \rangle = \lim_{i\to\infty} \langle \gamma f(x^*) - \mu F x^*, \ x_{n_i} - x^* \rangle. \tag{23}$$

Because $\{x_{n_i}\}$ is bounded, we may assume that $x_{n_i} \rightharpoonup z$. In terms of Lemma 2.2 and (20), we conclude that $z \in Fix(\phi)$.

Now, let us show that $z \in VI(C, B)$. Let $w_n = P_C(x_n - \delta_n B x_n)$, it follows from the definition of $\{y_n\}$ that

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ &\quad = \| \beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) P_C(x_{n+1} - \delta_{n+1} B x_{n+1}) - \beta_n x_n - (1 - \beta_n) P_C(x_n - \delta_n B x_n) \| \\ &\quad = \| \beta_{n+1}(x_{n+1} - x_n) + (\beta_{n+1} - \beta_n) x_n + (1 - \beta_{n+1}) P_C(x_{n+1} - \delta_{n+1} B x_{n+1}) \\ &\qquad - (1 - \beta_{n+1}) P_C(x_n - \delta_{n+1} B x_n) + (1 - \beta_{n+1}) P_C(x_n - \delta_{n+1} B x_n) - (1 - \beta_n) P_C(x_n - \delta_n B x_n) \| \\ &\quad \leq \beta_{n+1} \| x_{n+1} - x_n \| + | \beta_{n+1} - \beta_n | \| x_n \| \\ &\qquad + (1 - \beta_{n+1}) \| P_C(x_{n+1} - \delta_{n+1} B x_{n+1}) - P_C(x_n - \delta_{n+1} B x_n) \| \\ &\qquad + \| P_C(x_n - \delta_{n+1} B x_n) - P_C(x_n - \delta_n B x_n) \| \\ &\qquad + \| \beta_n P_C(x_n - \delta_n B x_n) - \beta_{n+1} P_C(x_n - \delta_{n+1} B x_n) \|] \\ &\quad \leq \beta_{n+1} \| x_{n+1} - x_n \| + | \beta_{n+1} - \beta_n | \| x_n \| + (1 - \beta_{n+1}) \| x_{n+1} - x_n \| \\ &\qquad + | \delta_{n+1} - \delta_n | \| B x_n \| + \| \beta_n P_C(x_n - \delta_n B x_n) - \beta_n P_C(x_n - \delta_{n+1} B x_n) \\ &\qquad + \beta_n P_C(x_n - \delta_{n+1} B x_n) - \beta_{n+1} P_C(x_n - \delta_{n+1} B x_n) \| \\ &\quad \leq \beta_{n+1} \| x_{n+1} - x_n \| + | \beta_{n+1} - \beta_n | \| x_n \| + (1 - \beta_{n+1}) \| x_{n+1} - x_n \| \\ &\qquad + | \delta_{n+1} - \delta_n | \| B x_n \| + \beta_n | \delta_{n+1} - \delta_n | \| B x_n \| + | \beta_{n+1} - \beta_n | \| P_C(x_n - \delta_{n+1} B x_n) \| \\ &\quad = \| x_{n+1} - x_n \| + | \beta_{n+1} - \beta_n | \| x_n \| + (1 + \beta_n) | \delta_{n+1} - \delta_n | \| B x_n \| \\ &\qquad + | \beta_{n+1} - \beta_n | \| P_C(x_n - \delta_{n+1} B x_n) \| . \end{aligned}$$

