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Connectedness of approximate solutions set for vector equilibrium problems in Hausdorff topological vector spaces

Bin Chen[1], Qing-you Liu[2], Zhi-bin Liu[3] and Nan-jing Huang[1*]

* Correspondence:
nanjinghuang@hotmail.com
[1]Department of Mathematics,
Sichuan University, Chengdu,
Sichuan 610064, P.R. China
Full list of author information is
available at the end of the article

Abstract

In this paper, a scalarization result of ε-weak efficient solution for a vector equilibrium problem (VEP) is given. Using this scalarization result, the connectedness of ε-weak efficient and ε-efficient solutions sets for the VEPs are proved under some suitable conditions in real Hausdorff topological vector spaces. The main results presented in this paper improve and generalize some known results in the literature.

Keywords: vector equilibrium problem, scalarization method, ε-weak efficient solution, ε-efficient solution, connectedness

1 Introduction

Let K be a nonempty subset of a real Hausdorff topological vector space E, and $f : K \times K \to R$ a bifunction such that $f(x, x) \geq 0$ for all $x \in K$. Then, the scalar equilibrium problem consists in finding $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K.$$

It provides a unifying framework for many important problems, such as optimization problems, variational inequality problems, complementary problems, minimax inequality problems, Nash equilibrium problems, and fixed point problems, and has been widely applied to study problems arising in economics, mechanics, and engineering science (see [1]). On the other hand, several operations research problems are formulated with a multicriteria consideration. These are vector optimization problems, vector variational inequality and complementarity problems and vector equilibrium problems (VEPs). Recently, the VEP has received much attention by many authors because it provides a unified model including vector optimization problems, vector variational inequality problems, vector complementarity problems and vector saddle point problems as special cases (see, for example, [2-24] and the references therein).

It is well known that another important problem for VEPs is to study the topological properties of solutions set. Among its topological properties, the connectedness is of interest. Recently, Lee et al. [25], Cheng [26] have studied the connectedness of weak efficient solutions set for vector variational inequalities in finite dimensional Euclidean space. Gong [27-29] has studied the connectedness of the various solutions set for VEPs in infinite dimension space. Chen et al. [30] studied the connectedness and the

compactness of the weak efficient solutions set for set-valued VEPs and the set-valued vector Hartman-Stampacchia variational inequalities in normed linear space. Gong and Yao [31] have studied the connectedness of the set of efficient solutions for generalized systems. Zhong et al. [32] have studied the connectedness and path-connectedness of solutions set for symmetric VEPs. However, the connectedness of approximate solutions set for VEPs remained unstudied.

In this paper, we show a scalarization result of ε-weak efficient solution for a VEP. Using this scalarization result, we discuss the connectedness of ε-weak efficient and ε-efficient solutions sets for VEPs under some suitable conditions in real Hausdorff topological vector spaces. The main results presented in this paper generalize some known results due to Gong [27] and Gong and Yao [31].

2 Preliminaries

Throughout this paper, let X and Y be two real Hausdorff topological vector spaces and A a nonempty subset of X. Let $F : A \times A \to Y$ be a mapping and C be a closed convex pointed cone in Y. The cone C induces a partial ordering in Y, defined by

$$z_1 \leq z_2 \quad \text{if and only if} \quad z_2 - z_1 \in C.$$

Let

$$C^* = \{f \in Y^* : f(\gamma) \geq 0 \quad \text{for all } \gamma \in C\}$$

be the dual cone of C.

Denote the quasi-interior of C^* by $C^{\#}$, that is,

$$C^{\#} = \{f \in Y^* : f(\gamma) > 0 \quad \text{for all } \gamma \in C \backslash \{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\text{cone}(D) = \{td : t \geq 0,\ d \in D\}.$$

A nonempty convex subset B of the convex cone C is called a base of C if

$$C = \text{cone}(B) \quad \text{and} \quad 0 \notin cl(B).$$

It is easy to see that $C^{\#} \neq \varnothing$ if and only if C has a base.

