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Fixed point theory for multivalued ϕ-contractions

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Abstract

The purpose of this paper is to present a fixed point theory for multivalued ϕ-contractions using the following concepts: fixed points, strict fixed points, periodic points, strict periodic points, multivalued Picard and weakly Picard operators; data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, limit shadowing property of a multivalued operator, set-to-set operatorial equations and fractal operators. Our results generalize some recent theorems given in Petruşel and Rus (The theory of a metric fixed point theorem for multivalued operators, Proc. Ninth International Conference on Fixed Point Theory and its Applications, Changhua, Taiwan, July 16-22, 2009, 161-175, 2010).

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1 Introduction

Let X be a nonempty set. Then, we denote

$$P(X) := \{Y \subset X | Y \neq \emptyset\},\ P_{cl}(X) := \{Y \in P(X) | Y \text{ is closed}\}.$$

If $T : Y \subseteq X \to P(X)$ is a multivalued operator, then $F_T := \{x \in Y \mid x \in T(x)\}$ denotes the fixed point set T, while $(S\,F)_T := \{x \in Y \mid \{x\} = T(x)\}$ is the strict fixed point set of T.

Recall now two important notions, see [1] for details. A mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\phi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\phi(t) < t$, for each $t > 0$, $\phi(0) = 0$ and ϕ is continuous in 0.

A comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ having the property that $t - \phi(t) \to +\infty$, as $t \to +\infty$ is said to be a strict comparison function.

Moreover, a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strong comparison function if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$.

If (X, d) is a metric space, then we denote by H the Pompeiu-Hausdorff generalized metric on $P_{cl}(X)$. Then, $T : X \to P_{cl}(X)$ is called a multivalued ϕ-contraction, if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function, and for all $x_1, x_2 \in X$, we have that

$$H(T(x_1), T(x_2)) \leq \varphi(d(x_1, x_2)).$$

The purpose of this paper is to present a fixed point theory for multivalued ϕ-contractions in terms of the following:

- fixed points, strict fixed points, periodic points ([2-17]);
- multivalued weakly Picard operators ([18]);
- multivalued Picard operators ([19]);
- data dependence of the fixed point set ([18,20-22]);
- sequence of multivalued operators and fixed points ([23,24]);
- Ulam-Hyers stability of a multivalued fixed point equation ([25]);
- well-posedness of the fixed point problem ([26,27]);
- limit shadowing property of a multivalued operator ([28]);
- set-to-set operatorial equations ([29-31]);
- fractal operators ([32-40]).

2 Notations and basic concepts

Throughout this paper, the standard notations and terminologies in non-linear analysis are used, see for example Kirk and Sims [41], Petruşel [42], Rus et al. [18,43]. See also [44-52].

Let X be a nonempty set. Then, we denote

$$\mathcal{P}(X) := \{Y | Y \text{ is a subset of } X\}, \quad P(X) := \{Y \in \mathcal{P}(X) | Y \text{ is nonempty}\}.$$

Let (X, d) be a metric space. Then $\delta(Y) := \sup \{d(a, b) | \ a, b \in Y\}$ and

$$P_b(X) := \{Y \in P(X) | \delta(Y) < +\infty\}, \quad P_{cl}(X) := \{Y \in P(X) | Y \text{ is closed}\},$$
$$P_{cp}(X) := \{Y \in P(X) | Y \text{ is compact}\}, \quad P_{op}(X) := \{Y \in P(X) | Y \text{ is open}\}.$$

Let $T : X \to P(X)$ be a multivalued operator. Then, the operator $\hat{T} : P(X) \to P(X)$defined by

$$\hat{T}(Y) := \bigcup_{x \in Y} T(x), \quad \text{for } Y \in P(X)$$

is called the fractal operator generated by T.

For the continuity of concepts with respect to multivalued operators, we refer to [44,45], etc.

It is known that if (X, d) is a metric spaces and $T : X \to P_{cp}(X)$, then the following conclusions hold:

(a) if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;

(b) the continuity of T implies the continuity of $\hat{T} : P_{cp}(X) \to P_{cp}(X)$. A sequence of successive approximations of T starting from $x \in X$ is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in X with $x_0 = x$, $x_{n+1} \in T(x_n)$, for $n \in \mathbb{N}$.