Using the last inequality, we get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \| \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n - \alpha_{n-1} \gamma f(x_{n-1}) - (I - \alpha_{n-1} \mu F) T_{\mu_{n-1}} y_{n-1} \| \\
&= \| \alpha_n \gamma f(x_n) - \alpha_n \gamma f(x_{n-1}) + (\alpha_n - \alpha_{n-1}) \gamma f(x_{n-1}) \\
&\quad + (I - \alpha_n \mu F) T_{\mu_n} y_n - (I - \alpha_n \mu F) T_{\mu_{n-1}} y_{n-1} \\
&\quad + (I - \alpha_n \mu F) T_{\mu_{n-1}} y_{n-1} - (I - \alpha_{n-1} \mu F) T_{\mu_{n-1}} y_{n-1} \| \\
&\le \alpha_n \gamma \alpha \| x_n - x_{n-1} \| + | \alpha_n - \alpha_{n-1} | \gamma \| f(x_{n-1}) \| \\
&\quad + (1 - \alpha_n \tau) \| T_{\mu_n} y_n - T_{\mu_{n-1}} y_{n-1} \| + | \alpha_n - \alpha_{n-1} | \mu \| F T_{\mu_{n-1}} y_{n-1} \| \\
&\le \alpha_n \gamma \alpha \| x_n - x_{n-1} \| + | \alpha_n - \alpha_{n-1} | \gamma \| f(x_{n-1}) \| + (1 - \alpha_n \tau) \| y_n - y_{n-1} \| \\
&\quad + (1 - \alpha_n \tau) \| T_{\mu_n} y_{n-1} - T_{\mu_{n-1}} y_{n-1} \| + | \alpha_n - \alpha_{n-1} | \mu \| F T_{\mu_{n-1}} y_{n-1} \| \\
&\le \alpha_n \gamma \alpha \| x_n - x_{n-1} \| + | \alpha_n - \alpha_{n-1} | \gamma \| f(x_{n-1}) \| + (1 - \alpha_n \tau) \| x_n - x_{n-1} \| \\
&\quad + (1 - \alpha_n \tau) | \beta_n - \beta_{n-1} | \| x_{n-1} \| + (1 - \alpha_n \tau)(1 + \beta_{n-1}) | \delta_n - \delta_{n-1} | \| B x_{n-1} \| \\
&\quad + (1 - \alpha_n \tau) | \beta_n - \beta_{n-1} | \| P_C(x_{n-1} - \delta_n B x_{n-1}) \| \\
&\quad + (1 - \alpha_n \tau) \| T_{\mu_n} y_{n-1} - T_{\mu_{n-1}} y_{n-1} \| + | \alpha_n - \alpha_{n-1} | \mu \| F T_{\mu_{n-1}} y_{n-1} \| .
\end{aligned}$$

Thus, for some large enough constant $M > 0$, we have

$$\begin{aligned}
\| x_{n+1} - x_n \| \le\ & (1 - \alpha_n(\tau - \gamma \alpha)) \| x_n - x_{n-1} \| \\
& + [| \alpha_n - \alpha_{n-1} | + | \beta_n - \beta_{n-1} | + | \delta_n - \delta_{n-1} | + \| \mu_n - \mu_{n-1} \|] M.
\end{aligned}$$

Therefore, using condition B_3 and Lemma 2.5, we get

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{24}$$

Let $p \in \mathcal{F}$, from (11) and deiinition of $\{y_n\}$, we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&= \| \beta_n x_n + (1 - \beta_n) P_C(x_n - \delta_n B x_n) - p\|^2 \\
&= \| \beta_n (x_n - p) + (1 - \beta_n)(P_C(x_n - \delta_n B x_n) - P_C(p - \delta_n B p))\|^2 \\
&\le \beta_n \| x_n - p\|^2 + (1 - \beta_n) \| (x_n - p) - \delta_n (B x_n - B p))\|^2 \\
&= \beta_n \| x_n - p\|^2 + (1 - \beta_n) \| x_n - p\|^2 + \delta_n^2 (1 - \beta_n) \| B x_n - B p\|^2 \\
&\quad - 2\delta_n (1 - \beta_n) \langle x_n - p,\ B x_n - B p \rangle \\
&\le \| x_n - p\|^2 + \delta_n^2 (1 - \beta_n) \| B x_n - B p\|^2 - 2\delta_n (1 - \beta_n) \beta \| B x_n - B p\|^2 \\
&= \| x_n - p\|^2 + \delta_n (1 - \beta_n)(\delta_n - 2\beta) \| B x_n - B p\|^2.
\end{aligned} \tag{25}$$

