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Fixed point theorems for some new nonlinear mappings in Hilbert spaces

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Abstract

In this paper, we introduced two new classes of nonlinear mappings in Hilbert spaces. These two classes of nonlinear mappings contain some important classes of nonlinear mappings, like nonexpansive mappings and nonspreading mappings. We prove fixed point theorems, ergodic theorems, demiclosed principles, and Ray's type theorem for these nonlinear mappings.
Next, we prove weak convergence theorems for Moudafi's iteration process for these nonlinear mappings. Finally, we give some important examples for these new nonlinear mappings.

Keywords: nonspreading mapping, fixed point, demiclosed principle, ergodic theorem, nonexpansive mapping

1 Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Then, a mapping $T : C \to C$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. The class of nonexpansive mappings is important, and there are many well-known results in the literatures. From literatures, we observe the following fixed point theorems for nonexpansive mappings in Hilbert spaces.

In 1965, Browder [1] gave the following demiclosed principle for nonexpansive mappings in Hilbert spaces.

Theorem 1.1. [1] Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of C into itself, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $Tw = w$.

In 1971, Pazy [2] gave the following fixed point theorems for nonexpansive mappings in Hilbert spaces.

Theorem 1.2. [2] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping. Then, $\{T^n x\}$ is a bounded sequence for some $x \in C$ if and only if $F(T) \neq \emptyset$.

In 1975, Baillon [3] gave the following nonlinear ergodic theorem in a Hilbert space.

Theorem 1.3. [3] Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping. Then, the following conditions are equivalent:

(i) $F(T) \neq \emptyset$;

(ii) for any $x \in C$, $S_n x := \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly to an element of C.

In fact, if $F(T) \neq \emptyset$, then $S_n x \rightharpoonup \lim_{n\to\infty} PT^n x$ for each $x \in C$, where P is the metric projection of H onto $F(T)$.

In 1980, Ray [4] gave the following result in a real Hilbert space.

Theorem 1.4. [4] Let C be a nonempty closed convex subset of a real Hilbert space H. Then, the following conditions are equivalent.

(i) Every nonexpansive mapping of C into itself has a fixed point in C;

(ii) C is bounded.

On the other hand, a mapping $T : C \to C$ is said to be firmly nonexpansive [5] if

$$||Tx - Ty||^2 \leq \langle x - y, Tx - Ty\rangle$$

for all $x, y \in C$, and it is an important example of nonexpansive mappings in a Hilbert space.

In 2008, Kohsaka and Takahashi [6] introduced nonspreading mapping and obtained a fixed point theorem for a single nonspreading mapping and a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces. A mapping $T : C \to C$ is called nonspreading [6] if

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2$$

for all $x, y \in C$. Kohsaka and Takahashi [6] extended Theorem 1.2 for nonspreading mapping in Hilbert spaces. In 2010, Takahashi [7] extended Ray's type theorem for nonspreading mapping in Hilbert spaces. Iemoto and Takahashi [8] also extended the demiclosed principles for nonspreading mappings. Recently, Takahashi and Yao [9] proved the following nonlinear ergodic theorem for nonspreading mappings in Hilbert spaces.

Furthermore, Takahashi and Yao [9] also introduced two nonlinear mappings in Hilbert spaces. A mapping $T : C \to C$ is called a *TJ*-1 mapping [9] if

$$2||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2$$

for all $x, y \in C$. A mapping $T : C \to C$ is called a *TJ*-2 [9] mapping if

$$3||Tx - Ty||^2 \leq 2||Tx - y||^2 + ||Ty - x||^2$$

for all $x, y \in C$. For these two nonlinear mappings, *TJ*-1 and *TJ*-2 mappings, Takahashi and Yao [9] also gave similar results to the above theorems.

Motivated by the above works, we introduce two nonlinear mappings in Hilbert spaces.

Definition 1.1. Let C be a nonempty closed convex subset of a Hilbert space H. We say $T : C \to C$ is an asymptotic nonspreading mapping if there exists two functions $\alpha : C \to [0, 2)$ and $\beta : C \to [0, k]$, $k < 2$, such that

(A1) $2||Tx\text{-}Ty||^2 \leq \alpha(x)||Tx\text{-}y||^2 + \beta(x)||Ty\text{-}x||^2$ for all $x, y \in C$;

(A2) $0 < \alpha(x) + \beta(x) \leq 2$ for all $x \in C$.

Remark 1.1. The class of asymptotic nonspreading mappings contains the class of nonspreading mappings and the class of *TJ*-2 mappings in a Hilbert space. Indeed, in Definition 1.1, we know that

(i) if $\alpha(x) = \beta(x) = 1$ for all $x \in C$, then T is a nonspreading mapping;
(ii) if $\alpha(x) = \frac{4}{3}$ and $\beta(x) = \frac{2}{3}$ for all $x \in C$, then T is a *TJ*-2 mapping.

Definition 1.2. Let C be a nonempty closed convex subset of a Hilbert space H. We say $T : C \to C$ is an asymptotic *TJ* mapping if there exists two functions $\alpha : C \to [0, 2]$ and $\beta : C \to [0, k]$, $k < 2$, such that

(B1) $2\|Tx - Ty\|^2 \leq \alpha(x)\|x - y\|^2 + \beta(x)\|Tx - y\|^2$ for all $x, y \in C$;

(B2) $\alpha(x) + \beta(x) \leq 2$ for all $x \in C$.

Remark 1.2. The class of asymptotic *TJ* mappings contains the class of *TJ*-1 mappings and the class of nonexpansive mappings in a Hilbert space. Indeed, in Definition 1.2, we know that

(i) if $\alpha(x) = 2$ and $\beta(x) = 0$ for each $x \in C$, then T is a nonexpansive mapping;
(ii) if $\alpha(x) = \beta(x) = 1$ for each $x \in C$, then T is a *TJ*-1 mapping.