Throughout this paper, we always assume that $\text{int}C \neq \varnothing$. Let e be a fixed point in $\text{int}C$ and we set

$$C' = \{f \in C^* \backslash \{0\} : f(e) = 1\},$$
$$C'' = \{f \in C^{\#} : f(e) = 1\}.$$

Now, we give the concepts of ε-weak efficient solution, ε-efficient solution, and ε-f efficient solution for the VEP.

Definition 2.1. A vector $x \in A$ satisfying

$$F(x, \gamma) \notin -\text{int}C - \varepsilon e \quad \text{for all } \gamma \in A,$$

is called a ε-weak efficient solution to the VEP. Denote by $V_{\varepsilon\text{-}W}(A, F)$ the set of all ε-weak efficient solutions to the VEP.

Definition 2.2. A vector $x \in A$ satisfying

$$F(x, \gamma) \notin -C \backslash \{0\} - \varepsilon e \quad \text{for all } \gamma \in A,$$

is called a ε-efficient solution to the VEP. Denote by $V_\varepsilon (A, F)$ the set of all ε-efficient solutions to the VEP.

Definition 2.3. Let $f \in C^*\backslash\{0\}$. A vector $x \in A$ is called a ε-f efficient solution to the VEP if

$$f(F(x, \gamma)) \geq -\varepsilon \text{ for all } \gamma \in A,$$

Denote by $V_{\varepsilon\text{-}f} (A, F)$ the set of all ε-f efficient solutions to the VEP.

Definition 2.4. [33] Let G be a set-valued map from a topological space W to another topological space Q. We say that G is

(i) upper semicontinuous at $x_0 \in W$ if, for any neighborhood $U(G(x_0))$ of $G(x_0)$, there is a neighborhood $U (x_0)$ of x_0 such that $G(x) \subset U (G(x_0))$ for all $x \in U(x_0)$;

(ii) upper semicontinuous on W if it is upper semicontinuous at each $x \in W$;

(iii) lower semicontinuous at $x_0 \in W$ if, for any net $\{x_\alpha : \alpha \in I\}$ converging to x_0 and for any $y_0 \in G(x_0)$, there exists a net $y_\alpha \in G(x_\alpha)$ that converges to y_0;

(iv) lower semicontinuous on W if it is lower semicontinuous at each $x \in W$;

(v) continuous on W if it is both upper semicontinuous and lower semicontinuous on W;

(vi) closed if $\text{Graph}(G) = \{(x, y) : x \in W, y \in G(x)\}$ is a closed subset in $W \times Q$.

3 Scalarization

Lemma 3.1. Suppose $F (x, A) + C$ is a convex set for each $x \in A$. Then,

$$V_{\varepsilon-W}(A, F) = \bigcup_{f \in C'} V_{\varepsilon-f}(A, F).$$

Proof. We first prove that

$$\bigcup_{f \in C'} V_{\varepsilon-f}(A, F) \subset V_{\varepsilon-W}(A, F).$$

In fact, letting $x_0 \in \bigcup_{f \in C'} V_{\varepsilon-f}(A, F)$, then there exists $f \in C'$,

$$f(F(x_0, \gamma)) \geq -\varepsilon, \quad \text{for all } \gamma \in A. \tag{3.1}$$

We claim that $x_0 \in V_{\varepsilon\text{-}W} (A, F)$. If not, then there exists $y_0 \in A$ such that $F (x_0, y_0) \in$ -int$C$ - εe. Thus, we have

$$f(F(x_0, y_0) + \varepsilon e) < 0$$

and so

$$f(F(x_0, y_0)) < -f(\varepsilon e) = -\varepsilon,$$

which is a contradiction to (3.1).