If $T : Y \subseteq X \to P(X)$, then $F_T := \{x \in Y \mid x \in T(x)\}$ denotes the fixed point set T, while $(SF)_T := \{x \in Y \mid \{x\} = T(x)\}$ is the strict fixed point set of T. By $Graph(T) := \{(x, y) \in Y \times \times : y \in T(x)\}$, we denote the graphic of the multivalued operator T.

If $T : X \to P(X)$, then $T^0 := 1_X$, $T^1 := T, \ldots, T^{n+1} = T \circ T^n$, $n \in \mathbb{N}$ denote the iterate operators of T.

By definition, a periodic point for a multivalued operator $T : X \to P_{cp}(X)$ is an element $p \in X$ such that $p \in F_{T^m}$, for some integer $m \geq 1$, i.e., $p \in \hat{T}^m(\{p\})$ for some integer $m \geq 1$.

The following (generalized) functionals are used in the main sections of the paper.

The gap functional

$$(1)\ D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D(A,B) = \begin{cases} \inf\{d(a,b) | a \in A,\ b \in B\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{otherwise} \end{cases}$$

The excess generalized functional

$$(2)\ \rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\rho(A,B) = \begin{cases} \sup\{D(a,B) | a \in A\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset \\ +\infty, & B = \emptyset \neq A \end{cases}$$

The Pompeiu-Hausdorff generalized functional.

$$(3)\ H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$H(A,B) = \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \neq B \\ 0, & A = \emptyset = B \\ +\infty, & \text{otherwise} \end{cases}$$

For other details and basic results concerning the above notions, see, for example, [2,41,44-50].

We recall now the notion of multivalued weakly Picard operator.

Definition 2.1. (Rus et al. [18]) Let (X, d) be a metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that:

(i) $x_0 = x$, $x_1 = y$;
(ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of T.

Definition 2.2. Let (X, d) be a metric space and $T : X \to P(X)$ be a MWP operator. Then, we define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x, y) = \{ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x, y) that converges to $z \}$.

Definition 2.3. Let (X, d) be a metric space and $T : X \to P(X)$ a MWP operator. Then, T is said to be a ψ-multivalued weakly Picard operator (briefly ψ-MWP operator) if and only if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous in $t = 0$ and increasing function such that $\psi(0) = 0$, and there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x, y)) \leq \psi(d(x, y)), \quad \text{for all } (x, y) \in Graph\ (T).$$

In particular, if $\psi(t) := ct$, for each $t \in \mathbb{R}_+$ (for some $c > 0$), then T is called c-MWP operator, see Petrușel and Rus [26]. See also [53,54].

We recall now the notion of multivalued Picard operator.

Definition 2.4. Let (X, d) be a complete metric space and $T : X \to P(X)$. By definition, T is called a multivalued Picard operator (briefly MP operator) if and only if:

(i) $(SF)_T = F_T = \{x^*\}$;
(ii) $T^n(x) \xrightarrow{H} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

For basic notions and results on the theory of weakly Picard and Picard operators, see [42,43,53,54].

The following lemmas will be useful for the proof of the main results.

Lemma 2.5. ([1,18]) *Let (X, d) be a metric space and $A, B \in P_{cl}(X)$. Suppose that there exists $\eta > 0$ such that for each $a \in A$ there exists $b \in B$ such that $d(a, b) \le \eta$] and for each $b \in B$ there exists $a \in A$ such that $d(a, b) \le \eta$]. Then, $H(A, B) \le \eta$.*

Lemma 2.6. ([1,18]) *Let (X, d) be a metric space and $A, B \in P_{cl}(X)$. Then, for each $q > 1$ and for each $a \in A$ there exists $b \in B$ such that $d(a, b) < qH(A, B)$.*

Lemma 2.7. (Generalized Cauchy's Lemma) (Rus and Şerban [55]) *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function and $(b_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, such that $\lim_{n\to+\infty} b_n = 0$. Then,*

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \varphi^{n-k}(b_k) = 0.$$

The following result is known in the literature as Matkowski-Rus's theorem (see [1]).