On the other hand, the following iteration process is known as Mann's type iteration process [10] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval [0, 1].

In 2007, Moudafi [11] studied weak convergence theorems for two nonexpansive mappings T_1, T_2 of C into itself, where C is a closed convex subset of a Hilbert space H. They considered the following iterative process:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \beta_n T_1 x_n + (1 - \beta_n)T_2 x_n \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n \end{cases}$$

for all $n \in N$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] and $F(T_1) \cap F(T_2) \neq \emptyset$. In 2009, Iemoto and Takahashi [8] also considered this iterative procedure for T_1 is a nonexpansive mapping and T_2 is nonspreading mapping of C into itself.

Motivated by the works in [8,11], we also consider this iterative process for asymptotic nonspreading mappings and asymptotic *TJ* mappings.

In this paper, we study asymptotic nonspreading mappings and asymptotic *TJ* mappings. We prove fixed point theorems, ergodic theorems, demiclosed principles, and Ray's type theorem for asymptotic nonspreading mappings and asymptotic *TJ* mappings. Our results generalize recent results of [1-4,6-9]. Next, we prove weak convergence theorems for Moudafi's iteration process for asymptotic nonspraeding mappings and asymptotic *TJ* mappings. Finally, we give some important examples for these new nonlinear mappings.

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$, respectively. We denote the strongly convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [12], for each $x, y \in H$ and $\lambda \in 0[1]$, we have

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2.$$

Let ℓ^∞ be the Banach space of bounded sequences with the supremum norm. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = ||\ \mu\ || = 1$, where $e = (1, 1, 1, ...)$. For $x = (x_1, x_2, x_3, ...)$, the value $\mu(x)$ is also denoted by $\mu_n(x_n)$. A Banach limit on ℓ^∞ is an invariant mean, that is, $\mu_n(x_n) = \mu_n(x_{n+1})$. If μ is a Banach limit on ℓ^∞, then for $x = (x_1, x_2, x_3, ...) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n x_n \le \limsup_{n\to\infty} x_n.$$

In particular, if $x = (x_1, x_2, x_3, ...) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(x) = \mu_n x_n = a$. For details, we can refer [13].

Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a mapping, and let $F(T)$ denote the set of fixed points of T. A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $||x - Ty|| \le ||x - y||$ for all $x \in F(T)$ and $y \in C$. It is well known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is a closed and convex set [14]. Hence, if $T : C \to C$ is an asymptotic nonspreading mapping (resp., asymptotic TJ mapping) with $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping and this implies that $F(T)$ is a nonempty closed convex subset of C.

Proposition 2.1. Let C be a nonempty closed convex subset of a Hilbert space H. Let α, β be the same as in Definition 1.1. Then, $T : C \to C$ is an asymptotic nonspreading mapping if and only if

$$\begin{aligned}
&||Tx - Ty||^2 \\
&\le \frac{\alpha(x) - \beta(x)}{2-\beta(x)}\|Tx - x\|^2 + \frac{\alpha(x)\|x-y\|^2}{2-\beta(x)} + \frac{2\langle Tx - x, \alpha(x)(x-y) + \beta(x)(Ty - x)\rangle}{2-\beta(x)}
\end{aligned}$$

for all $x, y \in C$.

Proof. We have that for $x, y \in C$,

$$\begin{aligned}
&2||Tx - Ty||^2 \\
&\le \alpha(x)||Tx - y||^2 + \beta(x)||Ty - x||^2 \\
&= \alpha(x)||Tx - x||^2 + 2\alpha(x)\langle Tx - x, x - y\rangle + \alpha(x)||x - y||^2 \\
&+\beta(x)||Ty - Tx||^2 + 2\beta(x)\langle Ty - Tx, Tx - x\rangle + \beta(x)||Tx - x||^2 \\
&= (\alpha(x) + \beta(x))||Tx - x||^2 + \beta(x)||Ty - Tx||^2 + \alpha(x)||x - y||^2 \\
&+2\alpha(x)\langle Tx - x, x - y\rangle + 2\beta(x)\langle Ty - x + x - Tx, Tx - x\rangle \\
&= (\alpha(x) - \beta(x))||Tx - x||^2 + \beta(x)||Ty - Tx||^2 + \alpha(x)||x - y||^2 \\
&+\langle Tx - x, 2\alpha(x)(x - y) + 2\beta(x)(Ty - x)\rangle.
\end{aligned}$$

And this implies that

$$\begin{aligned}&||Tx - Ty||^2\\&\leq \frac{\alpha(x) - \beta(x)}{2 - \beta(x)}||Tx - x||^2 + \frac{\alpha(x)||x - y||^2}{2 - \beta(x)} + \frac{2\langle Tx - x, \alpha(x)(x - y) + \beta(x)(Ty - x)\rangle}{2 - \beta(x)}.\end{aligned}$$

Hence, the proof is completed. □

Remark 2.1. If $\alpha(x) = \beta(x) = 1$ for all $x \in C$, then Proposition 2.1 is reduced to Lemma 3.2 in [8].

In the sequel, we need the following lemmas as tools.

Lemma 2.1. [13] Let C be a nonempty closed convex subset of a Hilbert space H. Let P be the metric projection from H onto C. Then for each $x \in H$, we know that $\langle x - Px, Px - y\rangle \geq 0$ for all $y \in C$.

Lemma 2.2. [15] Let D be a nonempty closed convex subset of a real Hilbert space H. Let P be the matric projection of H onto D, and let $\{x_n\}_{n\in \mathbb{N}}$ in H. If $||x_{n+1} - u|| \leq ||x_n - u||$ for all $u \in D$ and $n \in \mathbb{N}$. Then, $\{Px_n\}$ converges strongly to an element of D.

