Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some results on a general iterative method for *k*-strictly pseudo-contractive mappings

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Abstract

Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, and $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ for some $0 \leq k < 1$. Let $F : C \to C$ be a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$ and $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0, 1)$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in (0, 1). It is proved that under appropriate control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ generated by the iterative scheme $x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n$, where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$ and P_C is the metric projection of H onto C, converges strongly to $q \in F(T)$, which solves the variational inequality $\langle \mu Fq - \gamma f(q), q - p\rangle \leq 0$ for $p \in F(T)$.

MSC: 47H09, 47H05, 47H10, 47J25, 49M05

Keywords: Iterative schemes, *k*-strictly pseudo-contractive mapping, Nonexpansive mapping, Fixed points, Contraction, κ-Lipschitzian, η-strongly monotone operator, Variational inequality, Hilbert space

1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that $||f(x) - f(y)|| \leq \alpha||x - y||$, $x, y \in C$. A mapping $T : C \to H$ is said to be *k-strictly pseudo-contractive* if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad x, y \in C,$$

and $F(T)$ denote the set of fixed points of the mapping T; that is, $F(T) = \{x \in C : Tx = x\}$.

Note that the class of k-strictly pseudo-contractions includes the class of non-expansive mappings T on C (that is, $||Tx - Ty|| \leq ||x - y||$, $x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent of the class of k-strictly pseudo-contractive mappings (see [1-3]). The class of pseudo-contraction is one of the

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most important classes of mappings among nonlinear mappings. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudo-contractions, see, for example, [4-7] and references therein.

For nonexpansive mappings, one recent way to study them is to construct the iterative scheme, the so-called viscosity iteration method: more precisely, for a nonexpansive mapping T, a contraction f with the contractive constant $\alpha \in (0, 1)$, and $\alpha_n \in (0, 1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

This iterative scheme was first introduced by Moudafi [8].

In particular, under the control conditions on $\{\alpha_n\}$

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$;

(C3) $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$; or,

(C4) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$,

Xu [9] proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality:

$$\langle q - f(q),\ q - p\rangle \leq 0, \quad p \in F(T).$$

Recall that an operator A is strongly positive on H if there exists a constant $\bar{\gamma} > 0$ with the property:

$$\langle Ax,\ x\rangle \geq \bar{\gamma}||x||^2, \quad x \in H.$$

In 2006, as the viscosity approximation method, Marino and Xu [10] considered the following iterative method: for a strongly positive bounded linear operator A on H with constant $\bar{\gamma} > 0$, a nonexpansive mapping T on H, a contraction $f : H \to H$ with the contractive constant $\alpha \in (0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\gamma > 0$,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n\gamma f(x_n), \quad n \geq 0. \tag{1.2}$$

They proved that if the sequence $\{\alpha_n\}$ satisfies the conditions (C1), (C2), and (C3) (or (C1), (C2), and (C4)), then the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*,\ x - x^*\rangle \geq 0, \quad x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax,\ x\rangle - h(x),$$

where h is a potential function for γf.

In 2010, in order to improve the corresponding results of Cho et al. [5] as well as Marino and Xu [10] by removing the condition (C3), Jung [6] studied the following composite iterative scheme for the class of k-strictly pseudo-contractive mappings.

Theorem J. *Let H be a Hilbert space, C be a closed convex subset of H such that $C \pm C \subset C$, $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$, for some $0 \leq k < 1$. Let A be a strongly positive bounded linear operator on C with constant $\bar{\gamma} \in (0, 1)$ and $f : C \to C$ be a contraction with the contractive constant $\alpha \in (0, 1)$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* (0, 1) *satisfying the conditions* (C1), (C2) *and the condition* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Let* $\{x_n\}$ *be a sequence in C generated by*

$$\begin{cases} x_0 \in C \\ y_n = \beta_n x_n + (1-\beta_n) P_C S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \ge 0, \end{cases}$$

where $S : C \to H$ *is a mapping defined by* $Sx = kx + (1 - k)Tx$ *and* P_C *is the metric projection of H onto C. Then* $\{x_n\}$ *converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality related to the linear operator A:*

$$\langle \gamma f(q) - Aq,\ p - q \rangle \le 0, \quad p \in F(T).$$

On the other hand, a mapping $F : H \to H$ is called κ-Lipschitzian if there exists a positive constant κ such that

$$||Fx - Fy|| \le \kappa ||x - y||, \quad x,\ y \in H. \tag{1.3}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy,\ x - y \rangle \ge \eta ||x - y||^2, \quad x,\ y \in H. \tag{1.4}$$