Using (25), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \| \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n - p\|^2 \\
&= \| \alpha_n (\gamma f(x_n) - \mu F T_{\mu_n} y_n) + (T_{\mu_n} y_n - p)\|^2 \\
&= \alpha_n^2 \| \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \| T_{\mu_n} y_n - p \| \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p \rangle \\
&\le \alpha_n^2 \| \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \| y_n - p\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p \rangle^2 \\
&\le \alpha_n^2 \| \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \| x_n - p\|^2 \\
&\quad + \delta_n (1 - \beta_n)(\delta_n - 2\beta) \| B x_n - B p\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p \rangle \\
&= \alpha_n^2 \| \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \| x_n - p\|^2 \\
&\quad + \delta_n (\delta_n - 2\beta_n) \| B x_n - B p\|^2 - \delta_n \beta_n (\delta_n - 2\beta_n) \| B x_n - B p\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p \rangle.
\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}
&-\delta_n(\delta_n - 2\beta_n) \parallel Bx_n - Bp\|^2 \\
&\quad \le \alpha_n^2 \parallel \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + [\| x_n - p \| + \| x_{n+1} - p \|] \parallel x_{n+1} - x_n \| \\
&\qquad - \delta_n \beta_n(\delta_n - 2\beta_n) \parallel Bx_n - Bp\|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p\rangle.
\end{aligned}$$

Hence, using condition B_1 and (24), we get

$$\lim_{n\to\infty} \| Bx_n - Bp \| = 0. \tag{27}$$

From (8), we have

$$\begin{aligned}
&\|w_n - p\|^2 \\
&\quad = \| P_C(x_n - \delta_n Bx_n) - P_C(p - \delta_n Bp)\|^2 \\
&\quad \le \langle (x_n - \delta_n Bx_n) - (p - \delta_n Bp), w_n - p\rangle \\
&\quad = \frac{1}{2}[\| (x_n - \delta_n Bx_n) - (p - \delta_n Bp)\|^2 + \| w_n - p\|^2 \\
&\qquad - \| (x_n - \delta_n Bx_n) - (p - \delta_n Bp) - (w_n - p)\|^2] \\
&\quad \le \frac{1}{2}[\| x_n - p\|^2 + \| w_n - p\|^2 \\
&\qquad - \| (x_n - \delta_n Bx_n) - (p - \delta_n Bp) - (w_n - p)\|^2] \\
&\quad = \frac{1}{2}[\| x_n - p\|^2 + \| w_n - p\|^2 - \| x_n - w_n\|^2 \\
&\qquad + 2\delta_n \langle x_n - w_n, Bx_n - Bp\rangle - \delta_n^2 \parallel Bx_n - Bp\|^2].
\end{aligned}$$

So we obtain

$$\begin{aligned}
\| w_n - p\|^2 \le \| x_n - p\|^2 - \| x_n - w_n\|^2 \\
+ 2\delta_n \langle x_n - w_n, \ Bx_n - Bp\rangle - \delta_n^2 \parallel Bx_n - Bp\|^2.
\end{aligned} \tag{28}$$

It follows from (26) and (28) that

$$\begin{aligned}
&\|x_{n+} - p\|^2 \\
&\quad \le \alpha_n^2 \parallel \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \| y_n - p\|^2 \\
&\qquad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p\rangle \\
&\quad \le \alpha_n^2 \parallel \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \| \beta_n x_n + (1-\beta_n) P_C(x_n - \delta_n Bx_n) - p\|^2 \\
&\qquad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p\rangle \\
&\quad \le \alpha_n^2 \parallel \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \beta_n \parallel x_n - p\|^2 + (1-\beta_n) \parallel w_n - p\|^2 \\
&\qquad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p\rangle \\
&\quad \le \alpha_n^2 \parallel \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + \beta_n \parallel x_n - p\|^2 + (1-\beta_n) \parallel x_n - p\|^2 \\
&\qquad - (1-\beta_n) \parallel x_n - w_n\|^2 + 2\delta_n(1-\beta_n)\langle x_n - w_n, \ Bx_n - Bp\rangle - \delta_n^2(1-\beta_n) \parallel Bx_n - Bp\|^2 \\
&\qquad + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, T_{\mu_n} y_n - p\rangle.
\end{aligned}$$