Following the similar argument as in the proof of Theorem 3.1.5 [13], we get the following result.

Lemma 2.3. Let C be a nonempty closed convex subset of a real Hilbert space H, and let μ be a Banach limit. Let $\{x_n\}$ be a sequence with $x_n \rightharpoonup w$. If $x \neq w$, then $\mu_n||x_n - w|| < \mu_n||x_n - x||$ and $\mu_n||x_n - w||^2 < \mu_n||x_n - x||^2$.

Lemma 2.4. [9] Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^nx\}$ is bounded and

$$\mu_n||T^nx - Ty||^2 \leq \mu_n||T^nx - y||^2, \quad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

3 Main results

In this section, we study the fixed point theorems, ergodic theorems, demiclosed principles, and Ray's type theorems for asymptotic nonspreading mappings and for asymptotic TJ mappings in Hilbert spaces.

3.1: Fixed point theorems

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic nonspreading mapping. Then, the following conditions are equivalent.

(i) $\{T^nx\}$ is bounded for some $x \in C$;
(ii) $F(T) \neq \emptyset$.

Proof. In fact, we only need to show that (i) implies (ii). Let $x_0 = x$. For each $n \in \mathbb{N}$, let $x_n := Tx_{n-1}$. Clearly, $\{x_n\}$ is a bounded sequence. Then for each $z \in C$,

$$\begin{aligned}\mu_n||x_n - Tz||^2 &= \mu_n||x_{n+1} - Tz||^2\\&\leq \mu_n\left(\frac{\alpha(z)}{2}||Tz - x_n||^2 + \frac{\beta(z)}{2}||Tx_n - z||^2\right)\\&= \frac{\alpha(z)}{2}\mu_n||x_n - Tz||^2 + \frac{\beta(z)}{2}\mu_n||Tx_n - z||^2\\&= \frac{\alpha(z)}{2}\mu_n||x_n - Tz||^2 + \frac{\beta(z)}{2}\mu_n||x_n - z||^2.\end{aligned}$$

Hence,

$$\beta(z)\mu_n||x_n - Tz||^2 \le (2 - \alpha(z))\mu_n||x_n - Tz||^2 \le \beta(z)\mu_n||x_n - z||^2,$$

and this implies that $\mu_n||x_n - Tz||^2 \le \mu_n||x_n - z||^2$. By Lemma 2.4, $F(T) \neq \emptyset$. □

Since the class of asymptotic nonspreading mappings contains the class of nonspreading mappings, we get the following result by Theorem 3.1.

Corollary 3.1. [6] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonspreading mapping. Then, $\{T^n x\}$ is bounded for some $x \in C$ if and only if $F(T) \neq \emptyset$.

Theorem 3.2. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic *TJ* mapping. Then, the following conditions are equivalent.

(i) $\{T^n x\}$ is bounded for some $x \in C$;
(ii) $F(T) \neq \emptyset$.

Proof. In fact, we only need to show that (i) implies (ii). Let $x_0 = x$. For each $n \in \mathbb{N}$, let $x_n := Tx_{n-1}$. Clearly, $\{x_n\}$ is a bounded sequence. Then for each $z \in C$,

$$\begin{aligned}\mu_n||x_n - Tz||^2 &= \mu_n||Tx_n - Tz||^2 \\ &\le \mu_n\left(\frac{\alpha(z)}{2}||x_n - z||^2 + \frac{\beta(z)}{2}||Tz - x_n||^2\right) \\ &\le \frac{\alpha(z)}{2}\mu_n||x_n - z||^2 + \frac{\beta(z)}{2}\mu_n||x_n - Tz||^2.\end{aligned}$$

And this implies that

$$\frac{\alpha(z)}{2}\mu_n||x_n - Tz||^2 \le \left(1 - \frac{\beta(z)}{2}\right)\mu_n||x_n - Tz||^2 \le \frac{\alpha(z)}{2}\mu_n||x_n - z||^2.$$

Hence $\mu_n||x_n - Tz||^2 \le \mu_n||x_n - z||^2$. By Lemma 2.4, $F(T) \neq \emptyset$. □

Theorem 3.2 generalizes Theorem 1.2 since the class of asymptotic *TJ* mappings contains the class of nonexpansive mappings. By Theorems 3.1 and 3.2, we also get the following result as special cases, respectively.

Corollary 3.2. [9] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a *TJ*-2 mapping, i.e., $3||Tx - Ty||^2 \le 2||Tx - y||^2 + ||Ty - x||^2$ for all $x, y \in C$. Then, $\{T^n x\}$ is bounded for some $x \in C$ if and only if $F(T) \neq \emptyset$.

Corollary 3.3. [9] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a *TJ*-1 mapping, i.e., $2||Tx - Ty||^2 \le ||x - y||^2 + ||Tx - y||^2$ for all $x, y \in C$. Then, $\{T^n x\}$ is bounded for some $x \in C$ if and only if $F(T) \neq \emptyset$.

Theorem 3.3. Let C be a bounded closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic nonspreading mapping (respectively, asymptotic TJ mapping). Then, $F(T) \neq \emptyset$.

By Theorem 3.3, we also get the following well-known result.

Corollary 3.4. Let C be a nonempty bounded closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping. Then, $F(T) \neq \emptyset$.

3.2: Demiclosed principles

Lemma 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a mapping. Let $\{x_n\}$ be a bounded sequence in C with $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Then, $\mu_n||x_n - x||^2 = \mu_n||Tx_n - x||^2$ for each $x \in C$.