From the definitions, we note that a strongly positive bounded linear operator A is a $||A||$-Lipschitzian and $\bar{\gamma}$-strongly monotone operator.

In 2001, Yamada [11] introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = (I - \mu\lambda_n F) S x_n, \quad n \ge 1, \tag{1.5}$$

where $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$, $\eta > 0$, $0 < \mu < \frac{2\eta}{\kappa^2}$ and $S : H \to H$ is a nonexpansive mapping, and proved that if $\{\lambda_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle F\tilde{x},\ x - \tilde{x} \rangle \ge 0, \quad x \in F(S).$$

In 2010, by combining the iterative method (1.2) with the Yamada's method (1.5), Tian [12] considered the following general iterative method.

Theorem T1. *Let H be a Hilbert space,* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with* $\kappa > 0$ *and* $\eta > 0$, *and* $S : H \to H$ *be a nonexpansive mapping with* $F(S) \neq \emptyset$. *Let* $f : H \to H$ *be a contraction with the contractive constant* $\alpha \in (0, 1)$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$. *Let* $\{\alpha_n\}$ *be a sequence in* (0, 1) *satisfying the conditions* (C1), (C2) *and* (C3) *(or* (C1), (C2) *and* (C4)*). Let* $\{x_n\}$ *be a sequence in H generated by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) S x_n, \quad n \ge 0. \end{cases}$$

Then $\{x_n\}$ *converges strongly to a fixed point* $\tilde{x}$ *of S, which is the unique solution of the following variational inequality related to the operator F*:

$$\langle \mu F\tilde{x} - \gamma f(\tilde{x}), \tilde{x} - z \rangle \le 0, \quad z \in F(S). \tag{1.6}$$

In this paper, motivated by the above-mentioned results, we consider the following general iterative scheme for strictly pseudo-contractive mapping: for C a closed convex subset of H such that $C \pm C \subset C$, k-strictly pseudo-contractive mapping $T : C \to H$ with $F(T) \neq \emptyset$, a contraction $f : C \to C$ with the contractive constant $\alpha \in (0, 1)$, $\mu > 0$ and $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n, \quad n \geq 0, \end{cases} \tag{IS}$$

where $S : C \to H$ is a mapping defined by $Sx = kx+(1 - k)Tx$, P_C is the metric projection of H onto C, and $F : C \to C$ is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0$ and $\eta > 0$. Under certain different control conditions on $\{\alpha_n\}$, we establish the strong convergence of the sequence $\{x_n\}$ generated by (IS) to a fixed point of T, which is a solution of the variational inequality (1.6) related to the operator F. By removing the condition (C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on $\{\alpha_n\}$, the main results improve, develop and complement the corresponding results of Tian [12] as well as Cho et al. [5], Jung [6] and Marino and Xu [10]. Our results also improve the corresponding results of Halpern [13], Moudafi [8], Wittmann [14] and Xu [9].

2 Preliminaries and lemmas

Throughout this paper, when $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$||x - P_C(x)|| \leq ||x - y||$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive.

In a Hilbert space H, we have

$$||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y\rangle \quad \text{for } x, y \in H. \tag{2.1}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$$

holds for every $y \in H$ with $y \neq x$.

We need the following lemmas for the proof of our main results.