We next prove that

$$V_{\varepsilon-W}(A, F) \subset \bigcup_{f \in C'} V_{\varepsilon-f}(A, F).$$

Let $x \in V_{\varepsilon\text{-}W} (A, F)$. Then, $F (x, A) \cap (\text{-int}C - \varepsilon e) = \varnothing$. Since C is a pionted convex cone, we have

$$(F(x, A) + C + \varepsilon e) \cap (-\text{int}C) = \varnothing.$$

By assumption, we know that $F(x, A) + C$ is a convex set. Using the separation theorem for convex sets, there exists some $g \in Y^*\backslash\{0\}$, such that

$$\inf\{g\,(F(x,\ y) + c + \varepsilon e)\ : y \in A,\ c \in C\} \geq \sup\{g\,(-c) : c \in C\}. \tag{3.2}$$

From (3.2), we get $g \in C^*\backslash\{0\}$ and so

$$g(F(x,\ y)) \geq -g(\varepsilon e) \quad \text{for all } y \in A,$$

where $e \in \text{int} C$ with $g(e) > 0$. Letting $f = \dfrac{g}{g(e)}$, then $f(F(x, y)) \geq -\varepsilon$, for all $y \in A$ and $f(e) = 1$. Thus, $f \in C'$ and so

$$x \in \bigcup_{f \in C'} V_{\varepsilon-f}(A,\ F).$$

This completes the proof.

Remark 3.1 When $\varepsilon = 0$, it is easy to see that

$$V_{\varepsilon-W}(A,\ F) = \bigcup_{f \in C'} V_{\varepsilon-f}(A,\ F) = \bigcup_{f \in C^*\backslash\{0\}} V_{\varepsilon-f}(A,\ F).$$

Therefore, Lemma 3.1 generalizes Lemma 2.1 in [27].

4 Existence of the solutions

Definition 4.1. The bifunction $F : A \times A \to Y$ is concave-like with respect to the first variable, if for $t \in [0, 1]$, the following condition is satisfied: For $x_1, x_2 \in A$, there exists $x_3 \in A$, such that

$$F(x_3,\ y) \in tF(x_1,\ y) + (1-t)F(x_2,\ y) + C, \quad \text{for all } y \in A.$$

The bifunction $F : A \times A \to Y$ is convex-like with respect to the second variable, if for $t \in [0, 1]$, the following condition is satisfied: for $y_1, y_2 \in A$, there exists $y_3 \in A$, such that

$$F(x,\ y_3) \in tF(x,\ y_1) + (1-t)F(x,\ y_2) - C, \quad \text{for all } x \in A.$$

Theorem 4.1. Let A be a nonempty compact subset of X, and $f \in C'$. Assume that the following conditions are satisfied:

(i) $F(x, x) \in C - \varepsilon e$, for all $x \in A$;

(ii) $F : A \times A \to Y$ is concave-like with respect to the first variable and convex-like with respect to the second variable;

(iii) For each fixed $y \in A$, the function $x \mapsto f(F(x, y))$ is upper semicontinuous on A. Then, $V_{\varepsilon-f}(A, F) \neq \varnothing$.

Proof. Define the set-valued map $G : A \to 2^A$ by

$$G(y) = \{x \in A,\ f(F(x,\ y)) \geq -\varepsilon\}$$

By assumption, $y \in G(y)$, for all $y \in A$, so $G(y) \neq \varnothing$. By assumption, we can see that $G(y)$ is a closed subset of A. Next, we prove that $\cap\{G(y) : y \in A\} \neq \varnothing$. Since A is a compact, we need to show that $\bigcap_{i=1}^{n} G(y_i) \neq \varnothing$ for any arbitrary chosen $y_1, ..., y_n$ in A. Suppose it is not true. Then, there exists a set $B = \{y_1, ..., y_n\} \subseteq A$ such that $\bigcap_{i=1}^{n} G(y_i) \neq \varnothing$. Thus, for any $x \in A$, there exists $y_i \in B$ such that $x \notin G(y_i)$. It follows

that

$$f(F(x, y_i)) + \varepsilon \ < 0$$

and so there exists $\eta_i > 0$ such that

$$f(F(x, y_i)) + \varepsilon \ < -\eta_i, \quad i = 1, 2, \ldots, n.$$

Since $x \mapsto f(F(x, y))$ is upper semicontinuous on A, we can choose $\eta > 0$ such that, for any $x \in A$, there exists $y_i \in B$ satisfying