Theorem 2.8 *Let (X, d) be a complete metric space and $f : X \to \times$ be a ϕ-contraction, i.e., $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and*

$$d(f(x), f(y)) \le \varphi(d(x, y)) \quad \text{for all } x,\ y \in X.$$

Then f is a Picard operator, i.e., f has a unique fixed point $x^* \in X$ and $\lim_{n\to+\infty} f^n(x) = x^*$, for all $\times \in X$.

Finally, let us recall the concept of H-convergence for sets. Let (X, d) be a metric space and $(A_n)_{n\in\mathbb{N}}$ be a sequence in $P_{cl}(X)$. By definition, we will write $A_n \xrightarrow{H} A^* \in P_{cl}(X)$ as $n \to \infty$ if and only if $H(A_n, A^*) \to 0$ as $n \to \infty$.

3 A fixed point theory for multivalued generalized contractions

Our first result concerns the case of multivalued ϕ-contractions.

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued ϕ-contraction. Then, we have:*

(i) (Existence of the fixed point) T is a MWP operator;
(ii) If additionally $\phi(qt) \le q\phi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$, then T is a ψ-MWP operator, with $\psi(t) := t + s(t)$, for each $t \in \mathbb{R}_+$ (where $s(t) := \sum_{n=1}^{\infty} \varphi^n(t)$);
(iii) (Data dependence of the fixed point set) Let $S : X \to P_{cl}(X)$ be a multivalued ϕ-contraction and $\eta > 0$ be such that $H(S(x), T(x)) \le \eta$, for each $\times \in X$. Suppose that $\phi(qt) \le q\phi(t)$ for every $t \in \mathbb{R}_+$ (where $q > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$. Then, $H(F_S, F_T) \le \psi(\eta)$;

(iv) (sequence of operators) Let T, $T_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be multivalued ϕ-contractions such that $T_n(x) \xrightarrow{H} T(x)$ as $n \to +\infty$, uniformly with respect to each $x \in X$. Then, $F_{T_n} \xrightarrow{H} F_T$ as $n \to +\infty$.

If, moreover $T(x) \in P_{cp}(X)$, for each $x \in X$, then we additionally have:

(v) (generalized Ulam-Hyers stability of the inclusion $x \in T(x)$) Let $\varepsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \varepsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \psi(\varepsilon)$;
(vi) T is upper semicontinuous, $\hat{T} : (P_{cp}(X), H) \to (P_{cp}(X), H)$, $\hat{T}(Y) := \bigcup_{x \in Y} T(x)$ is a set-to-set ϕ-contraction and (thus) $F_{\hat{T}} = \{A_T^*\}$;
(vii) $T^n(x) \xrightarrow{H} A_T^*$ as $n \to +\infty$, for each $x \lfloor X$;
(viii) $F_T \subset A_T^*$ and F_T is compact;
(ix) $A_T^* = \bigcup_{n \in \mathbb{N}^*} T^n(x)$, for each $x \in F_T$.

Proof. (i) This is Węgrzyk's Theorem, see [56].

(ii) Let $x_0 \in X$ and $x_1 \in T(x_0)$ be arbitrarily chosen. We may suppose that $x_0 \neq x_1$. Denote $t_0 := d(x_0, x_1) > 0$. Then, for any $q > 1$ there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < qH(T(x_0), T(x_1)) \leq q\phi(t_0)$. We may again suppose that $x_1 \neq x_2$. Thus, $\phi(d(x_1, x_2)) < \phi(q\phi(t_0))$. Next, there exists $x_3 \in T(x_2)$ such that $T(x_2)) \leq \dfrac{\varphi(q\varphi(t_0))}{\varphi(d(x_1,x_2))}\varphi(d(x_1,x_2)) \leq q\varphi^2(t_0)$,
$T(x_2)) \leq \dfrac{\varphi(q\varphi(t_0))}{\varphi(d(x_1,x_2))}\varphi(d(x_1,x_2)) \leq q\varphi^2(t_0)$. By an inductive procedure, we obtain a sequence of successive approximations for T starting from $(x_0, x_1) \in Graph(T)$ such that

$$d(x_n, x_{n+1}) \leq q\varphi^n(t_0), \quad \text{for each } n \in \mathbb{N}^*.$$