Lemma 2.1 [15]. *Let H be a Hilbert space and C be a closed convex subset of H. If T is a k-strictly pseudo-contractive mapping on C, then the fixed point set F(T) is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.2 [15]. *Let H be a Hilbert space and C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$. Then $F(P_C T) = F(T)$.*

Lemma 2.3 [15]. *Let H be a Hilbert space, C be a closed convex subset of H, and $T : C \to H$ be a k-strictly pseudo-contractive mapping. Define a mapping $S : C \to H$ by $Sx = \lambda x + (1 - \lambda) Tx$ for all $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

The following Lemmas 2.4 and 2.5 can be obtained from the Proposition 2.6 of Acedo and Xu [4].

Lemma 2.4. *Let H be a Hilbert space and C be a closed convex subset of H. For any* $N \geq 1$, *assume that for each* $1 \leq i \leq N$, $T_i : C \to H$ *is a* k_i*-strictly pseudo-contractive mapping for some* $0 \leq k_i < 1$. *Assume that* $\{\eta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$. *Then* $\sum_{i=1}^N \eta_i T_i$ *is a nonself-k-strictly pseudo-contractive mapping with* $k = \max\{k_i : 1 \leq i \leq N\}$.

Lemma 2.5. *Let* $\{T_i\}_{i=1}^N$ *and* $\{\eta_i\}_{i=1}^N$ *be given as in Lemma 2.4. Suppose that* $\{T_i\}_{i=1}^N$ *has a common fixed point in C. Then* $F(\sum_{i=1}^N \eta_i T_i) = \bigcap_{i=1}^N F(T_i)$.

Lemma 2.6 [16,17]. *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n \delta_n + r_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^\infty \lambda_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^\infty \lambda_n \delta_n < \infty$,

(iii) $r_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^\infty r_n = \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.7 [18]. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space E and* $\{\gamma_n\}$ *be a sequence in* $[0, 1]$ *which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n$ *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty} (||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then $\lim_{n\to\infty} ||z_n - x_n|| = 0$.

Lemma 2.8. *In a Hilbert space H, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle, \quad x, y \in H.$$

Lemma 2.9. *Let C be a nonempty closed convex subset of a Hilbert space H such that* $C \pm C \subset C$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with* $\kappa > 0$ *and* $\eta > 0$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < t < \rho < 1$. *Then* $S := \rho I - t\mu F : C \to C$ *is a contraction with contractive constant* $\rho - t\tau$, *where* $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2) < 1$ *with* $t < \frac{1}{\tau}$.

Proof. From (1.3), (1.4) and (2.1), we have

$$\begin{aligned}\|Sx - Sy\|^2 &= ||\rho(x - y) - t\mu(Fx - Fy)||^2 \\ &= \rho^2||x - y||^2 + t^2\mu^2||Fx - Fy||^2 - 2t\rho\mu\langle Fx - Fy, x - y\rangle \\ &\leq \rho^2||x - y||^2 + t^2\mu^2\kappa^2||x - y|| - 2t\rho\mu\eta||x - y||^2 \\ &< \rho^2||x - y||^2 + t\rho\mu^2\kappa^2||x - y|| - 2t\rho\mu\eta||x - y||^2 \\ &= (\rho^2 - t\rho\mu(2\eta - \mu\kappa^2))||x - y||^2 \\ &< (\rho - t\tau)^2||x - y||^2,\end{aligned}$$

where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2) < 1$, and so

$$||Sx - Sy|| < (\rho - t\tau)||x - y||.$$

Hence S is a contraction with contractive constant $\rho - t\tau$. □

3 Main results

We need the following result for the existence of solutions of a certain variational inequality, which is slightly an improvement of Theorem 3.1 of Tian [12].

Theorem T2. *Let H be a Hilbert space, C be a closed convex subset of H such that C ± C ⊂ C, and* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with* $\kappa > 0$ *and* $\eta > 0$. *Let* $f : C \to C$ *be a contraction with the contractive constant* $\alpha \in (0, 1)$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ *and* $\tau < 1$. *Let* x_t *be a fixed point of a contraction* $St \ni x \alpha\, t\gamma f(x) + (I - t\mu F)Tx$ *for* $t \in (0, 1)$ *and* $t < \frac{1}{\tau}$. *Then* $\{x_t\}$ *converges strongly to a fixed point* $\tilde{x}$ *of T as* $t \to 0$, *which solves the following variational inequality:*

$$\langle \mu F\tilde{x} - \gamma f(\tilde{x}), \tilde{x} - p\rangle \leq 0, \quad p \in F(T).$$

Equivalently, we have $P_{F(T)}(I - \mu F + \gamma f)\tilde{x} = \tilde{x}$.

