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Another weak convergence theorems for accretive mappings in banach spaces

Satit Saejung*, Kanokwan Wongchan and Pongsakorn Yotkaew

* Correspondence: saejung@kku.ac.th
Department of Mathematics, Faculty of Science, Khon kaen University, Khon kaen 40002, Thailand

Abstract

We present two weak convergence theorems for inverse strongly accretive mappings in Banach spaces, which are supplements to the recent result of Aoyama et al. [Fixed Point Theory Appl. (2006), Art. ID 35390, 13pp.].

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1. Introduction

Let E be a real Banach space with the dual space E^*. We write $\langle x, x^* \rangle$ for the value of a functional $x^* \in E^*$ at $x \in E$. The *normalized duality mapping* is the mapping $J : E \to 2E^*$ given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\} \quad (x \in E).$$

In this paper, we assume that E is *smooth*, that is, $\lim_{t\to 0}\frac{||x+ty||-||x||}{t}$ exists for all $x, y \in E$ with $||x|| = ||y|| = 1$. This implies that J is single-valued and we do consider the singleton Jx as an element in E^*. For a closed convex subset C of a (smooth) Banach space E, the *variational inequality problem* for a mapping $A : C \to E$ is the problem of finding an element $u \in C$ such that

$$\langle Au, J(v - u)\rangle \geq 0 \quad \text{for all } v \in C.$$

The set of solutions of the problem above is denoted by $S(C, A)$. It is noted that if $C = E$, then $S(C, A) = A^{-1}0 := \{x \in E : Ax = 0\}$. This problem was studied by Stampacchia (see, for example, [1,2]). The applicability of the theory has been expanded to various problems from economics, finance, optimization and game theory.

Gol'shteĭn and Tret'yakov [3] proved the following result in the finite dimensional space $\mathbb{R}^N$.

Theorem 1.1. *Let $\alpha > 0$, and let $A : \mathbb{R}^N \to \mathbb{R}^N$ be an α-inverse strongly monotone mapping, that is, $\langle Ax - Ay, x - y\rangle \geq \alpha||Ax - Ay||^2$ for all $x, y \in \mathbb{R}^N$. Suppose that $\{x_n\}$ is a sequence in $\mathbb{R}^N$ defined iteratively by $x_1 \in \mathbb{R}^N$ and*

$$x_{n+1} = x_n - \lambda_n Ax_n,$$

where $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$. If $A^{-1} 0 \neq \emptyset$, then $\{x_n\}$ converges to some element of $A^{-1}0$.

The result above was generalized to the framework of Hilbert spaces by Iiduka et al. [4]. Note that every Hilbert space is uniformly convex and 2-uniformly smooth (the related definitions will be given in the next section). Aoyama et al. [[5], Theorem 3.1] proved the following result.

Theorem 1.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the uniform smoothness constant K, and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$ and let $A : C \to E$ be an α-inverse strongly accretive mapping with $S(C, A) \neq \emptyset$. Suppose that $\{x_n\}$ is iteratively defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where $\{\alpha_n\} \subset [b, c] \subset (0, 1)$ and $\{\lambda_n\} \subset [a, \alpha/K^2] \subset (0, \alpha/K^2]$. Then, $\{x_n\}$ converges weakly to some element of $S(C, A)$.

Motivated by the result of Aoyama et al., we prove two more convergence theorems for α-inverse strongly accretive mappings in a Banach space, which are supplements to Theorem 1.2 above. The first one is proved without the presence of the uniform convexity, while the last one is proved in uniformly convex space with some different control conditions on the parameters.

The paper is organized as follows: In Section 2, we collect some related definitions and known fact, which are referred in this paper. The main results are presented in Section 3. We start with some common tools in proving the main results in Section 3.1. In Section 3.2, we prove the first weak convergence theorem without the presence of uniform convexity. The second theorem is proved in uniformly convex Banach spaces in Section 3.3.