$$f(F(x, y_i)) + \varepsilon \ + \eta < 0.$$

Define $g : A \to R^n$ by

$$g(x) = (-f(F(x, y_1)) - \varepsilon \ - \eta, \ -f(F(x, y_2)) - \varepsilon \ - \eta, ..., \ -f(F(x, y_n)) - \varepsilon \ - \eta),$$

where $x \in A$. We get

$$g(x) \notin \mathrm{int} R^n_+, \quad \text{for all } x \in A. \tag{4.1}$$

Since $f \in C'$ and $F(x, y)$ is concave-like with respect to the first variable, we can see that, for $t \in [0, 1]$, $x_1, x_2 \in A$, there exists $x_3 \in A$ such that

$$g(x_3) \in tg(x_1) + (1 - t)g(x_2) - C.$$

This shows that $g(A) + R^n_+$ is a convex set. It follows from (4.1) that

$$0 \notin g(A) + \mathrm{int} R^n_+.$$

By the separation theorem of convex sets (see, for example, [34]), we can find $t_1, ..., t_n \geq 0$ with $\sum_{i=1}^{n} t_i = 1$ such that

$$0 \leq \sum_{i=1}^{n} t_i(-f(F(x, y_i)) - \varepsilon \ - \eta), \quad \text{for all } x \in A.$$

It follows that

$$\sum_{i=1}^{n} t_i f(F(x, y_i)) \leq -\varepsilon - \eta, \quad \text{for all } x \in A.$$

By assumption, there exists $y \in A$ such that

$$F(x, y) \in \sum_{i=1}^{n} t_i(F(x, y_i)) - C, \quad \text{for all } x \in A.$$

Since $f \in C'$, we have

$$f(F(x, y)) \leq f\left(\sum_{i=1}^{n} t_i(F(x, y_i))\right), \quad \text{for all } x \in A.$$

So $f(F(x, y)) \leq -\varepsilon - \eta < -\varepsilon$, for all $x \in A$. Setting $x = y$, it follows that

$$f(F(y, y)) < -\varepsilon.$$

On the other hand, by the assumption,

$$f(F(y, y)) \geq -\varepsilon.$$

This is a contradiction. Therefore, $\cap\{G\ (y) : y \in A\} \neq \varnothing$, and so there exists $x \in \cap\{G\ (y) : y \in A\}$. This means that $V_{\varepsilon\text{-}f}\ (A, F) \neq \varnothing$. This completes the proof.

5 Connectedness of the solutions set

In this section, we discuss the connected results of ε-weak efficient solutions set and ε-efficient solutions set.

Definition 5.1. Let A be a convex set. The bifunction $F : A \times A \to Y$ is C-concave with respect to the first variable, if for $t \in [0, 1]$, $x_1, x_2 \in A$,

$$F(tx_1 + (1 - t)x_2, y) \in tF(x_1, y) + (1 - t)tF(x_2, y) + C.$$

It is clear that when A is a convex set, $F : A \times A \to Y$ is C-concave with respect to the first variable, then it is concave-like about the first variable.

Theorem 5.1 *Let A be a nonempty compact convex subset of X, and $f \in C'$. Assume that the following conditions are satisfied.*

(i) $F\ (x, x) \in C - \varepsilon e$, for all $x \in A$;
(ii) $F : A \times A \to Y$ is C-concave with respect to the first variable and convex-like with respect to the second variable;
(iii) For each fixed $y \in A$, the function $x \mapsto f\ (F\ (x, y))$ is upper semicontinuous on A;
(iv) $D = \{F\ (x, y) : x, y \in A\}$ is a bounded set of Y.

Then, $V_{\varepsilon\text{-}W}\ (A, F)$ is a connected set.