Proof. Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, $\{Tx_n\}$ is also a bounded sequence. For each $x \in C$ and $n \in \mathbb{N}$, we know that

$$|\langle Tx_n - x_n, x_n - x\rangle| \leq ||Tx_n - x_n|| \cdot ||x_n - x||.$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, we get $\lim_{n\to\infty} \langle Tx_n - x_n, x_n - x\rangle = 0$. Hence, for each $x \in C$, we know that

$$||Tx_n - x||^2 = ||Tx_n - x_n||^2 + 2\langle Tx_n - x_n, x_n - x\rangle + ||x_n - x||^2.$$

And this implies that $\mu_n||Tx_n - x||^2 = \mu_n||x_n - x||^2$ for each $x \in C$. □

Theorem 3.4. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic nonspreading mapping. Let $\{x_n\}$ be a sequence in C with $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $x_n \rightharpoonup w \in C$. Then, $Tw = w$.

Proof. Let $\phi : X \to [0, \infty)$ be defined by

$$\varphi(x) := \mu_n||x_n - x||^2$$

for each $x \in C$. Since $x_n \rightharpoonup w$, $\{x_n\}$ is a bounded sequence. Clearly, $\{Tx_n\}$ is a bounded sequence. By Lemma 3.1,

$$\mu_n||x_n - x||^2 = \mu_n||Tx_n - x||^2 \text{for each } x \in C.$$

Next, we want to show that $Tw = w$. If not, then $Tw \neq w$. By Lemma 2.3, $0 \leq \phi(w) < \phi(Tw)$, and

$$\begin{aligned}\mu_n||x_n - Tw||^2 &= \mu_n||Tx_n - Tw||^2\\ &\leq \mu_n\left(\frac{\alpha(w)}{2}||Tw - x_n||^2 + \frac{\beta(w)}{2}||Tx_n - w||^2\right)\\ &= \frac{\alpha(w)}{2}\mu_n||x_n - Tw||^2 + \frac{\beta(w)}{2}\mu_n||Tx_n - w||^2.\end{aligned}$$

Hence,

$$\beta(w)\varphi(Tw) \leq (2 - \alpha(w))\varphi(Tw) \leq \beta(w)\varphi(w).$$

If $\beta(w) > 0$, then $\phi(Tw) \leq \phi(w)$. And this leads to a contradiction. If $\beta(w) = 0$, then $\phi(Tw) = 0$. This leads to a contradiction. Therefore, $Tw = w$. □

Theorem 3.5. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic TJ mapping. Let $\{x_n\}$ be a sequence in C with $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and $x_n \rightharpoonup w \in C$. Then, $Tw = w$.

Proof. Let $\phi : X \to [0, \infty)$ be defined by

$$\varphi(x) := \mu_n||x_n - x||^2$$

for each $x \in C$. Since $x_n \rightharpoonup w$, $\{x_n\}$ is a bounded sequence. Clearly, $\{Tx_n\}$ is a bounded sequence. By Lemma 3.1,

$$\mu_n||x_n - x||^2 = \mu_n||Tx_n - x||^2 \text{for each } x \in C.$$

Next, we want to show that $Tw = w$. If not, then $0 \leq \phi(w) < \phi(Tw)$. Hence,

$$\begin{aligned}\mu_n||x_n - Tw||^2 &= \mu_n||Tx_n - Tw||^2 \\ &\leq \mu_n\left(\frac{\alpha(w)}{2}||x_n - w||^2 + \frac{\beta(w)}{2}||Tw - x_n||^2\right) \\ &\leq \frac{\alpha(w)}{2}\mu_n||x_n - w||^2 + \frac{\beta(w)}{2}\mu_n||x_n - Tw||^2.\end{aligned}$$

And this implies that

$$\left(1 - \frac{\beta(w)}{2}\right)\mu_n||x_n - Tw||^2 \leq \frac{\alpha(w)}{2}\mu_n||x_n - w||^2.$$

So, $\mu_n||x_n - Tw||^2 \leq \mu_n||x_n - w||^2 \leq \mu_n||x_n - Tw||^2$. And this leads to a contradiction. Therefore, $Tw = w$. □

Theorem 3.5 generalizes Theorem 1.1 since the class of asymptotic *TJ* mappings contains the class of nonexpansive mappings. Furthermore, we have the following results as special cases of Theorems 3.4 and 3.5, respectively.

Corollary 3.5. [8] Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonspreading mapping of C into itself, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $Tw = w$.

Corollary 3.6. [9] Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a *TJ*-1 mapping of C into itself, and let $\{x_n\}$ be a sequence in C. If $x_n \rightharpoonup w$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $Tw = w$.

3.3: Ergodic theorems

Theorem 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic nonspreading mapping. Let α and β be the same as in Definition 1.1. Suppose that $\alpha(x)/\beta(x) = r > 0$ for all $x \in C$. Then, the following conditions are equivalent.

(i) $F(T) \neq \emptyset$;

(ii) for any $x \in C$, $S_nx = \frac{1}{n}\sum_{k=0}^{n-1} T^kx$ converges weakly to an element in C.

In fact, if $F(T) \neq \emptyset$, then $S_nx \rightharpoonup \lim_{n\to\infty} PT^nx$ for each $x \in C$, where P is the metric projection of H onto $F(T)$.