2. Definitions and related known fact

Let E be a real Banach space. If $\{x_n\}$ is a sequence in E, we denote *strong convergence* of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and *weak convergence* by $x_n \rightharpoonup x$. Denote by $\omega_w(\{x_n\})$ the set of weakly sequential limits of the sequence $\{x_n\}$, that is, $\omega_w(\{x_n\}) = \{p :$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p\}$. It is known that if $\{x_n\}$ is a bounded sequence in a reflexive space, then $\omega_w(\{x_n\}) = \emptyset$.

The space E is said to be *uniformly convex* if for each $\varepsilon \in (0, 2)$ there exists $\delta > 0$ such that for any $x, y \in U := \{z \in E : ||z|| = 1\}$

$$||x - y|| \geq \varepsilon \text{ implies } ||x + y||/2 \leq 1 - \delta.$$

The following result was proved by Xu.

Lemma 2.1 ([6]). *Let E be a uniformly convex Banach space, and let $r > 0$. Then, there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$||\alpha x + (1 - \alpha)y||^2 \leq \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)g(||x - y||)$$

for all $\alpha \in [0, 1]$ and $x, y \in B_r := \{z \in E : ||z|| \leq r\}$.

The space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.1}$$

exists for all $x, y \in U$. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$.

Let C be a nonempty subset of a smooth Banach space E and $\alpha > 0$. A mapping $A : C \to E$ is said to be *α-inverse strongly accretive* if

$$\langle Ax - Ay, J(x - y)\rangle \geq \alpha||Ax - Ay||^2 \tag{2.2}$$

for all $x, y \in C$. It follows from (2.2) that A is *$\frac{1}{\alpha}$-Lipschitzian*, that is,

$$||Ax - Ay|| \leq \frac{1}{\alpha}||x - y|| \quad \text{for all } x, y \in C.$$

A Banach space E is *2-uniformly smooth* if there is a constant $c > 0$ such that $\rangle_E(\tau) \leq c\tau^2$ for all $\tau > 0$ where

$$\varrho_E(\tau) = \sup\left\{\frac{1}{2}(||x + \tau y|| + ||x - \tau y||) - 1 : x, y \in U\right\}.$$

In this case, we say that a real number $K > 0$ is a *2-uniform smoothness constant* of E if the following inequality holds for all $x, y \in E$:

$$||x + y||^2 \leq ||x||^2 + 2\langle y, Jx\rangle + 2||Ky||^2.$$

Note that every 2-uniformly smooth Banach space has the Fréchet differentiable norm and hence it is reflexive.

The following observation extracted from Lemma 2.8 of [5] plays an important role in this paper.

Lemma 2.2. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E with a 2-uniform smoothness constant K. Suppose that $A : C \to E$ is an α-inverse strongly accretive mapping. Then, the following inequality holds for all $x, y \in C$ and $\lambda \in \mathbb{R}$:*

$$||(I - \lambda A)x - (I - \lambda A)y||^2 \leq ||x - y||^2 + 2\lambda(K^2\lambda - \alpha)||Ax - Ay||^2,$$

where I is the identity mapping. In particular, if $\lambda \in [0, \frac{\alpha}{K^2}]$, then $I - \lambda A$ is nonex pansive, that is, $||(I - \lambda A)x - (I - \lambda A)y|| \leq ||x - y||$ for all $x, y \in C$.

Let C be a subset of a Banach space E. A mapping $Q : E \to C$ is said to be:

(i) *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$;
(ii) a *retraction* if $Q^2 = Q$.

It is known that a retraction Q from a smooth Banach space E onto a nonempty closed convex subset C of E is sunny and nonexpansive if and only if $\langle x-Qx, J(Qx-y)\rangle \geq 0$ for all $x \in E$ and $y \in C$. In this case, Q is uniquely determined. Using this result, Aoyama et al. obtained the following result. Recall that, for a mapping $T : C \to E$, the set of *fixed points* of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$.