Proof. Define a set-valued mapping $H : C' \to 2^A$ by

$$H(f) = V_{\varepsilon\text{-}f}(A, F), \ f \in C'.$$

By Theorem 4.1, we know that, for each $f \in C'$, $V_{\varepsilon\text{-}f}\ (A, F) \neq \varnothing$. It is easy to see that C' is a convex set and so is connected. Next, we prove that, for each $f \in C'$, $H\ (f)$ is a connected set. Let $x_1, x_2 \in H\ (f)$, we have $x_1, x_2 \in A$, and

$$f(F(x_i, y)) \geq -\varepsilon, \quad \forall y \in A, \ i = 1, 2. \tag{5.1}$$

Because $F : A \times A \to Y$ is C-concave with respect to the first variable, for each fixed $y \in A$, $t \in [0, 1]$,

$$F(tx_1 + (1 - t)x_2, y) \in tF(x_1, y) + (1 - t)F(x_2, y) + C.$$

Hence

$$tf(F(x_1, y)) + (1 - t)f(F(x_2, y)) \leq f(F(tx_1 + (1 - t)x_2, y)).$$

It follows from (5.1) that

$$f(F(tx_1 + (1 - t)x_2, y)) \geq -\varepsilon.$$

Hence

$$tx_1 + (1 - t)x_2 \in H(f)$$

and so $H\ (f)$ is a convex set. Thus, it is a connected set.

Next, we show that H is upper semicontinuous on C'. Since A is compact, we only need to show that H is closed. Let $\{(f_\alpha, x_\alpha) : \alpha \in I\} \subset Graph\ (H)$ be a net such that $(f_\alpha, x_\alpha) \to (f, x)$, where $f_\alpha \to f$ means that $\{f_\alpha\}$ converges to f with respect to the strong topology $\beta\ (Y^*, Y)$ in Y^*. Since C' is a closed set and A is a compact set, we know that $(f, x) \in C' \times A$. From $x_\alpha \in H\ (f_\alpha) = V_{\varepsilon\text{-} f_\alpha}\ (A, F)$, we have

$$f_\alpha(F(x_\alpha, y)) \geq -\varepsilon. \tag{5.2}$$

For any $\delta > 0$,

$$U = \left\{ y^* \in Y^* : \sup_{u \in D} |y^*(u)| < \delta \right\}$$

is a neighborhood of 0 with respect with to $\beta\ (Y^*, Y)$. Since $f_\alpha \to f$, there exists $\alpha_0 \in I$ such that $f_\alpha - f \in U$, for all $\alpha \geq \alpha_0$. It follows that

$$\sup_{u \in D} |(f_\alpha - f)(u)| < \delta, \quad \text{whenever } \alpha \geq \alpha_0.$$

Therefore, for any $y \in A$,

$$|(f_\alpha - f)(F(x_\alpha, y))| < \delta, \quad \text{whenever } \alpha \geq \alpha_0$$

and so

$$\lim_{\alpha}(f_\alpha(F(x_\alpha, y)) - f(F(x_\alpha, y))) = 0, \quad \text{for all } y \in A. \tag{5.3}$$

Because $x \mapsto f\ (F\ (x, y))$ is upper semicontinuous on A, then

$$\limsup f(F(x_\alpha, y)) \leq f(F(x, y)). \tag{5.4}$$

From (5.2), (5.3) and (5.4), we have

$$\begin{aligned}
-\varepsilon &\leq \limsup f_\alpha(F(x_\alpha, y)) \\
&\leq \lim_{\alpha}(f_\alpha(F(x_\alpha, y)) - f(F(x_\alpha, y))) + \limsup f(F(x_\alpha, y)) \\
&\leq f(F(x, y)).
\end{aligned}$$

Hence, $x \in H\ (f) = V_{\varepsilon\text{-}f}\ (A, F)$. By Theorem 3.1 in [35], $\bigcup_{f \in C'} V_{\varepsilon-f}(A, F)$ is a connected set. Because $F : A \times A \to Y$ is convex-like with respect to the second variable, we have $F\ (x, A) + C$ is a convex set, by Lemma 3.1,

$$V_{\varepsilon-W}(A,\ F) = \bigcup_{f \in C'} V_{\varepsilon-f}(A,\ F)$$

is a connected set. This completes the proof.

Next, we give an example to illustrate Theorem 5.1.