Which implies that

$$\begin{aligned}
\| x_n - w_n\|^2 \le \alpha_n^2 \parallel \gamma f(x_n) - \mu F T_{\mu_n} y_n\|^2 + [\| x_n - p \| + \| x_{n+1} - p \|] \parallel x_{n+1} - x_n \| \\
+ \beta_n \parallel x_n - w_n\|^2 + 2\delta_n(1-\beta_n) \parallel x_n - w_n \| \| Bx_n - Bp \| \\
- \delta_n^2(1-\beta_n) \parallel Bx_n - Bp\|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu F T_{\mu_n} y_n, \ T_{\mu_n} y_n - p\rangle.
\end{aligned}$$

Therefore, using condition B_1, (24) and (27), we get

$$\lim_{n\to\infty} \| x_n - w_n \| = 0. \tag{29}$$

Let $U : H \to 2^H$ be a set-valued mapping is defined by

$$Ux = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

where $N_C x$ is the normal cone to C at $x \in C$. Since B is relaxed, β-inverse strongly monotone. Thus, U is maximal monotone see [19]. Let $(x, y) \in G(U)$, hence $y - Bx \in N_C x$. Since $w_n = P_C(x_n - \zeta_n B x_n)$ therefore, $\langle x - w_n, y - Bx\rangle \geq 0$. On the other hand from $w_n = P_C(x_n - \zeta_n B x_n)$, we have

$$\langle x - w_n, w_n - (x_n - \delta_n B x_n)\rangle \geq 0,$$

that is

$$\left\langle x - w_n, \frac{w_n - x_n}{\delta_n} + Bx_n \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
&\langle x - w_{n_i}, y\rangle \\
&\quad \geq \langle x - w_{n_i}, Bx\rangle \\
&\quad \geq \langle x - w_{n_i}, Bx\rangle - \left\langle x - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{\delta_{n_i}} + Bx_{n_i}) \right\rangle \\
&\quad = \left\langle x - w_{n_i}, Bx - \frac{w_{n_i} - x_{n_i}}{\delta_{n_i}} - Bx_{n_i} \right\rangle \\
&\quad = \langle x - w_{n_i}, Bx - Bw_{n_i}\rangle + \langle x - w_{n_i}, Bw_{n_i} - Bx_{n_i}\rangle - \left\langle x - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{\delta_{n_i}} \right\rangle \\
&\quad \geq \langle x - w_{n_i}, Bw_{n_i} - Bx_{n_i}\rangle - \left\langle x - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{\delta_{n_i}} \right\rangle \\
&\quad \geq \langle x - w_{n_i}, Bw_{n_i} - Bx_{n_i}\rangle - \| x - w_{n_i} \| \left\| \frac{w_{n_i} - x_{n_i}}{\delta_{n_i}} \right\|.
\end{aligned}$$

Noting that $\lim_{i\to\infty} \| w_{n_i} - x_{n_i} \| = 0, x_{n_i} \to z$, $x_{n_i} \to z$ and B is $\frac{1}{\beta}$-lipschitzian, we obtain

$$\langle x - z, y\rangle \geq 0.$$

Since U is maximal monotone, we have $z \in U^{-1}0$, and hence $z \in VI(C, B)$. Therefore, $z \in \mathcal{F}$.

Since $x_{n_i} \to z$ from (21) and (23), we have

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - \mu F x^*, x_n - x^*\rangle \leq 0.$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. By Lemmas 2.4, 2.6, and (14), we have