Lemma 2.3 ([5]). *Let C be a nonempty closed convex subset of a smooth Banach space*

E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let $A : C \to E$ be a mapping. Then, for each $\lambda > 0$,

$$S(C, A) = F(Q_C(I - \lambda A)).$$

The space E is said to satisfy *Opial's condition* if

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

whenever $x_n \rightharpoonup x \in E$ and $y \in E$ satisfy $x \neq y$. The following results are known from theory of nonexpansive mappings. It should be noted that Oplial's condition and the Fréchet differentiability of the norm are independent in uniformly convex space setting.

Lemma 2.4 ([7], [8]). *Let C be a nonempty closed convex subset of a Banach space. E. Suppose that E is uniformly convex or satisfies Opial's condition. Suppose that T is a nonexpansive mapping of C into itself. Then, I - T is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in C such that* $x_n \rightharpoonup p$ *and* $x_n - Tx_n \to 0$, *then* $p = Tp$.

Lemma 2.5 ([9]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm. Suppose that* $\{T_n\}_{n=1}^{\infty}$ *is a sequence of nonexpansive mappings of C into itself with* $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $x \in C$ *and* $S_n = T_n T_{n-1} \cdots T_1$ *for all* $n \geq 1$. *Then, the set*

$$\bigcap_{n=1}^{\infty} \overline{\text{co}}\{S_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n)$$

consists of at most one element, where $\overline{\text{co}}D$ *is the closed convex hull of D.*

The following two lemmas are proved in the absence of uniform convexity, and they are needed in Section 3.2.

Lemma 2.6 ([10]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space and* $\{\alpha_n\}$ *be a real sequence in* $[0, 1]$ *such that* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *Suppose that* $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n$ *for all* $n \geq 1$. *If* $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$, *then* $x_n - y_n \to 0$.

Lemma 2.7 ([11]). *Let* $\{z_n\}$ *and* $\{w_n\}$ *be sequences in a Banach space and* $\{\alpha_n\}$ *be a real sequence in* $[0, 1]$. *Suppose that* $z_{n+1} = \alpha_n z_n + (1 - \alpha_n) w_n$ for all $n \geq 1$. *If the following properties are satisfied:*

(i) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$ *and* $\liminf_{n\to\infty} \alpha_n > 0$;
(ii) $\lim_{n\to\infty} ||z_n|| = d$ *and* $\limsup_{n\to\infty} ||w_n|| \leq d$;
(iii) *the sequence* $\left\{\sum_{i=1}^{n} (1 - \alpha_i) w_i\right\}$ *is bounded*;

then $d = 0$.

We also need the following simple but interesting results.

Lemma 2.8 ([12]). *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of nonnegative real numbers. If* $\sum_{n=1}^{\infty} b_n < \infty$ *and* $a_{n+1} \leq a_n + b_n$ *for all* $n \geq 1$, *then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.9 ([13]). *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of nonnegative real numbers. If* $\sum_{n=1}^{\infty} a_n b_n < \infty$ *and* $\sum_{n=1}^{\infty} a_n b_n < \infty$, *then* $\liminf_{n\to\infty} b_n = 0$.

3. Main results

From now on, we assume that

- E is 2-uniformly smooth Banach space with a 2-uniform smoothness constant K;

- C is a nonempty closed convex subset of E;
- Q_C is a sunny nonexpansive retraction from E onto C;
- $A : C \to E$ is an α-inverse strongly accretive mapping with $S(C, A) \neq \emptyset$ and $\alpha > 0$.

Suppose that $\{x_n\}$ is iteratively defined by

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen,} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C(x_n - \lambda_n Ax_n) \quad (n \geq 1), \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \frac{\alpha}{K^2}]$. For convenience, we write $y_n \equiv Q_C\,(x_n - \lambda_n Ax_n)$.

3.1. Some properties of the sequence $\{x_n\}$ for weak convergence theorems

We start with some propositions, which are the common tools for proving the main results in the next two subsections.