Denote by

$$s_n(t) := \sum_{k=1}^{n} \varphi^k(t), \quad \text{for each } t > 0.$$

Then, $d(x_n, x_{n+p}) \leq q(\phi^n(t_0) + ... + \phi^{n+p-1}(t_0))$, for each $n, p \in \mathbb{N}^*$. If we set $s_0(t) := 0$ for each $t \in \mathbb{R}_+$, then

$$d(x_n, x_{n+p}) \leq q(s_{n+p-1}(t_0) - s_{n-1}(t_0)), \quad \text{for each } n,\ p \in \mathbb{N}^*. \tag{3.1}$$

By (3.1) we get that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence it is convergent in (X, d) to some $x^* \in X$. Notice that, by the ϕ-contraction condition, we immediately get that $Graph(T)$ is closed in $X \times X$. Hence, $x^* \in F_T$. Then, by (3.1) letting $p \to +\infty$, we obtain that

$$d(x_n, x^*) \leq q(s(t_0) - s_{n-1}(t_0)), \quad \text{for each } n \in \mathbb{N}^*. \tag{3.2}$$

If we put $n = 1$ in (3.2), we obtain that $d(x_1, x^*) \leq qs(t_0)$. Hence,

$$d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq t_0 + qs(t_0). \tag{3.3}$$

Finally, letting $q \searrow 1$ in (3.3), we get that

$$d(x_0, x^*) \leq t_0 + s(t_0) = \psi(t_0) = \psi(d(x_0, x_1)). \tag{3.4}$$

Notice that, ψ is increasing (since ϕ is), $\psi(0) = 0$ and, since $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$, ψ is continuous in $t = 0$.

These, together with (3.4), prove that T is a ψ-MWP operator.

(iii) Let $x_0 \in F_S$ be arbitrary chosen. Then, by (ii), we have that

$$d(x_0, t^{\infty}(x_0, x_1)) \leq \psi(d(x_0, x_1)), \quad \text{for each } x_1 \in T(x_0).o$$

Let $q > 1$ be arbitrary. Then, there exists $x_1 \in T(x_0)$ such that $d(x_0, x_1) < qH(S(x_0), T(x_0))$. Then

$$d(x_0, t^{\infty}(x_0, x_1)) \leq \psi(qH(S(x_0), T(x_0))) \leq q\psi(H(S(x_0), T(x_0))) \leq q\psi(\eta).$$

By a similar procedure we can prove that, for each $y_0 \in F_T$, there exists $y_1 \in S(y_0)$ such that

$$d(y_0, s^{\infty}(y_0, y_1)) \leq q\psi(\eta).$$

By the above relations and using Lemma 2.5, we obtain that

$$H(F_S, F_T) \leq q\psi(\eta), \quad \text{where } q > 1.$$

Letting $q \searrow 1$, we get the conclusion.

(iv) Let $\varepsilon > 0$. Since $T_n(x) \xrightarrow{H} T(x)$ as $n \to +\infty$, uniformly with respect to each $x \in X$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\sup_{x \in X} H(T_n(x), T(x)) < \varepsilon, \quad \text{for each } n \geq N_\varepsilon.$$

Then, by (iii) we get that $H(F_{T_n}, F_T) \leq \psi(\varepsilon)$, for each $n \geq N_\varepsilon$. Since ψ is continuous in 0 and $\psi(0) = 0$, we obtain that $F_{T_n} \xrightarrow{H} F_T$.

(v) Let $\varepsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \varepsilon$. Then, since $T(x)$ is compact, there exists $y \in T(x)$ such that $d(x, y) \leq \varepsilon$. By the proof of (i), we have that

$$d(x, t^{\infty}(x, y)) \leq \psi(d(x, y)).$$

Since $x^* := t^{\infty}(x, y) \in F_T$, we get the desired conclusion $d(x, x^*) \leq \psi(\varepsilon)$.