$$\begin{aligned}
&\|x_{n+1} - x^* \|^2 \\
&= \| \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n - x^* \|^2 \\
&= \| \alpha_n \gamma f(x_n) - \alpha_n \mu F x^* + (I - \alpha_n \mu F) T_{\mu_n} y_n \\
&\quad - (I - \alpha_n \mu F) x^* \|^2 \\
&\le \| (I - \alpha_n \mu F) T_{\mu_n} y_n - (I - \alpha_n \mu F) x^* \|^2 \\
&\quad + 2\alpha_n \langle \gamma f(x_n) - \mu F x^*, x_{n+1} - x^* \rangle \\
&\le (1 - \alpha_n \tau)^2 \| y_n - x^* \|^2 + 2\alpha_n \langle \gamma f(x_n) - \mu F x^*, x_{n+1} - x^* \rangle \qquad (30)\\
&\le (1 - \alpha_n \tau)^2 \| y_n - x^* \|^2 + 2\alpha_n \langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\quad + 2\alpha_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle. \\
&\le (1 - \alpha_n \tau)^2 \| y_n - x^* \|^2 + \alpha_n \gamma \alpha [\| x_n - x^* \|^2 + \| x_{n+1} - x^* \|^2] \\
&\quad + 2\alpha_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle. \\
&\le (1 - \alpha_n \tau)^2 \| x_n - x^* \|^2 + \alpha_n \gamma \alpha [\| x_n - x^* \|^2 + \| x_{n+1} - x^* \|^2] \\
&\quad + 2\alpha_n \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle.
\end{aligned}$$

So from (30), we reach the following

$$\begin{aligned}
\| x_{n+1} - x^*\|^2 \le & \frac{1 + \alpha^2\tau^2 - 2\alpha_n\tau + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha} \| x_n - x^*\|^2 \\
& + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha} \langle \gamma f(x^*) - \mu F x^*,\ x_{n+1} - x^* \rangle \\
\le & (1 - \alpha_n \frac{2(\tau - \gamma\alpha) - \alpha_n\tau^2}{1 - \alpha_n\gamma\alpha}) \| x_n - x^*\|^2 \\
& + \alpha_n \frac{2(\tau - \gamma\alpha) - \alpha_n\tau^2}{1 - \alpha_n\gamma\alpha} \frac{2}{2(\tau - \gamma\alpha) - \alpha_n\tau^2} \langle \gamma f(x^*) - \mu F x^*,\ x_{n+1} - x^* \rangle
\end{aligned}$$

It follows that

$$\| x_{n+1} - x^*\|^2 \le (1 - b_n) \| x_n - x^*\|^2 + b_n c_n, \qquad (31)$$

where

$$b_n = \alpha_n \frac{2(\tau - \gamma\alpha) - \alpha_n\tau^2}{1 - \alpha_n\gamma\alpha},\ c_n = \frac{2}{2(\tau - \gamma\alpha) - \alpha_n\tau^2} \langle \gamma f(x^*) - \mu F x^*, x_{n+1} - x^* \rangle$$

Since $\alpha_n \to o$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have $\sum_{n=0}^{\infty} b_n = \infty$ and by (22), we get $\limsup_{n\to\infty} c_n \le 0$. Consequently, applying Lemma 2.5, to (31), we conclude that $x^n \to x^*$. Since $|| y_n - x^* || \le || x_n - x^* ||$, we have $y^n \to x^*$. □

Corollary 3.2 *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ *and B be as in Theorem 3.1. Let T a nonexpansive mapping of C into C such that* $\mathcal{F} = \mathrm{Fix}(T) \cap VI(C, B) \neq \emptyset$. *Suppose* $x_0 \in H$ *and* $\{x_n\}$ *and* $\{y_n\}$ *be generated by the iteration algorithm*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) P_C(x_n - \delta_n B x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) \frac{n}{n-1} \int_0^{\frac{n-1}{n}} T(t) y_n dt, \quad n \ge 0. \end{cases}$$

Then $\{x_n\}$ *and* $\{y_n\}$ *convergence strongly to* x^* *which is the unique solution of the systems of variational inequalities*:

$$\begin{cases} \langle (\mu F - \gamma f)x^*,\ x - x^* \rangle \geq 0, \forall x \in \mathcal{F}, \\ \langle Bx^*,\ y - x^* \rangle \geq 0 \qquad\qquad \forall y \in C, \end{cases}$$