Proposition 3.1. *If $p \in S(C, A)$, then $\lim_{n\to\infty} ||x_n - p||$ exists, and hence, the sequences $\{x_n\}$ and $\{Ax_n\}$ are both bounded.*

Proof. Let $p \in S(C, A)$. By the nonexpansiveness of $Q_C\,(I - \lambda_n A)$ for all $n \geq 1$ and Lemma 2.3, we have

$$||y_n - p|| = ||Q_C(I - \lambda_n A)x_n - (Q_C(I - \lambda_n A)p|| \leq ||x_n - p||$$

for all $n \geq 1$. This implies that

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_n(x_n - p) + (1-\alpha_n)(y_n - p)|| \\ &\leq \alpha_n ||x_n - p|| + (1-\alpha_n)||y_n - p|| \\ &\leq \alpha_n ||x_n - p|| + (1-\alpha_n)||x_n - p|| = ||x_n - p|| \end{aligned}$$

for all $n \geq 1$. Therefore, $\lim_{n\to\infty} ||x_n - p||$ exists, and hence, the sequence $\{x_n\}$ is bounded. Since A is $\frac{1}{\alpha}$-Lipschitzian, we have $\{Ax_n\}$ is bounded. The proof is finished.

Proposition 3.2. *The following inequality holds*:

$$||y_{n+1} - y_n|| \leq ||x_{n+1} - x_n|| + |\lambda_{n+1} - \lambda_n|\; ||Ax_n||$$

for all $n \geq 1$.

Proof. Since $Q_C\,(I - \lambda_{n+1}A)$ and Q_C are nonexpansive, we have

$$\begin{aligned} ||y_{n+1} - y_n|| &= ||Q_C(I - \lambda_{n+1}A)x_{n+1} - Q_C(I - \lambda_n A)x_n|| \\ &\leq ||Q_C(I - \lambda_{n+1}A)x_{n+1} - Q_C(I - \lambda_{n+1}A)x_n|| \\ &\quad + ||Q_C(I - \lambda_{n+1}A)x_n - Q_C(I - \lambda_n A)x_n|| \\ &\leq ||x_{n+1} - x_n|| + ||(I - \lambda_{n+1}A)x_n - (I - \lambda_n A)x_n|| \\ &= ||x_{n+1} - x_n|| + |\lambda_{n+1} - \lambda_n|\; ||Ax_n||. \end{aligned}$$

□

Proposition 3.3. *Suppose that E is a reflexive Banach space such that either it is uniformly convex or it satisfies Opial's condition. Suppose that $\{x_n\}$ is a bounded sequence of C satisfying $x_n - Q_C\,(I - \lambda_n A)x_n \to 0$ and $\{\lambda_n\} \subset [a, \frac{\alpha}{K^2}] \subset (0, \frac{\alpha}{K^2}]$.*

Then, $\{x_n\}$ converges weakly to some element of $S(C, A)$.

Proof. Suppose that E is a uniformly convex Banach space or a reflexive Banach space satisfying Opial's condition. Then, $\omega_w(\{x_n\}) \neq \emptyset$. We first prove that $\omega_w(\{x_n\}) \subset S(C, A)$. To see this, let $z \in \omega_w(\{x_n\})$. Passing to a subsequence, if necessary, we assume that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup z$ and $\lambda_{n_k} \to \lambda \in [a, \frac{\alpha}{K^2}]$. We observe that

$$\begin{aligned} ||x_{n_k} - Q_C(I-\lambda A)x_{n_k}|| &\leq ||x_{n_k} - y_{n_k}|| + ||y_{n_k} - Q_C(I-\lambda A)x_{n_k}|| \\ &\leq ||x_{n_k} - y_{n_k}|| + ||(I-\lambda_{n_k}A)x_{n_k} - (I-\lambda A)x_{n_k}|| \\ &= ||x_{n_k} - y_{n_k}|| + |\lambda_{n_k} - \lambda|\ ||Ax_{n_k}||. \end{aligned}$$

This implies that $x_{n_k} - Q_C(I-\lambda A)x_{n_k} \to 0$. By the nonexpansiveness of $Q_C(I - \lambda A)$, Lemmas 2.3 and 2.4, we obtain that $z \in F(Q_C(I - \lambda A)) = S(C, A)$. Hence $\omega_w(\{x_n\}) \subset S(C, A)$.