(vi) (Andres-Górniewicz [39], Chifu and Petruşel [40].) By the ϕ-contraction condition, one obtain that the operator T is H-upper semicontinuos. Since $T(x)$ is compact, for each $x \in X$, we know that T is upper semicontinuous if and only if T is H-upper semicontinuous. We will prove now that

$$H(T(A), T(B)) \leq \varphi(H(A, B)), \quad \text{for each } A,\ B \in P_{cp}(X).$$

For this purpose, let $A, B \in P_{cp}(X)$ and let $u \in T(A)$. Then, there exists $a \in A$ such that $u \in T(a)$. For $a \in A$, by the compactness of the sets A, B there exists $b \in B$ such that

$$d(a,b) \leq H(A,B). \tag{3.5}$$

Then, we have $D(u, T(B)) \leq D(u, T(b)) \leq H(T(a), T(b)) \leq \phi(d(a, b))$. Hence, by the above relation and by (3.5) we get

$$\rho(T(A), T(B)) \leq \varphi(d(a,b)) \leq \varphi(H(A,B)). \tag{3.6}$$

By a similar procedure, we obtain

$$\rho(T(B), T(A)) \leq \varphi(d(a,b)) \leq \varphi(H(A,B)). \tag{3.7}$$

Thus, (3.6) and (3.7) together imply that

$$H(T(A), T(B)) \leq \varphi(H(A,B)).$$

Hence, $\hat{T}$ is a self-ϕ-contraction on the complete metric space $(P_{cp}(X), H)$. By the ϕ-contraction principle for singlevalued operators (see Theorem 2.8), we obtain:

(a) $F_{\hat{T}} = \{A_T^*\}$

and

(b) $\hat{T}^n(A) \xrightarrow{H} A_T^*$ as $n \to +\infty$, for each $A \in P_{cp}(X)$.

(vii) By (vi)-(b) we get that $T^n(\{x\}) = \hat{T}^n(\{x\}) \xrightarrow{H} A_T^*$ as $n \to +\infty$, for each $x \in X$.

(viii)-(ix) (Chifu and Petruşel [40].) Let $x \in F_T$ be arbitrary. Then, $x \in T(x) \subset T^2(x) \subset \ldots \subset T^n(x) \subset \ldots$ Hence $x \in T^n(x)$, for each $n \in \mathbb{N}^*$. Moreover, $\lim\limits_{n\to+\infty} T^n(x) = \bigcup_{n\in\mathbb{N}*} T^n(x)$. By (vii), we immediately get that $A_T^* = \bigcup_{n\in\mathbb{N}*} T^n(x)$. Hence, $x \in \bigcup_{n\in\mathbb{N}*} T^n(x) = A_T^*$. The proof is complete. ■

A second result for multivalued ϕ-contractions is as follows.

Theorem 3.2. *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued ϕ-contraction with $(SF)_T \neq \emptyset$. Then, the following assertions hold:*

(x) $F_T = (SF)_T = \{x^*\}$;

(xi) If, additionally $T(x)$ is compact for each $\times \in X$, then $F_{T^n} = (SF)_{T^n} = \{x*\}$for $n \in \mathbb{N}^*$;

(xii) If, additionally $T(x)$ is compact for each $\times \in X$, then $T^n(x) \xrightarrow{H} \{x^*\}$as $n \to +\infty$, for each $x \in X$;

(xiii) Let $S : X \to P_{cl}(X)$ be a multivalued operator and $\eta > 0$ such that $F_S \neq \emptyset$ and $H(S(x), T(x)) \leq \eta$, for each $\times \in X$. Then, $H(F_S, F_T) \leq \beta(\eta)$, where $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $\beta(\eta) := \sup\{t \in \mathbb{R}_+ | \ t - \phi(t) \leq \eta\}$;

(xiv) Let $T_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be a sequence of multivalued operators such that $F_{T_n} \neq \emptyset$for each $n \in \mathbb{N}$ and $T_n(x) \xrightarrow{H} T(x)$as $n \to +\infty$, uniformly with respect to $\times \in X$. Then, $F_{T_n} \xrightarrow{H} F_T$as $n \to +\infty$.