Proof. (ii)) ⇒ (i): Take any $x \in C$ and let x be fixed. Then, $S_nx \rightharpoonup v$ for some $v \in C$. Then, $v \in F(T)$. Indeed, for any $y \in C$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}0 &\leq \alpha(T^{k-1}x)||T^kx - y||^2 + \beta(T^{k-1}x)||Ty - T^{k-1}x||^2 - 2||T^kx - Ty||^2 \\ &\leq \alpha(T^{k-1}x)\{||T^kx - Ty||^2 + 2\langle T^kx - Ty, Ty - y\rangle + ||Ty - y||^2\} \\ &\quad + \beta(T^{k-1}x)||Ty - T^{k-1}x||^2 - (\alpha(T^{k-1}x) + \beta(T^{k-1}x))||T^kx - Ty||^2 \\ &= \beta(T^{k-1}x)(||Ty - T^{k-1}x||^2 - ||T^kx - Ty||^2) + 2\alpha(T^{k-1}x)\langle T^kx - Ty, Ty - y\rangle \\ &\quad + \alpha(T^{k-1}x)||Ty - y||^2.\end{aligned}$$

Hence,

$$||T^k x - Ty||^2 - ||T^{k-1}x - Ty||^2 \leq 2r\langle T^k x - Ty, Ty - y\rangle + r||Ty - y||^2.$$

Summing up these inequalities with respect to $k = 1, 2, ..., n - 1$,

$$\begin{aligned}
&-||x - Ty||^2 \\
&\quad \leq \ ||T^{n-1}x - Ty||^2 - ||x - Ty||^2 \\
&\quad \leq (n-1)r||Ty - y||^2 + \ 2r\langle(\sum_{k=1}^{n-1} T^k x) - (n-1)Ty, Ty - y\rangle \\
&\quad = (n-1)r||Ty - y||^2 + \ 2r\langle nS_n x - x - (n-1)Ty, Ty - y\rangle.
\end{aligned}$$

Dividing this inequality by n, we have

$$\frac{-||x - Ty||^2}{n} \leq r||Ty - y||^2 + \ 2ry\left\langle S_n x - \frac{x}{n} - \frac{(n-1)Ty}{n}, Ty - y\right\rangle.$$

Letting $n \to \infty$, we obtain

$$0 \leq r||Ty - y||^2 + \ 2r\langle v - Ty, Ty - y\rangle.$$

Since y is any point of C and $r > 0$, let $y = v$ and this implies that $Tv = v$.

(i)⇒ (ii): Take any $x \in C$ and $u \in F(T)$, and let x and u be fixed. Since T is an asymptotic nonspreading mapping, $||T^n x - u|| \leq ||T^{n-1}x - u||$ for each $n \in \mathbb{N}$. By Lemma 2.2, $\{PT^n x\}$ converges strongly to an element p in $F(T)$. Then for each $n \in \mathbb{N}$,

$$||S_n x - u|| \ \leq \frac{1}{n}\sum_{k=0}^{n-1} ||T^k x - u|| \ \leq \ ||x - u||.$$

So, $\{S_n x\}$ is a bounded sequence. Hence, there exists a subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ and $v \in C$ such that $S_{n_i} x \rightharpoonup v$. As the above proof, $Tv = v$.

By Lemma 2.1, for each $k \in \mathbb{N}$, $\langle T^k x - PT^k x, PT^k x - u\rangle \geq 0$. And this implies that

$$\begin{aligned}
\langle T^k x - PT^k x, u - p\rangle &\leq \langle T^k x - PT^k x, PT^k x - p\rangle \\
&\leq ||T^k x - PT^k x|| \cdot ||PT^k x - p|| \\
&\leq ||T^k x - p|| \cdot ||PT^k x - p|| \\
&\leq ||x - p|| \cdot ||PT^k x - p||.
\end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing n, we have

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x, u - p\right\rangle \leq \frac{||x - p||}{n}\sum_{k=0}^{n-1} ||PT^k x - p||.$$

Since $S_{n_i} x \rightharpoonup v$ and $PT^k x \to p$, we get $\langle v - p, u - p\rangle \leq 0$. Since u is any point of $F(T)$, we know that $v = p$.

Furthermore, if $\{S_{n_j} x\}$ is a subsequence of $\{S_n x\}$ and $S_{n_j} \rightharpoonup w$, then $w = p$ by following the same argument as in the above proof. Therefore, $S_n x \rightharpoonup p = \lim_{n\to\infty} PT^n x$, and the proof is completed. □

Theorem 3.7. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an asymptotic TJ mapping. Let α and β be the same as in

Definition 1.2. Suppose that $\beta(x)/\alpha(x) = r > 0$ for all $x \in C$. Then, the following conditions are equivalent.

(i) $F(T) \neq \emptyset$;

(ii) for any $x \in C$, $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly to an element in C.

In fact, if $F(T) \neq \emptyset$, then $S_n x \rightharpoonup \lim_{n\to\infty} PT^n x$ for each $x \in C$, where P is the metric projection of H onto $F(T)$.

Proof. The proof of Theorem 3.7 is similar to the proof of Theorem 3.6, and we only need to show the following result.

Take any $x \in C$ and let x be fixed. Then, $S_n x \rightharpoonup v$ for some $v \in C$. Then, $v \in F(T)$. Indeed, for any $y \in C$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \alpha(T^{k-1}x)||T^{k-1}x - y||^2 + \beta(T^{k-1}x)||T^k x - y||^2 - 2||T^k x - Ty||^2 \\ &= \alpha(T^{k-1}x)||T^{k-1}x - y||^2 + \beta(T^{k-1}x)||T^k x - Ty||^2 + 2\beta(T^{k-1}x)\langle T^k x - Ty, Ty - y\rangle \\ &\quad + \beta(T^{k-1}x)||Ty - y||^2 - 2||T^k x - Ty||^2 \\ &\leq \alpha(T^{k-1}x)(||T^{k-1}x - y||^2 - ||T^k x - Ty||^2) + 2\beta(T^{k-1}x)\langle T^k x - Ty, Ty - y\rangle \\ &\quad + \beta(T^{k-1}x)||Ty - y||^2. \end{aligned}$$

And this implies that

$$||T^k x - Ty||^2 - ||T^{k-1}x - Ty||^2 \leq 2r\langle T^k x - Ty, Ty - y\rangle + r||Ty - y||^2.$$

And following the same argument as the proof of Theorem 3.6, we get Theorem 3.7. □

By Theorems 3.6 and 3.7, we get the following result.