We next prove that $\omega_w(\{x_n\})$ is exactly a singleton in the following cases.

Case 1: E is uniformly convex. We follow the idea of Aoyama et al. [5] in this case. For any $n \geq 1$, we define a nonexpansive mapping $T_n : C \to C$ by

$$T_n = \alpha_n I + (1-\alpha_n)Q_C(I-\lambda_n A).$$

We get that $x_{n+1} = T_n T_{n-1} \cdots T_1 x_1$ for all $n \geq 1$. It follows from Lemma 2.3 that $S(C,A) = \bigcap_{n=1}^{\infty} F(Q_C(I-\lambda_n A)) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Applying Lemma 2.5, since every 2-uniformly smooth Banach space has Fréchet differentiable norm, gives

$$\bigcap_{n=1}^{\infty} \overline{co}\{x_m : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n)$$

consists of at most one element. But we know that

$$\emptyset \neq \omega_w(\{x_n\}) \subset \bigcap_{n=1}^{\infty} \overline{co}\{x_m : m \geq n\} \cap S(C,A) \subset \bigcap_{n=1}^{\infty} \overline{co}\{x_m : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

Therefore, $\omega_w(\{x_n\})$ is a singleton.

Case 2: E satisfies Opial's condition. Suppose that p and q are two different elements of $\omega_w(\{x_n\})$. There are subsequences $\{x_{n_k}\}$ and $\{x_{m_j}\}$ of $\{x_n\}$ such that

$$x_{n_k} \rightharpoonup p \quad \text{and} \quad x_{m_j} \rightharpoonup q.$$

Since p and q also belong to $S(C, A)$, both limits $\lim_{n\to\infty} ||x_n - p||$ and $\lim_{n\to\infty} ||x_n - q||$ exist. Consequently, by Opial's condition,

$$\begin{aligned} \lim_{k\to\infty} ||x_{n_k} - p|| &< \lim_{k\to\infty} ||x_{n_k} - q|| = \lim_{j\to\infty} ||x_{m_j} - q|| \\ &< \lim_{j\to\infty} ||x_{m_j} - p|| = \lim_{k\to\infty} ||x_{n_k} - p||. \end{aligned}$$

This is a contradiction. Hence, $\omega_w(\{x_n\})$ is a singleton, and the proof is finished. □

Remark 3.4. There exists a reflexive Banach space such that it satisfies Opial's condition but it is not uniformly convex. In fact, we consider $E = \mathbb{R}^2$ with the norm $||(x, y)|| = |x| + |y|$ for all $(x, y) \in \mathbb{R}^2$. Note that E is finite dimensional, and hence it is reflexive and satisfies Opial's condition. To see that E is *not* uniformly convex, let $x = (1, 0)$ and $y = (0, 1)$, it follows that $||x - y|| = ||(1, -1)|| = 2$ and $||x + y||/2 = ||(1/2, 1/2)|| = 1 \nleq 1 - \delta$ for all $\delta > 0$.

3.2. Convergence results without uniform convexity

In this subsection, we make use of Lemmas 2.6 and 2.7 to show that $x_n - y_n \to 0$ under the additional restrictions on the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$.

Proposition 3.5. *Suppose that* $\{\alpha_n\} \subset [c, d] \subset (0, 1)$ *and* $\lambda_{n+1} - \lambda_n \to 0$. *Then,* $x_n - y_n \to 0$.

Proof. We will apply Lemma 2.6. Let us rewritten the iteration as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n.$$

It follows from Proposition 3.1 that $\{x_n\}$ and $\{Ax_n\}$ are bounded. Then, $\{y_n\} = \{(I - \lambda_n A) x_n\}$ is bounded. Since $\lambda_{n+1} - \lambda_n \to 0$, it is a consequence of Proposition 3.2 that

$$\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le \limsup_{n\to\infty} |\lambda_{n+1} - \lambda_n| \, ||Ax_n|| = 0.$$

Since all the requirements of Lemma 2.6 are satisfied, $x_n - y_n \to 0$. □

Proposition 3.6. *Suppose that* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy the following properties*:

(i) $\{a_n\} \subset [c, 1) \subset (0, 1)$ *and* $\sum_{n=1}^{\infty} (1-\alpha_n) = \infty$;

(ii) $\dfrac{\lambda_{n+1} - \lambda_n}{1-\alpha_n} \to 0$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then, $x_n - y_n \to 0$.