Example 5.1 *Let* $X = R$, $Y = R^2$, $C = R_+^2 = \{(x_1,\ x_2) : x_1 \geq 0,\ x_2 \geq 0\}$, $\varepsilon = 2$, $e = (2, 3)$, *and* $A = [0, 2]$. *Let*

$$F(x,\ y) = (y - x^2 - 2, y - x^2 - 4), \quad \forall x,\ y \in A.$$

Then, F satisfies all conditions of Theorem 5.1. It is easy to see that $V_{\varepsilon-W}(A,\ F) = \left[0,\ \sqrt{2}\right]$. *Clearly,* $V_{\varepsilon\text{-}W}\ (A, F)$ *is a nonempty connected set.*

Definition 5.2. The bifunction $F : A \times A \to Y$ is ε-C strictly monotone if, for any $x, y \in A$, $x \neq y$,

$$F(x, y) + F(y, x) \in -\text{int } C - 2\varepsilon e.$$

Theorem 5.2 *Suppose that all conditions of Theorem 5.1 are satisfied and $F : A \times A \to Y$ is ε-C strictly monotone. Then, $V_{\varepsilon\text{-}W}(A, F)$ is a path connected set.*

Proof. Define the set-valued mapping $H : C' \to 2^A$ by

$$H(f) = V_{\varepsilon-f}(A, F), \ f \in C'.$$

By Theorem 4.1, we know that, for each $f \in C'$, $V_{\varepsilon\text{-}f}(A, F) \neq \varnothing$. Furthermore, because $F : A \times A \to Y$ is ε-C strictly monotone, it is easy to see that, for each $f \in C'$, $H(f) = V_{\varepsilon\text{-}f}(A, F)$ is a single point set. From the proof of Theorem 5.1, we know that H is upper semicontinuous on C' and so it is continuous on C'. Since C' is a convex set, so it is a path connected set. Hence,

$$V_{\varepsilon-W}(A, F) = H(C') = \bigcup_{f \in C'} H(f) = \bigcup_{f \in C'} V_{\varepsilon-f}(A, F)$$

is a path connected set. This completes the proof.

Next, we give an example to illustrate Theorem 5.2.

Example 5.2 *Let $X = R$, $Y = R^2$, $C = R_+^2 = \{(x_1, x_2) : x_1 \geq 0, \ x_2 \geq 0\}$, $\varepsilon = -1$, $e = (1, 2)$, and $A = [-1, 1]$. Let*

$$F(x, y) = (x(y - x) + 1, \ x(y - x) + 2), \quad \forall x, \ y \in A.$$

Then, F satisfies all the conditions of Theorem 5.2. It is easy to see that $V_{\varepsilon\text{-}W}(A, F) = \{0\}$. Clearly, $V_{\varepsilon\text{-}W}(A, F)$ is a nonempty path connected set.

Next, we give a lemma before we give the connectedness theorem of ε-efficient solutions set.

Lemma 5.1. Suppose that all conditions of Theorem 5.2 are satisfied, then

$$\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \subset V_\varepsilon(A, F) \subset cl \left(\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \right).$$

Proof. By Theorem 4.1, we know that, for each $f \in C'$, $V_{\varepsilon\text{-}f}(A, F) \neq \varnothing$. By definition, we have

$$\bigcup_{f \in C'} V_{\varepsilon-f}(A, F) \subset V_\varepsilon(A, F) \subset V_{\varepsilon-W}(A, F). \tag{5.5}$$

From Lemma 3.1, we have

$$V_{\varepsilon-W}(A, F) = \bigcup_{f \in C'} V_{\varepsilon-f}(A, F). \tag{5.6}$$

By (5.5) and (5.6), we have

$$\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \subset V_\varepsilon(A, F) \subset \bigcup_{f \in C'} V_{\varepsilon-f}(A, F).$$

Next, we prove that

$$\bigcup_{f \in C'} V_{\varepsilon-f}(A, F) \subset cl \left(\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \right).$$

Define the set-valued mapping: $H : C' \to 2^A$ by

$$H(f) = V_{\varepsilon-f}(A, F), \ f \in C'.$$

From Theorem 5.2, we know that $H(f)$ is a single-valued mapping, and H is continuous on C'.