(xv) (Well-posedness of the fixed point problem with respect to D) If $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\times$ such that $D(x_n, T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$as $n \to \infty$;

(xvi) (Well-posedness of the fixed point problem with respect to H) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in × such that $H(x_n, T(x_n)) \to 0$ as $n \to \infty$, then $x_n \xrightarrow{d} x^*$as $n \to \infty$;
(xvii) (Limit shadowing property of the multivalued operator) Suppose additionally that ϕ is a sub-additive function. If $(y_n)_{n\in\mathbb{N}}$ is a sequence in × such that $D(y_{n+1}, T(y_n)) \to 0$ as $n \to \infty$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations for T, such that $d(x_n, y_n) \to 0$ as $n \to \infty$.

Proof. (x) Let $x^* \in (SF)_T$. Notice first that $(SF)_T = \{x^*\}$. Indeed, if $y \in (SF)_T$ with $y \neq x^*$, then $d(x^*, y) = H(T(x^*), T(y)) \leq \phi(d(x^*, y))$. By the properties of ϕ, we immediately get that $y = x^*$. Suppose now that $y \in F_T$. Then,

$$d(x^*, y) = D(T(x^*), y) \leq H(T(x^*), T(y)) \leq \varphi(d(x^*, y)).$$

Thus, $y = x^*$. Hence, $F_T \subset (SF)_T$. Since $(SF)_T \subset F_T$, we get that $(SF)_T = F_T$.

(xi) Notice first that $x^* \in (SF)_{T^n} \subset F_{T^n}$, for each $n \in \mathbb{N}^*$. Consider $y \in (SF)_{T^n}$, for arbitrary $n \in \mathbb{N}^*$. Then, by (vi) we have that

$$d(x^*, y) = H(T^n(x^*), T^n(y)) \leq \varphi(H(T^{n-1}(x^*), T^{n-1}(y))) \leq \cdots \leq \varphi^n(d(x^*, y)).$$

Thus, $y = x^*$ and $(SF)_{T^n} = \{x^*\}$. Consider now $y \in F_{T^n}$. Then, we have

$$\begin{aligned} d(x^*, y) &= D(T^n(x^*), y) \leq H(T^n(x^*), T^n(y)) \\ &\leq \varphi(H(T^{n-1}(x^*), T^{n-1}(y))) \leq \cdots \leq \varphi^n(d(x^*, y)). \end{aligned}$$

Thus, $y = x^*$ and hence $T^n(x) \xrightarrow{H} \{x^*\}$.

(xii) Let $x \in X$ be arbitrarily chosen. Then, we have

$$\begin{aligned} &H(T^n(x), x^*) = H(T^n(x), T^n(x^*)) \leq \varphi(H(T^{n-1}(x), \\ &T^{n-1}(x^*))) \leq \cdots \leq \varphi(^n d(x, x^*)) \to 0 \text{ as } n \to +\infty. \end{aligned}$$

(xiii) Let $y \in F_S$. Then,

$$d(y, x^*) \leq H(S(y), x^*) \leq H(S(y), T(y)) + H(T(y), x^*) \leq \eta + \varphi(d(y, x^*)).$$

Thus, $d(y, x^*) \leq \beta(\eta)$. The conclusion follows now by the following relations

$$H(F_S, F_T) = \sup_{y\in F_S} d(y, x^*) \leq \beta(\eta).$$

(xiv) follows by (xiii).
(xv) ([26,27]) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(x_n, T(x_n)) \to 0$ as $n \to \infty$. Then,

$$\begin{aligned} d(x_n, x^*) &\leq D(x_n, T(x_n)) + H(T(x_n), T(x^*)) \\ &\leq D(x_n, T(x_n)) + \varphi(d(x_n, x^*)). \end{aligned}$$

Then

$$d(x_n, x^*) \leq \beta(D(x_n, T(x_n))) \to 0 \text{ as } n \to +\infty.$$

(xvi) follows by (xv).

(xvii) Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X such that $D(y_{n+1}, T(y_n)) \to 0$ as $n \to \infty$. Then, there exists $u_n \in T(y_n)$, $n \in \mathbb{N}$ such that $d(y_{n+1}, u_n) \to 0$ as $n \to +\infty$.