Proof. We will apply Lemma 2.7. From the iteration, we have

$$z_{n+1} = \alpha_n z_n + (1-\alpha_n) w_n,$$

where $z_n \equiv x_n - y_n$ and $w_n \equiv \dfrac{y_n - y_{n+1}}{1-\alpha_n}$. Using Proposition 3.2, we obtain

$$\begin{aligned} ||z_{n+1}|| &\le \alpha_n ||z_n|| + ||y_n - y_{n+1}|| \\ &\le \alpha_n ||z_n|| + ||x_{n+1} - x_n|| + |\lambda_{n+1} - \lambda_n| \, ||Ax_n|| \\ &= \alpha_n ||z_n|| + (1-\alpha_n)||z_n|| + |\lambda_{n+1} - \lambda_n| \, ||Ax_n|| \\ &= ||z_n|| + |\lambda_{n+1} - \lambda_n| \, ||Ax_n||. \end{aligned}$$

It follows from $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| \, ||Ax_n|| < \infty$ and Lemma 2.8 that $d := \lim_{n\to\infty} ||z_n||$ exists. We next prove that $\limsup_{n\to\infty} ||w_n|| \le d$. Again, by Proposition 3.2, we get

$$\begin{aligned} \limsup_{n\to\infty} ||w_n|| &= \limsup_{n\to\infty} \frac{||y_n - y_{n+1}||}{1-\alpha_n} \\ &\le \lim_{n\to\infty} ||z_n|| + \limsup_{n\to\infty} \frac{|\lambda_{n+1} - \lambda_n|}{1-\alpha_n} ||Ax_n|| = d. \end{aligned}$$

Finally, for all $n \ge 1$, we have

$$\sum_{i=1}^{n} (1-\alpha_i) w_i = \sum_{i=1}^{n} (y_i - y_{i+1}) = y_1 - y_{n+1}.$$

Hence, the sequence $\left\{\sum_{i=1}^{n} (1-\alpha_i) w_i\right\}$ is bounded. It follows then that $d = 0$. □

We now have the following weak convergence theorems without uniform convexity.

Theorem 3.7. *Let E be a 2-uniformly smooth Banach space satisfying Opial's condition. Let C be a nonempty closed convex subset of E. Let* Q_C *be a sunny nonexpansive retraction from E onto C and* $A : C \to E$ *be an* α*-inverse strongly accretive mapping with* $S(C, A) \neq \emptyset$ *and* $\alpha > 0$. *Suppose that* $\{x_n\}$ *is iteratively defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{\lambda_n\} \subset [a, \frac{\alpha}{K^2}] \subset (0, \frac{\alpha}{K^2}]$ *satisfy one of the following conditions:*

(i) $\{\alpha_n\} \subset [c, d] \subset (0, 1)$ *and* $\lambda_{n+1} - \lambda_n \to 0$;

(ii) $\{\alpha_n\} \subset [c, 1) \subset (0, 1)$, $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$, *and* $\frac{\lambda_{n+1}-\lambda_n}{1-\alpha_n} \to 0$.

Then, $\{x_n\}$ *converges weakly to an element in* $S(C, A)$.

Proof. Note that every 2-uniformly smooth Banach space is reflexive. The result follows from Propositions 3.3, 3.5 and 3.6. □

Remark 3.8. Conditions (i) and (ii) in Theorem 3.7 are not comparable.

(1) If $\alpha_n \equiv \frac{1}{2}$ and $\{\lambda_n\}$ is a sequence in $(0, \frac{\alpha}{K^2}]$ such that $\lambda_n - \lambda_{n+1} \to 0$ and $0 < \liminf_{n\to\infty} \lambda_n < \limsup_{n\to\infty} \lambda_n < 1$, then $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy condition (i) but fail condition (ii).