Let $x \in \bigcup_{f \in C'} V_{\varepsilon-f}(A, F)$. Then, there exists $f \in C'$, such that

$$\{x\} = V_{\varepsilon-f}(A, F) = H(f).$$

Let $g \in C''$, and set

$$f_n = \frac{n-1}{n} f + \frac{1}{n} g = f + \frac{1}{n}(g - f).$$

Then, $f_n \in C^{\#}$ and $f_n(e) = 1$. Thus, $f_n \in C''$.

Next, we show that $\{f_n\}$ converges to f with respect to the topology $\beta(Y^*, Y)$. For any neighborhood of 0 with respect to $\beta(Y^*, Y)$, there exist bounded subsets $B_i \in Y$ $(i = 1, 2,..., m)$ and $\delta > 0$ such that

$$\bigcap_{i=1}^{m} \left\{ f \in Y^* : \sup_{\gamma \in B_i} |f(\gamma)| < \delta \right\} \subset U.$$

Since B_i is bounded and $g - f \in Y^*$, it is easy to see that $|(g - f)(B_i)|$ is bounded for $i = 1, 2,..., m$. This implies that there exists N such that

$$\sup_{\gamma \in B_i} \left| \frac{1}{n}(g - f)(\gamma) \right| < \delta, \quad i = 1, 2, ..., m; \quad n \geq N.$$

Hence, $\frac{1}{n}(g - f) \in U$, that is $f_n - f \in U$. Hence, $\{f_n\}$ converges to f with respect to $\beta(Y^*, Y)$.

Since H is continuous on f, we have $H(f_n) \to H(f)$. Set $\{x_n\} = H(f_n)$, then

$$\{x_n\} = H(f_n) = V_{\varepsilon-f_n}(A, F) \subset \bigcup_{f \in C''} V_{\varepsilon-f}(A, F).$$

Because $\{x\} = H(f)$, we have $x_n \to x$. This implies that

$$x \in cl\left(\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \right).$$

Since $x \in \bigcup_{f \in C'} V_{\varepsilon-f}(A, F)$ is arbitrary, we have

$$\bigcup_{f \in C'} V_{\varepsilon-f}(A, F) \subset cl\left(\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \right).$$

Therefore,

$$\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \subset V_{\varepsilon}(A, F) \subset cl\left(\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \right).$$

This completes the proof.

Theorem 5.3. Suppose that all the conditions of Theorem 5.2 are satisfied. Then, $V_{\varepsilon}(A, F)$ is a connected set.

Proof. By Lemma 5.1, we have

$$\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \subset V_{\varepsilon}(A, F) \subset cl\left(\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) \right). \tag{5.7}$$

From Theorem 5.2, we can get

$$\bigcup_{f \in C''} V_{\varepsilon-f}(A, F)$$

is connected set and so (5.7) implies that $V_\varepsilon (A, F)$ is a connected set. This completes the proof.

Remark 5.1 When $\varepsilon = 0$, we can get

$$\bigcup_{f \in C''} V_{\varepsilon-f}(A, F) = \bigcup_{f \in c \neq} V_{\varepsilon-f}(A, F).$$

Therefore, Theorem 5.3 generalizes Theorem 2.2 in [31].

Next, we give an example to illustrate Theorem 5.3.

Example 5.3 *Let $X = R$, $Y = R$, $C = R_+$, $\varepsilon = 1$, $e = 1$, and $A = [1, 2]$. Let $F(x, y) = x(y - x) - 1$ for all $x, y \in A$. Then, it is easy to check that all the conditions of Theorem 5.3 are satisfied and*

$$V_\varepsilon(A, F) = \{1\}.$$

Clearly, $V_\varepsilon (A, F)$ is a nonempty connected set.

Author details
[1]Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China [2]State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation, Southwest Petroleum University, Chengdu, Sichuan 610500, PR China [3]Department of Applied Mathematics, Southwest Petroleum University, Chengdu, 610500, PR China

Authors' contributions
BC carried out the study of connectedness of ε-weak efficient and ε-efficient solutions sets for VEPs and drafted the manuscript. Q-YL participated in the design of the study. Z-BL gave some examples to show the main results. N-JH conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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