Corollary 3.7. [9,16] Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be any one of nonspreading mapping, *TJ*-1 mapping, and *TJ*-2 mapping. Then, the following conditions are equivalent.

(i) $F(T) \neq \emptyset$;

(ii) for any $x \in C$, $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges weakly to an element in C.

In fact, if $F(T) \neq \emptyset$, then $S_n x \rightharpoonup \lim_{n\to\infty} PT^n x$ for each $x \in C$, where P is the metric projection of H onto $F(T)$.

3.4 Ray's type theorems

Theorem 3.8. Let C be a nonempty closed convex subset of a real Hilbert space H. Then, the following conditions are equivalent.

(i) Every asymptotic *TJ* mapping of C into itself has a fixed point in C;

(ii) C is bounded.

Proof. (i)⇒ (ii): Suppose that every asymptotic *TJ* mapping of C into itself has a fixed point in C. Since the class of asymptotic *TJ* mappings contains the class of nonexpansive mappings, every nonexpansive mapping of C into itself has a fixed point in C. By Theorem 1.4, C is bounded. Conversely, by Theorem 3.3, it is easy to show that (ii) ⇒ (i). □

By Theorem 4.9 in [7] and Theorem 3.3, we get the following result.

Theorem 3.9. Let C be a nonempty closed convex subset of a real Hilbert space H. Then, C is bounded if and only if every asymptotic nonspreading mapping of C into itself has a fixed point in C.

3.5 Common fixed point theorems

Following the similar argument as the proof of Lemma 4.5 in [6], we get the following results. For details, we give the proof of Theorem 3.10.

Theorem 3.10. Let C be a nonempty bounded closed convex subset of a real Hilbert space H, and let $\{T_1, T_2, ..., T_N\}$ be a commutative finite family of asymptotic nonspreading mappings from C into itself. Then, $\{T_1, T_2, ..., T_N\}$ has a common fixed point.

Proof. The proof is given by induction with respect to N. We first show the case that $N = 2$. By Theorem 3.3, $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$. Furthermore, $F(T_1)$ and $F(T_2)$ are bounded closed convex subsets of C. Furthermore, $T_2(F(T_1)) \subseteq F(T_1)$. Indeed, if $u \in F(T_1)$, then $T_1T_2u = T_2T_1u = T_2u$. Hence, $T_2u \in F(T_1)$, and this implies that $T_2(F(T_1)) \subseteq F(T_1)$. Let $T_2' : F(T_1) \to F(T_1)$ be defined by $T_2'(x) := T_2(x)$ for each $x \in F(T_1)$. Clearly, $T_2' : F(T_1) \to F(T_1)$ is a asymptotic nonspreading mapping. By Theorem 3.3 again, there exists $\bar{x} \in F(T_1)$ such that $\bar{x} = T_2'(\bar{x}) = T_2(\bar{x})$. So, $\bar{x} \in F(T_1) \cap F(T_2)$.

Suppose that for some $n \geq 2$, $X = \cap_{k=1}^{n} F(T_k) \neq \emptyset$. Then, X is a nonempty bounded closed convex subset of C. Let $T_{n+1}' : X \to X$be defined by $T_{n+1}'(x) = T_{n+1}(x)$ for each $x \in X$. Clearly, T_{n+1}' is an asymptotic nonspreading mapping. By Theorem 3.3 again, we know that $X \cap F(T_{n+1}) \neq \emptyset$. That is, $\cap_{k=1}^{n+1} F(T_k) \neq \emptyset$. And the proof is completed. □

Corollary 3.8. [6] Let C be a nonempty bounded closed convex subset of a real Hilbert space H, and let $\{T_1, T_2, ..., T_N\}$ be a commutative finite family of non-spreading mappings from C into itself. Then, $\{T_1, T_2, ..., T_N\}$ has a common fixed point.

Theorem 3.11. Let C be a nonempty bounded closed convex subset of a real Hilbert space H, and let $\{T_1, T_2, ..., T_N\}$ be a commutative finite family of asymptotic TJ mappings from C into itself. Then, $\{T_1, T_2, ..., T_N\}$ has a common fixed point.

4 Weak convergence theorem for common fixed point

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T_i : C \to C$, $i = 1, 2$, be any one of asymptotic nonspreading mapping and asymptotic TJ mapping. Let $\Im = F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in (0, 1). Let $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := a_n x_n + (1 - a_n)(b_n T_1 x_n + (1 - b_n) T_2 x_n). \end{cases}$$

Assume that $\liminf_{n\to\infty} a_n(1 - a_n) > 0$ and $\liminf_{n\to\infty} b_n(1 - b_n) > 0$. Then, $x_n \rightharpoonup w$ for some $w \in \Im$.