Now, we study the strong convergence result for a general iterative scheme (IS).

Theorem 3.1. *Let H be a Hilbert space, C be a closed convex subset of H such that C ± C ⊂ C, and* $T : C \to H$ *be a k-strictly pseudo-contractive mapping with* $F(T) \neq \emptyset$ *for some* $0 \leq k < 1$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with* $\kappa > 0$ *and* $\eta > 0$. *Let* $f : C \to C$ *be a contraction with the contractive constant* $\alpha \in (0, 1)$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ *and* $\tau < 1$. *Let* $f\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0, 1)$ *which satisfy the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\Sigma_{n=0}^{\infty} \alpha_n = \infty$;

(B) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta n < 1$.

Let $\{x_n\}$ *be a sequence in C generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n, \quad n \geq 0, \end{cases}$$

where $S : C \to H$ *is a mapping defined by* $Sx = kx + (1 - k)Tx$ *and* P_C *is the metric projection of H onto C. Then* $\{x_n\}$ *converges strongly to* $q \in F(T)$, *which solves the following variational inequality:*

$$\langle \mu F q - \gamma f(q),\ q - p\rangle \leq 0, \quad p \in F(T).$$

Proof. First, from the condition (C1), without loss of generality, we assume that $\alpha_n \tau < 1$, $\frac{2\alpha_n(\tau - \gamma\alpha)}{1 - \alpha_n \alpha\gamma} < 1$ and $\alpha_n < (1 - \beta_n)$ for $n \geq 0$.

We divide the proof several steps:

Step 1. We show that $||x_n - p|| \leq \max\left\{||x_0 - p||, \frac{||\gamma f(p) - \mu F p||}{\tau - \gamma\alpha}\right\}$ for all $n \geq 0$ and all $p \in F(T) = F(S)$. Indeed, let $p \in F(T)$. Then from Lemma 2.9, we have

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_n(\gamma f(x_n) - \mu F p) + \beta_n(x_n - p) \\ &\quad + ((1 - \beta_n)I - \alpha_n \mu F)P_C S x_n - ((1 - \beta_n)I - \alpha_n \mu F)P_C S p|| \\ &\leq (1 - \beta_n - \alpha_n \tau)||x_n - p|| + \beta_n ||x_n - p|| + \alpha_n ||\gamma f(x_n) - \mu F p|| \\ &\leq (1 - \alpha_n \tau)||x_n - p|| + \alpha_n(||\gamma f(x_n) - \gamma f(p)|| + ||\gamma f(p) - \mu F p||) \\ &\leq (1 - (\tau - \gamma\alpha)\alpha_n)||x_n - p|| + (\tau - \gamma\alpha)\alpha_n \frac{||\gamma f(p) - \mu F p||}{\tau - \gamma\alpha} \\ &\leq \max\left\{||x_n - p||,\ \frac{||\gamma f(p) - \mu F p||}{\tau - \gamma\alpha}\right\}. \end{aligned}$$

Using an induction, we have $\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p)-\mu Fp\|}{\tau-\gamma\alpha}\right\}$. Hence, $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$, $\{P_C Sx_n\}$ and $\{FP_C Sx_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To this show, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \quad \text{for all } n \geq 0.$$

Observe that from the definition of z_n,

$$\begin{aligned} & z_{n+1} - z_n \\ = & \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ = & \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}\mu F)P_C Sx_{n+1}}{1-\beta_{n+1}} \\ & - \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n\mu F)P_C Sx_n}{1-\beta_n} \\ = & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n}\gamma f(x_n) \\ & + P_C Sx_{n+1} - P_C Sx_n + \frac{\alpha_n}{1-\beta_n}\mu FP_C Sx_n - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\mu FP_C Sx_{n+1} \\ = & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - \mu FP_C Sx_{n+1}) \\ & + \frac{\alpha_n}{1-\beta_n}(\mu FP_C Sx_n - \gamma f(x_n)) + P_C Sx_{n+1} - P_C Sx_n. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|f(x_{n+1})\| + \mu\|FP_C Sx_{n+1}\|) \\ & + \frac{\alpha_n}{1-\beta_n}(\mu\|FP_C Sx_n\| + \gamma\|f(x_n)\|). \end{aligned}$$