(2) If $\alpha_n \equiv \frac{n}{n+1}$ and $\lambda_n \equiv \lambda \in (0, \frac{\alpha}{K^2}]$, then $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy condition (ii) but fail condition (i).

Remark 3.9. Note that the Opial property and uniform convexity are independent. Theorem 3.7 is a supplementary to Theorem 3.1 of Aoyama et al. [5].

3.3. Convergence results in uniformly convex spaces

In this subsection, we prove two more convergence theorems in uniformly convex spaces, which are also a supplementary to Theorem 3.1 of Aoyama et al. [5]. Let us start with some propositions.

Proposition 3.10. *Assume that* E *is a uniformly convex Banach space. Suppose that* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy the following properties*:

(i) $\{\lambda_n\} \subset [a, \alpha/K^2] \subset (0, \alpha/K^2]$;

(ii) $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$ *and* $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$.

Then, $x_n - y_n \to 0$.

Proof. Let $p \in S(C, A)$. Note that $\lim_{n\to\infty} ||x_n - p||$ exists and hence both $\{x_n\}$ and $\{y_n\}$ are bounded. By the uniform convexity of E and Lemma 2.1, there exists a continuous and strictly increasing function g such that

$$\begin{aligned} ||x_{n+1}-p||^2 &= ||\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)||^2 \\ &\leq \alpha_n||x_n-p||^2+(1-\alpha_n)||y_n-p||^2-\alpha_n(1-\alpha_n)g(||x_n-y_n||) \\ &\leq \alpha_n||x_n-p||^2+(1-\alpha_n)||x_n-p||^2-\alpha_n(1-\alpha_n)g(||x_n-y_n||) \\ &= ||x_n-p||^2-\alpha_n(1-\alpha_n)g(||x_n-y_n||) \end{aligned}$$

for all $n \geq 1$. Hence, for each $m \geq 1$, we have

$$\sum_{n=1}^{m} \alpha_n(1-\alpha_n)g(||x_n - y_n||) \leq ||x_1 - p||^2 - ||x_{m+1} - p||^2.$$

In particular, $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)g(||x_n - y_n||) < \infty$. It follows from $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and Lemma 2.9 that $\liminf_{n\to\infty} g(||x_n - y_n||) = 0$. By the properties of the function g, we get that $\liminf_{n\to\infty} ||x_n - y_n|| = 0$. Finally, we show that $\lim_{n\to\infty} ||x_n - y_n||$ actually exists. To see this, we consider the following estimate obtained directly from Proposition 3.2:

$$\begin{aligned} ||x_{n+1} - y_{n+1}|| &\leq ||x_{n+1} - y_n|| + ||y_n - y_{n+1}|| \\ &\leq \alpha_n||x_n - y_n|| + ||x_{n+1} - x_n|| + |\lambda_{n+1} - \lambda_n|\ ||Ax_n|| \\ &= \alpha_n||x_n - y_n|| + (1-\alpha_n)||x_n - y_n|| + |\lambda_{n+1} - \lambda_n|\ ||Ax_n|| \\ &= ||x_n - y_n|| + |\lambda_{n+1} - \lambda_n|\ ||Ax_n||. \end{aligned}$$

The assertion follows since $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}|\ ||Ax_n|| < \infty$ and Lemma 2.8. □

Let us recall the concept of strongly nonexpansive sequences introduced by Aoyama et al. (see [14]). A sequence of nonexpansive mappings $\{T_n\}$ of C is called a *strongly nonexpansive sequence* if $x_n - y_n - (T_nx_n - T_ny_n) \to 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and $||x_n - y_n|| - ||T_nx_n - T_ny_n|| \to 0$. It is noted that if $\{T_n\}$ is a constant sequence, then this property reduces to the concept of strongly nonexpansive mappings studied by Bruck and Reich [15].