Proof. Take any $w \in \Im$ and let w be fixed. Then for each $n \in \mathbb{N}$, we have $||T_i x_n - w|| \leq ||x_n - w||$ for each $n \in \mathbb{N}$ and $i = 1, 2$. Hence,

$$\begin{aligned}
&||b_nT_1x_n + (1-b_n)T_2x_n - w||^2 \\
&\quad = b_n||T_1x_n - w||^2 + (1-b_n)||T_2x_n - w||^2 - b_n(1-b_n)||T_1x_n - T_2x_n||^2 \\
&\quad \leq b_n||T_1x_n - w||^2 + (1-b_n)||T_2x_n - w||^2 \\
&\quad \leq \ ||x_n - w||^2,
\end{aligned}$$

and

$$\begin{aligned}
&||x_{n+1} - w||^2 \\
&\quad = \ ||a_nx_n + (1-a_n)(b_nT_1x_n + (1-b_n)T_2x_n) - w||^2 \\
&\quad \leq a_n||x_n - w||^2 + \ (1-a_n)||b_nT_1x_n + (1-b_n)T_2x_n - w||^2 \\
&\quad -a_n(1-a_n)||(b_nT_1x_n + (1-b_n)T_2x_n) - x_n||^2 \\
&\quad \leq (1-a_n)||x_n - w||^2 + a_n||x_n - w||^2 - a_n(1-a_n)||(b_nT_1x_n + (1-b_n)T_2x_n) - x_n||^2 \\
&\quad = \ ||x_n - w||^2 - a_n(1-a_n)||(b_nT_1x_n + (1-b_n)T_2x_n) - x_n||^2.
\end{aligned}$$

Hence, $\{||x_n - w||\}$ is a nonincreasing sequence, and $\lim_{n\to\infty} ||x_n - w||$ exists. Besides, we know that

$$a_n(1-a_n)||(b_nT_1x_n + (1-b_n)T_2x_n) - x_n||^2 \leq \ ||x_n - w||^2 - ||x_{n+1} - w||^2.$$

And this implies that $\lim_{n\to\infty} ||(b_nT_1x_n + (1-b_n)T_2x_n) - x_n|| \ = 0$. Next, we also have

$$\begin{aligned}
||x_{n+1} - x_n|| \ &= \ ||a_nx_n + (1-a_n)(b_nT_1x_n + (1-b_n)T_2x_n) - x_n|| \\
&= (1-a_n)||b_nT_1x_n + (1-b_n)T_2x_n - x_n||,
\end{aligned}$$

and this implies that $\lim_{n\to\infty} ||x_{n+1} - x_n|| \ = 0$. Besides, we get:

$$\begin{aligned}
&b_n(1-b_n)||T_1x_n - T_2x_n||^2 \\
&\quad \leq \ ||x_n - w||^2 - ||b_nT_1x_n + (1-b_n)T_2x_n - w||^2 \\
&\quad \leq M(||x_n - w|| - ||b_nT_1x_n + (1-b_n)T_2x_n - w||) \\
&\quad \leq M||(x_n - w) - (b_nT_1x_n + (1-b_n)T_2x_n - w)|| \\
&\quad = M||b_nT_1x_n + (1-b_n)T_2x_n - x_n||.
\end{aligned}$$

Then $\lim_{n\to\infty} b_n(1-b_n)||T_1x_n - T_2x_n||^2 \ = 0$. Since $\liminf_{n\to\infty} b_n(1-b_n) > 0$, we get $\lim_{n\to\infty} ||T_1x_n - T_2x_n|| \ = 0$. We have that

$$\begin{aligned}
&||x_{n+1} - T_1x_n|| \\
&\quad = \ ||a_nx_n + (1-a_n)(b_nT_1x_n + (1-b_n)T_2x_n) - T_1x_n|| \\
&\quad = \ ||a_n(x_n - T_1x_n) + (1-a_n)(b_nT_1x_n + (1-b_n)T_2x_n - T_1x_n)|| \\
&\quad = \ ||a_n(x_n - T_1x_n) + (1-a_n)(1-b_n)(T_2x_n - T_1x_n)|| \\
&\quad \leq a_n||x_n - T_1x_n|| + (1-a_n)(1-b_n)||T_2x_n - T_1x_n|| \\
&\quad \leq a_n||x_n - x_{n+1}|| + a_n||x_{n+1} - T_1x_n|| + (1-a_n)(1-b_n)||T_2x_n - T_1x_n||.
\end{aligned}$$

And this implies that

$$(1-a_n)||x_{n+1} - T_1x_n|| \ \leq a_n||x_n - x_{n+1}|| + \ (1-a_n)(1-b_n)||T_2x_n - T_1x_n||.$$

Hence,

$$a_n(1-a_n)||x_{n+1} - T_1x_n|| \ \leq \ ||x_n - x_{n+1}|| + ||T_2x_n - T_1x_n||.$$

So, $\lim_{n\to\infty} a_n(1-a_n)||x_{n+1}-T_1x_n|| = 0$. By assumption, $\lim_{n\to\infty} ||x_{n+1}-T_1x_n|| = 0$, and this implies that

$$\lim_{n\to\infty} ||x_n - T_1x_n|| = \lim_{n\to\infty} ||x_n - T_2x_n|| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w \in C$. By Theorems 3.4 and 3.5, $T_1w = T_2w = w$.

If x_{n_j} is a subsequence of $\{x_n\}$ and $x_{n_j} \rightharpoonup u$, then $T_1u = T_2u = u$. Suppose that $u \neq w$. Then, we have:

$$\begin{aligned}
\liminf_{k\to\infty} ||x_{n_k} - w|| &< \liminf_{k\to\infty} ||x_{n_k} - u|| \\
&= \lim_{n\to\infty} ||x_n - u|| \\
&= \lim_{j\to\infty} ||x_{n_j} - u|| \\
&< \liminf_{j\to\infty} ||x_{n_j} - w|| \\
&= \lim_{n\to\infty} ||x_n - w|| = \liminf_{k\to\infty} ||x_{n_k} - w||.
\end{aligned}$$

And this leads to a contradiction. Then, $x_n \rightharpoonup w$, and the proof is completed. □

Remark 4.1. Theorem 4.1 generalizes Theorem 4.1 (iii) in [8]. Similarly, the following corollary generalizes Corollary 4.1 in [8].