From the condition (C1) and (B), it follows that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.7, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - P_C Sx_n\| = 0$. Indeed, since

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)P_C Sx_n,$$

we have

$$\begin{aligned} \|x_n - P_C Sx_n\| \leq & \|x_n - x_{n+1}\| + \|x_{n+1} - P_C Sx_n\| \\ \leq & \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - \mu FP_C Sx_n\| \\ & + \beta_n\|x_n - P_C Sx_n\|, \end{aligned}$$

that is,

$$\|x_n - P_C Sx_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - \mu FP_C Sx_n\|.$$

So, from the conditions (C1) and (B) and Step 2, it follows that

$$\lim_{n\to\infty} ||x_n - P_C S x_n|| = 0.$$

Step 4. We show that

$$\limsup_{n\to\infty} \langle \gamma f(q) - \mu F q,\ x_n - q\rangle \le 0,$$

where $q = \lim_{t\to 0} x_t$ being $x_t = t\gamma f(x_t) + (I - t\mu F)P_C S x_t$ for $0 < t < 1$ and $t < \frac{1}{\tau}$. We note that from Lemmas 2.2 and 2.3 and Theorem T2, $q \in F(T) = F(S)$ and q is a solution of a variational inequality

$$\langle \mu F q - \gamma f(q),\ q - p\rangle \le 0, \quad p \in F(T). \tag{3.1}$$

To show this, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty} \langle \gamma f(q) - \mu F q,\ x_{n_j} - q\rangle = \limsup_{n\to\infty} \langle \gamma f(q) - \mu F q,\ x_n - q\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup w$. Since $||x_n - P_C S x_n|| \to 0$ by Step 3, we obtain $w = P_C S w$. In fact, if $w \neq P_C S w$, then, by Opial condition,

$$\begin{aligned}\liminf_{j\to\infty} ||x_{n_j} - w|| &< \liminf_{j\to\infty} ||x_{n_j} - P_C S w|| \\ &\le \liminf_{j\to\infty} (||x_{n_j} - P_C S x_{n_j}|| + ||P_C S x_{n_j} - P_C S w||) \\ &\le \liminf_{j\to\infty} ||x_{n_j} - w||,\end{aligned}$$

which is a contradiction. Hence $w = P_C S w$. Since $F(P_C S) = F(S)$, from Lemma 2.3, we have $w \in F(T)$. Therefore, from (3.1), we conclude that

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(q) - \mu F q,\ x_n - q\rangle &= \lim_{j\to\infty} \langle \gamma f(q) - \mu F q,\ x_{n_j} - q\rangle \\ &= \langle \gamma f(q) - \mu F q,\ w - q\rangle \\ &\le 0.\end{aligned}$$

Step 5. We show that $\lim_{n\to\infty} ||x_n - q|| = 0$, where $q = \lim_{t\to 0} x_t$ being $x_t = t\gamma f(xt) + (I - t\mu F)P_C S x_t$ for $0 < t < 1$ and $t < \frac{1}{\tau}$, and q is a solution of a variational inequality

$$\langle \mu F q - \gamma f(q),\ q - p\rangle \le 0, \quad p \in F(T).$$

Indeed, from (IS), we have

$$\begin{aligned}x_{n+1} - q = {} & \alpha_n(\gamma f(x_n) - \mu F q) + \beta_n(x_n - q) \\ & + ((1-\beta_n)I - \alpha_n \mu F)P_C S x_n - ((1-\beta_n)I - \alpha_n \mu F)q.\end{aligned}$$