Proposition 3.11. *Assume that E is a uniformly convex Banach space and* $\{\lambda_n\} \subset (0, b] \subset (0, \alpha/K^2)$. *Then,* $\{Q_C (I - \lambda_nA)\}$ *is a strongly nonexpansive sequence.*

Proof. Notice first that Q_C is a strongly nonexpansive mapping (see [16,17]). Next, we prove that $\{I - \lambda_nA\}$ is a strongly nonexpansive sequence and then the assertion follows. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\{x_n - y_n\}$ is bounded and $||x_n - y_n|| - ||(I - \lambda_nA)x_n - (I - \lambda_nA)y_n|| \to 0$. It follows from Lemma 2.2 that

$$\begin{aligned} &\frac{2(\alpha - K^2b)}{b}||\lambda_nAx_n - \lambda_nAy_n||^2 \\ &\leq \frac{2(\alpha - K^2\lambda_n)}{\lambda_n}||\lambda_nAx_n - \lambda_nAy_n||^2 \\ &= 2\lambda_n(\alpha - K^2\lambda_n)||Ax_n - Ay_n||^2 \\ &\leq ||x_n - y_n||^2 - ||(I - \lambda_nA)x_n - (I - \lambda_nA)y_n||^2 \to 0. \end{aligned}$$

In particular, $\lambda_nAx_n - \lambda_nAy_n \to 0$ and hence

$$x_n - y_n - ((I - \lambda_nA)x_n - (I - \lambda_nA)y_n) = \lambda_nAx_n - \lambda_nAy_n \to 0.$$

Proposition 3.12. *Assume that E is a uniformly convex Banach space. Suppose that* $\alpha_n \equiv 0$ *and* $\{\lambda_n\} \subset (0, b] \subset (0, \alpha/K^2)$. *Then,* $x_n - y_n \to 0$.

Proof. Let us rewritten the iteration as follows:

$$x_{n+1} = Q_C(I - \lambda_nA)x_n \quad (n \geq 1).$$

Let $p \in S(C, A)$. Notice that $p = Q_C (I - \lambda_nA)p$ for all $n \geq 1$. Then, $\lim_{n\to\infty} ||x_n - p||$ exists, and hence,

$$||x_n - p|| - ||Q_C(I - \lambda_n A)x_n - p|| = ||x_n - p|| - ||x_{n+1} - p|| \to 0.$$

It follows from the preceding proposition that

$$x_n - Q_C(I - \lambda_n A)x_n = (x_n - p) - (Q_C(I - \lambda_n A)x_n - p) \to 0.$$

□

We now obtain the following weak convergence theorems in uniformly convex spaces.

Theorem 3.13. *Let E be a uniformly convex and 2-uniformly smooth Banach space. Let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and $A : C \to E$ be an α-inverse strongly accretive mapping with $S(C, A) \neq \emptyset$ and $\alpha > 0$. Suppose that $\{x_n\}$ is iteratively defined by*

$$\begin{cases} x_1 \in C \text{ arbitrarily chosen}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n A x_n) \quad (n \geq 1), \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [a, \frac{\alpha}{K^2}] \subset (0, \frac{\alpha}{K^2}]$ satisfy one of the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;
(ii) $\alpha_n \equiv 0$ *and* $\{\lambda_n\} \subset [a, b] \subset (0, \alpha/K^2)$.

Then, $\{x_n\}$ converges weakly to an element in $S(C, A)$.

Proof. The result follows from Propositions 3.3, 3.10 and 3.12. □

Remark 3.14. It is easy to see that conditions (i) and (ii) in Theorem 3.13 are not comparable.

Remark 3.15. Compare Theorem 3.13 to Theorem 1.2 of Aoyama et al., our result is a supplementary to their result. It is noted that, for example, our iteration scheme with $\alpha_n \equiv 0$ and $\lambda_n \equiv \alpha/(\alpha/K^2)$ is simpler than the one in Theorem 1.2.

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Authors' contributions
All authors contribute equally and significantly in this research work. All authors read and approved the final manuscript.

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