Corollary 4.1. Let C be a closed convex subset of a real Hilbert space H, and let $T: C \to C$ be any one of asymptotic nonspreading mapping and asymptotic TJ mapping. Suppose that $F(T) \neq \emptyset$. Let $\{a_n\}$ be a sequence in (0, 1). Let $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := (1-a_n)x_n + a_nTx_n. \end{cases}$$

If $\liminf_{n\to\infty} a_n(1-a_n) > 0$, then $x_n \rightharpoonup w$ for some $w \in F(T)$.

Proof. Let $T_1, T_2 : C \to C$ be defined by $T_1x = T_2x = Tx$ for each $x \in C$, and let $b_n = 1/2$ for each $n \in \mathbb{N}$. Then, Corollary 4.1 follows from Theorem 4.1. □

5 Examples

The following example shows that T is an asymptotic nonspreading mapping. But T is not a nonspreading mapping and not a TJ-2 mapping.

Example 5.1. Let $H = \mathbb{R}$, $C := [0, \infty)$, and let $T : C \to C$ be defined by

$$T(x) := \begin{cases} 0.8 & \text{if } 1 \leq x, \\ 0 & \text{if } 0 \leq x < 1, \end{cases}$$

for each $x \in \mathbb{R}$. Then, T is not a nonspreading mapping. Indeed, if $x = 1.1$ and $y = 0.6$, then

$$2||Tx - Ty||^2 = 1.28 > 1.25 = 0.04 + 1.21 = ||Tx - y||^2 + ||Ty - x||^2.$$

Furthermore, T is not a TJ-2 mapping. Indeed, if $x = 1.1$ and $y = 0.6$, then

$$2||Tx - Ty||^2 = 1.28 > 0.86 = \frac{4}{3} \times 0.04 + \frac{2}{3} \times 1.21 = \frac{4}{3}||Tx - y||^2 + \frac{2}{3}||Ty - x||^2.$$

However, T is an asymptotic nonspreading mapping. Indeed, let $\alpha : C \to [0, 2)$ and $\beta : C \to [0, 1.9)$ be defined by

$$\alpha(x) := \begin{cases} 0 & \text{if } 1 \leq x, \\ 1.28 & \text{if } 0 \leq x < 1, \end{cases}$$

and

$$\beta(x) := \begin{cases} 1.28 & \text{if } 1 \leq x, \\ 0 & \text{if } 0 \leq x < 1, \end{cases}$$

Now, we only need to consider the following two cases.

(a) If $x \geq 1$ and $0 \leq y < 1$, then $\alpha(x) = 0$, $\beta(x) = 1.28$, and

$$2||Tx - Ty||^2 = 1.28 \leq \beta(x) \cdot x^2 = \alpha(x)||Tx - y||^2 + \beta(x)||Ty - x||^2.$$

(b) If $0 \leq x < 1$ and $y \geq 1$, then $\alpha(x) = 1.28$, $\beta(x) = 0$, and

$$2||Tx - Ty||^2 = 1.28 \leq \alpha(x) \cdot y^2 = \alpha(x)||Tx - y||^2 + \beta(x)||Ty - x||^2.$$

Therefore, T is an asymptotic nonspreading mapping. □

Remark 5.1. Example 5.1 can be applied to demonstrate Theorems 3.1, 3.3, 3.4, and Corollary 4.1.

The following example shows that T is an asymptotic TJ mapping, but T is not a nonexpansive mapping.

Example 5.2. Let $H = \mathbb{R}$, $C := [0, 3]$, and let $T : C \to C$ be defined by

$$T(x) := \begin{cases} 0 & \text{if } x \neq 3, \\ 1 & \text{if } x = 3, \end{cases}$$

for each $x \in \mathbb{R}$. Then T is not a nonexpansive mapping. Indeed, if $x = 3$ and $y = 2.9$, then

$$||Tx - Ty||^2 = 1 > 0.01 = ||x - y||^2.$$

However, T is an asymptotic TJ mapping. Indeed, let $\alpha : C \to [0, 2)$ and $\beta : C \to [0, 1.9)$ be defined by

$$\alpha(x) := \begin{cases} 0 & \text{if } x \neq 3, \\ 1 & \text{if } x = 3, \end{cases}$$

and

$$\beta(x) := \begin{cases} \frac{1}{3} & \text{if } x \neq 3, \\ 1 & \text{if } x = 3, \end{cases}$$

Now, we only need to consider the following two cases.

(a) If $x \neq 3$ and $y = 3$, then $\alpha(x) = 0$, $\beta(x) = \frac{1}{3}$, and

$$2||Tx - Ty||^2 = 2 < 3 = \frac{1}{3} \times 9 = \alpha(x)||x - y||^2 + \beta(x)||Tx - y||^2.$$

(b) If $x = 3$ and $y \neq 3$, then $\alpha(x) = 1$, $\beta(x) = 1$, and

$$\begin{aligned}\alpha(x)||x - y||^2 + \beta(x)||Tx - y||^2 &= (3 - y)^2 + (1 - y)^2 \\ &= (y^2 - 6y + 9) + (y^2 - 2y + 1) \\ &= 2(y - 2)^2 + 2 \\ &\geq 2||Tx - Ty||^2.\end{aligned}$$

Therefore, T is an asymptotic TJ mapping. Note that T is a TJ-1 mapping. □

Remark 5.2. Example 5.2 can be applied to demonstrate Theorems 3.2, 3.3, 3.5, and Corollary 4.1. Furthermore, Examples 5.1 and 5.2 can also be applied to demonstrate Theorem 4.1.

6 Competing interests

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7 Authors' contributions

LJL: Problem resign, coordinator, discussion, revise the important part, and submit CSC: Responsible for the important results of asymptotic nonspreading mappings and asymptotic TJ mapping, discuss, draft. ZTY: responsible for giving the examples of this types of problems, discussion.

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