Applying Lemmas 2.8 and 2.9, we have

$$\begin{aligned}& ||x_{n+1} - q||^2 \\ \le {} & ||\beta_n(x_n - q) + ((1-\beta_n)I - \alpha_n\mu F)P_C S x_n - ((1-\beta_n)I - \alpha_n\mu F)P_C S q||^2 \\ & + 2\alpha_n \langle \gamma f(x_n) - \mu F q,\ x_{n+1} - q\rangle \\ \le {} & ((1-\beta_n - \alpha_n\tau)||x_n - q|| + \beta_n ||x_n - q||)^2 \\ & + 2\alpha_n\gamma\langle f(x_n) - f(q),\ x_{n+1} - q\rangle + 2\alpha_n\langle \gamma f(q) - \mu F q,\ x_{n+1} - q\rangle \\ \le {} & (1 - \tau\alpha_n)^2 ||x_n - q||^2 + 2\alpha_n\gamma\alpha ||x_n - q||\ ||x_{n+1} - q|| \\ & + 2\alpha_n\langle \gamma f(q) - \mu F q,\ x_{n+1} - q\rangle \\ \le {} & (1 - \tau\alpha_n)^2 ||x_n - q||^2 + \alpha_n\gamma\alpha(||x_n - q||^2 + ||x_{n+1} - q||^2) \\ & + 2\alpha_n\langle \gamma f(q) - \mu F q,\ x_{n+1} - q\rangle,\end{aligned}$$

that is,

$$\begin{aligned} \|x_{n+1}-q\|^2 &\le \frac{1-2\tau\alpha_n+\tau^2\alpha_n^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-\mu Fq,\ x_{n+1}-q\rangle \\ &= \left(1-\frac{2(\tau-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)\|x_n-q\|^2+\frac{\tau^2\alpha_n^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-\mu Fq,\ x_{n+1}-q\rangle \\ &\le \left(1-\frac{2(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\alpha_n\right)\|x_n-q\|^2+\frac{2(\tau-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\times \\ &\qquad \left(\frac{\tau^2\alpha_n}{2(\tau-\gamma\alpha)}M+\frac{1}{\tau-\gamma\alpha}\langle\gamma f(q)-\mu Fq,\ x_{n+1}-q\rangle\right) \\ &= (1-\lambda_n)\|x_n-q\|^2+\lambda_n\delta_n, \end{aligned}$$

where $M = \sup\{||x_n - q||2 : n \geq 0\}$, $\lambda_n = \frac{2(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\alpha_n$ and

$$\delta_n = \frac{\tau^2\alpha_n}{2(\tau-\gamma\alpha)}M + \frac{1}{\tau-\gamma\alpha}\langle\gamma f(q)-\mu Fq,\ x_{n+1}-q\rangle.$$

From the conditions (C1) and (C2) and Step 4, it is easy to see that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Hence, by Lemma 2.7, we conclude $x_n \to q$ as $n \to \infty$. This completes the proof. □

Remark 3.1. (1) Theorem 3.1 extends and develops Theorem 3.2 of Tian [12] from a nonexpansive mapping to a strictly pseudo-contractive mapping together with removing the condition (C3) $\Sigma_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$.

(2) Theorem 3.1 also generalizes Theorem 2.1 of Jung [6] as well as Theorem 2.1 of Cho et al. [5] and Theorem 3.4 of Marino and Xu [10] from a strongly positive bounded linear operator A to a κ-Lipschitzian and η-strongly monotone operator F.

(3) Theorem 3.1 also improves the corresponding results of Halpern [13], Moudafi [8], Wittmann [14] and Xu [9] as some special cases.