We shall prove that $d(y_n, x^*) \to 0$ as $n \to +\infty$. We successively have:

$$\begin{aligned} d(x^*, y_{n+1}) &\leq H(x^*, T(y_n)) + D(y_{n+1}, T(y_n)) \\ &\leq \varphi(d(x^*, y_n)) + D(y_{n+1}, T(y_n)) \\ &\leq \varphi(\varphi(d(x^*, y_{n-1})) + D(y_n, T(y_{n-1}))) + D(y_{n+1}, T(y_n)) \\ &\leq \varphi^2(d(x^*, y_{n-1})) + \varphi(D(y_n, T(y_{n-1}))) + D(y_{n+1}, T(y_n)) \\ &\leq \dots \leq \varphi^{n+1}(d(x^*, y_0)) + \varphi^n(D(y_1, T(y_0))) \\ &\quad + \dots + D(y_{n+1}, T(y_n)). \end{aligned}$$

By the generalized Cauchy's Lemma, the right-hand side tends to 0 as $n \to +\infty$. Thus, $d(x^*, y_{n+1}) \to 0$ as $n \to +\infty$.

On the other hand, by the proof of Theorem 3.1 (i)-(ii), we know that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from arbitrary $(x_0, x_1) \in Graph(T)$ which converge to a fixed point $x^* \in X$ of the operator T. Since the fixed point is unique, we get that $d(x_n, x^*) \to 0$ as $n \to +\infty$. Hence, for such a sequence $(x_n)_{n\in\mathbb{N}}$, we have

$$d(y_n, x_n) \leq d(y_n, x^*) + d(x^*, x_n) \to 0 \text{ as } n \to +\infty.$$

The proof is complete. ■

A third result for multivalued ϕ-contraction is the following.

Theorem 3.3. *Let (X, d) be a complete metric space and $T : X \to P_{cp}(X)$ be a multivalued ϕ-contraction such that $T(F_T) = F_T$. Then, we have:*

(xviii) $T^n(x) \xrightarrow{H} F_T$ as $n \to +\infty$, for each $x \in X$;

(xix) $T(x) = F_T$, for each $x \in F_T$;

(xx) If $(x_n)_{n\in\mathbb{N}} \subset X$ is a sequence such that $x_n \xrightarrow{d} x^* \in F_T$ as $n \to \infty$, then $T^n(x) \xrightarrow{H} F_T$ as $n \to +\infty$.

Proof. (xviii) By $T(F_T) = F_T$ and Theorem 3.1 (vi), we have that $F_T = A_T^*$. The conclusion follows by Theorem 3.1 (vii).

(xix) Let $x \in F_T$ be arbitrary. Then, $x \in T(x)$ and thus $F_T \subset T(x)$. On the other hand $T(x) \subset T(F_T) \subset F_T$. Thus, $T(x) = F_T$, for each $x \in F_T$.

(xx) Let $(x_n)_{n\in\mathbb{N}} \subset X$ is a sequence such that $x_n \xrightarrow{d} x^* \in F_T$ as $n \to +\infty$.

Then, we have:

$$H(T(x_n), F_T) = H(T(x_n), T(x^*)) \leq \varphi(d(x_n, x^*)) \to 0 \text{ as } n \to +\infty.$$

The proof is complete. ■

For compact metric spaces, we have:

Theorem 3.4. *Let* (X, d) *be a compact metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* ϕ*-contraction. Then, we have:*

(xxi) (Generalized well-posedness of the fixed point problem with respect to D) If $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* × *such that* $D(x_n, T(x_n)) \to 0$ *as* $n \to \infty$*, then there exists a subsequence* $(x_{n_i})_{i\in\mathbb{N}}$ *of* $(x_n)_{n\in\mathbb{N}} x_{n_i} \xrightarrow{d} x^* \in F_T$ *as* $i \to \infty$.

Proof. (xxi) Let $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n, T(x_n)) \to 0$ as $n \to \infty$. Let $(x_{n_i})_{i\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_i} \xrightarrow{d} x^*$ as $i \to \infty$. Then, there exists $y_{n_i} \in T(x_{n_i})$, $i \in \mathbb{N}$ such that $y_{n_i} \xrightarrow{d} x^*$ as $i \to \infty$. By the ϕ-contraction condition, we have that T has closed graph. Hence, $x^* \in F_T$. ■

Remark 3.1. For the particular case $\phi(t) = at$ (with $a \in [0, 1[$), for each $t \in \mathbb{R}_+$ see Petruşel and Rus [57].