Proof. Take $\lambda_n = \frac{n-1}{n}$, for $n \in \mathbb{N}$, we define $\mu_n(f) = \frac{1}{\lambda_n}\int_0^{\lambda_n} f(t)\mathrm{d}t$ for each $f \in C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ denotes the space of all real-valued bounded continuous functions on R^+ with supremum norm. Then, $\{\mu_n\}$ is regular sequence of means on $C(\mathbb{R}_+)$ such that

$$\| \mu_{n+1} - \mu_n \| \leq 2\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)$$

for more details, see [21]. Further, for each $y \in C$, we have

$$T_{\mu_n}y = \frac{1}{\lambda_n}\int_0^{\lambda_n} T(t)y\mathrm{d}t.$$

On the other hand

$$\begin{aligned} \sum_{n=1}^{\infty} \|\mu_{n+1} - \mu_n\| &\leq 2\sum_{n=1}^{\infty}\left(\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right) \\ &= 2\sum_{n=1}^{\infty}\left(\frac{\frac{n}{n+1} - \frac{n-1}{n}}{\frac{n}{n+1}}\right) \\ &= 2\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty \end{aligned}$$

Now, apply Theorem 3.1 to conclude the result. □

Corollary 3.3 *Let S, ϕ, X, $\{\mu_n\}$, $\mathcal{F}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ be as in Theorem 3.1. Let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$, ζ a number in $(0, \frac{\bar{\tau}}{\alpha})$, where $\bar{\tau} = \bar{\mu}(\bar{\gamma} - \frac{\bar{\mu}\|A\|^2}{2})$and $0 < \bar{\mu} < \frac{2\bar{\gamma}}{\|A\|^2}$. If $\{x_n\}$ and $\{y_n\}$ are generated by $x_0 \in C$ and*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)P_C(x_n - \delta_n A x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \bar{\mu} A)T_{\mu_n} y_n, \quad n \geq 0. \end{cases}$$

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly, as $n \to \infty$, to $x^* \in \mathcal{F}$, which is a unique solution of the variational inequalities:

$$\begin{cases} \langle (\mu F - \gamma f)x^*,\ x - x^* \rangle \geq 0, \forall x \in \mathcal{F}, \\ \langle Ax^*,\ y - x^* \rangle \geq 0 \qquad\qquad \forall y \in C. \end{cases}$$

Proof. Because A is strongly positive bounded linear operator on H with coefficient $\bar{\gamma}$, we have

$$\langle Ax - Ay,\ x - y \rangle \geq \bar{\gamma} \| x - y \|^2.$$

Therefore, A is $\bar{\gamma}$-strongly monotone.

On the other hand

$$\| Ax - Ay \| \leq \| A \| \| x - y \| .$$

Therefore,

$$\frac{\bar{\gamma}}{\| A \|^2} \| Ax - Ay \|^2 \leq \langle Ax - Ay,\ x - y \rangle.$$

Hence, A is $\frac{\bar{\gamma}}{\|A\|^2}$-inverse strongly monotone. Now apply Theorem 3.1 to conclude the result. □

Corollary 3.4 *Let $\{\alpha_n\}$, $\{\beta_n\}$ and B be as in Theorem 3.1. Let u, $x_0 \in C$ and $\{x_n\}$ and $\{y_n\}$ be generated by the iterative algorithm*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) P_C(x_n - \delta_n B x_n), \\ x_{n+1} = \alpha_n u + (I - \alpha_n \mu F) y_n, \qquad n \geq 0. \end{cases}$$

Then $\{x_n\}$ and $\{y_n\}$ convergence strongly to x^* which is the unique solution of the systems of variational inequalities:

$$\begin{cases} \langle (\mu F - \gamma f) x^*, \ x - x^* \rangle \geq 0, & \forall x \in \mathcal{F}, \\ \langle B x^*, \ y - x^* \rangle \geq 0 & \forall y \in C. \end{cases}$$

Proof. It is sufficient to take $f = \frac{1}{\gamma} u$ and $\phi = \{I\}$ in Theorem 3.1. □

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Authors' contributions
The authors have equitably contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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