Theorem 3.2. *Let H be a Hilbert space, C be a closed convex subset of H such that* $C \pm C \subset C$, *and* $T_i : C \to H$ *be a* k_i*-strictly pseudo-contractive mapping for some* $0 \leq k_i < 1$ *and* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with* $\kappa > 0$ *and* $\eta > 0$. *Let* $f : C \to C$ *be a contraction with the contractive constant* $\alpha \in (0, 1)$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ *and* $\tau < 1$. *Let* $\{\alpha_n\}$ *and* $\{\beta n\}$ *be sequences in* (0, 1) *which satisfy the conditions.*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$;

(B) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}$ *be a sequence in C generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\mu F)P_C S x_n, \quad n \geq 0, \end{cases}$$

where $S : C \to H$ *is a mapping defined by* $Sx = kx + (1-k)\sum_{i=1}^N \eta_i T_i x$ *with* $k = \max\{k_i : 1 \leq i \leq N\}$ *and* $\{\eta_i\}$ *is a positive sequence such that* $\sum_{i=1}^N \eta_i = 1$ *and* P_C *is the metric* *projection of H onto C. Then* $\{x_n\}$ *converges strongly to* $q \in F(T)$, *which solves the following variational inequality:*

$$\langle \mu Fq - \gamma f(q),\ q - p\rangle \leq 0, \quad p \in \bigcap_{i=1}^{N} F(T_i).$$

Proof. Define a mapping $T : C \to H$ by $Tx = \sum_{i=1}^{N} \eta_i T_i x$. By Lemmas 2.4 and 2.5, we conclude that $T : C \to H$ is a k-strictly pseudo-contractive mapping with $k = \max\{k_i : 1 \leq i \leq N\}$ and $F(T) = F(\sum_{i=1}^{N} \eta_i T_i) = \bigcap_{i=1}^{N} F(T_i)$. Then the result follows from Theorem 3.1 immediately. □

As a direct consequence of Theorem 3.2, we have the following result for nonexpansive mappings (that is, 0-strictly pseudo-contractive mappings).

Theorem 3.3. *Let H be a Hilbert space, C be a closed convex subset of H such that* $C \pm C \subset C$, $\{T_i\}_{i=1}^{N} : C \to H$ *be a finite family of nonexpansive mappings with* $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $F : C \to C$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with* $\kappa > 0$ *and* $\eta > 0$. *Let* $f : C \to C$ *be a contraction with the contractive constant* $\alpha \in (0, 1)$. *Let* $0 < \mu < \frac{2\eta}{\kappa^2}$, $0 < \gamma < \frac{\mu(\eta - \frac{\mu\kappa^2}{2})}{\alpha} = \frac{\tau}{\alpha}$ *and* $\tau < 1$. *Let* $\{\alpha_n\}$ *and* $\{\beta n\}$ *be sequences in* $(0, 1)$ *which satisfy the conditions.*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\Sigma_{n=0}^{\infty} \alpha_n = \infty$;

(B) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}$ *be a sequence in C generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F) P_C \sum_{i=1}^{N} \eta_i T_i x_n, \quad n \geq 0, \end{cases}$$

where $\{\eta_i\}_{i=1}^{N}$ *is a positive sequence such that* $\sum_{i=1}^{N} \eta_i = 1$ *and* P_C *is the metric projection of H onto C. Then* $\{x_n\}$ *converges strongly to a common fixed point q of* $\{T_i\}_{i=1}^{N}$, *which solves the following variational inequality:*

$$\langle \mu Fq - \gamma f(q),\ q - p\rangle \leq 0, \quad p \in \bigcap_{i=1}^{N} F(T_i).$$

Remark 3.2. (1) Theorems 3.2 and 3.3 also generalize Theorems 2.2 and 2.4 of Jung [6] from a strongly positive bounded linear operator A to a κ-Lipschitzian and η-strongly monotone operator F.

(2) Theorems 3.2 and 3.3 also improve and complement the corresponding results of Cho et al. [5] by removing the condition (C3) $\Sigma_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ together with using a κ-Lipschitzian and η-strongly monotone operator F.

(3) As in [19], we also can establish the result for a countable family $\{T_i\}$ of k_i-strict pseudo-contractive mappings with $0 \leq k_i < 1$.

4 Competing interests

The authors declare that they have no competing interests.

Acknowledgements

This study was supported by research funds from Dong-A University.

Received: 30 September 2010 Accepted: 2 August 2011 Published: 2 August 2011

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doi:10.1186/1687-1812-2011-24
Cite this article as: Jung: Some results on a general iterative method for *k*-strictly pseudo-contractive mappings. *Fixed Point Theory and Applications* 2011 **2011**:24.