Recall now that a self-multivalued operator $T : X \to P_{cl}(X)$ on a metric space (X, d) is called (ε, ϕ)-contraction if $\varepsilon > 0$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function and

$$x,\ y \in X \text{ with } x \neq y \text{ and } d(x, y) < \varepsilon \text{ implies } H(T(x), T(y)) \le \varphi(d(x, y)).$$

Then, for the case of periodic points we have the following results.

Theorem 3.5. *Let* (X, d) *be a metric space and* $T : X \to P_{cp}(X)$ *be a continuous* (ε, ϕ)*-contraction. Then, the following conclusions hold:*

(i) $\hat{T}^m : P_{cp}(X) \to P_{cp}(X)$ *is a continuous* (ε, ϕ)*-contraction, for each* $m \in \mathbb{N}^*$;
(ii) if, additionally, there exists some $A \in P_{cp}(X)$ *such that a sub-sequence* $(\hat{T}^m(A))_{m\in\mathbb{N}*}$ *of* $(\hat{T}^m(A))_{m\in\mathbb{N}*}$ *converges in* $(P_{cp}(X), H)$ *to some* $X^* \in P_{cp}(X)$*, then there exists* $x^* \in X^*$ *a periodic point for* T.

Proof. (i) By Theorem 3.1 (vi) we have that the operator $\hat{T}$ given by $\hat{T}(Y) := \bigcup_{x\in Y} T(x)$ maps $P_{cp}(X)$ to $P_{cp}(X)$ and it is continuous. By induction we get that $\hat{T}^m : P_{cp}(X) \to P_{cp}(X)$ and it is continuous. We will prove that $\hat{T}$ is a (ε, ϕ)-contraction., i.e., if $\varepsilon > 0$ and $A, B \in P_{cp}(X)$ are two distinct sets such that $H(A, B) < \varepsilon$, then $H(\hat{T}(A), \hat{T}(B)) \le \varphi(H(A, B))$. Notice first that, by the symmetry of the Pompoiu-Hausdorff metric we only need to prove that

$$\sup_{u\in\hat{T}(A)} D(u, \hat{T}(B)) \le \varphi(H(A, B)).$$

Let $u \in \hat{T}(A)$. Then, there exists $a_0 \in A$ such that $u \in T(a_0)$. It follows that

$$D(u, T(b)) \le H(T(a_0), T(b)), \quad \text{for every } b \in B.$$

Since $A, B \in P_{cp}(X)$, there exists $b_0 \in B$ such that $d(a_0, b_0) \le H(A, B) < \varepsilon$. Thus, by the (ε, ϕ)-contraction condition, we get

$$H(T(a_0), T(b_0)) \le \varphi(d(a_0, b_0)) \le \varphi(H(A, B)).$$

Hence

$$D(u, T(b)) \leq \varphi(H(A, B)).$$

Moreover, by the compactness of $\hat{T}(A)$ we get the conclusion, namely

$$\sup_{u \in \hat{T}(A)} D(u, \hat{T}(B)) \leq \varphi(H(A, B)).$$

For the case of arbitrary $m \in \mathbb{N}^*$, the proof of the fact that $\hat{T}^m$is a (ε, ϕ)-contraction easily follows by induction.

(ii) By (i) and the properties of the function ϕ, we get that $\hat{T}^m$ is an ε-contractive operator, i.e., if $\varepsilon > 0$ and $A, B \in P_{cp}(X)$ are two distinct sets such that $H(A, B) < \varepsilon$, then $H(\hat{T}^m(A), \hat{T}^m(B)) < H(A, B)$. Now the conclusion follows from Theorem 3.2 in [2]. ∎

Theorem 3.6. *Let (X, d) be a compact metric space and $T : X \to P_{cp}(X)$ be a continuous $(\varepsilon; \phi)$-contraction. Then, there exists $x^* \in X$ a periodic point for T.*

Proof. The conclusion follows by Theorem 3.5 (ii) and Corollary 3.3. in [2]. ∎

Remark 3.2. We also refer to [58,59] for some results of this type for multivalued operators of Reich's type.

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