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Convergence theorems of solutions of a generalized variational inequality

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Abstract

The convex feasibility problem (CFP) of finding a point in the nonempty intersection $\bigcap_{m=1}^{r} C_m$ is considered, where $r \geq 1$ is an integer and each C_m is assumed to be the solution set of a generalized variational inequality. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A_m, B_m : C \to H$ be relaxed cocoercive mappings for each $1 \leq m \leq r$. It is proved that the sequence $\{x_n\}$ generated in the following algorithm:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{m=1}^{r} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n), \quad n \geq 1,$$

where $u \in C$ is a fixed point, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{(1,n)}\}$, ..., and $\{\delta_{(r,n)}\}$ are sequences in (0, 1) and $\{\tau_m\}_{m=1}^{r}$, $\{\lambda_m\}_{m=1}^{r}$ are positive sequences, converges strongly to a solution of CFP provided that the control sequences satisfies certain restrictions.
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1. Introduction and Preliminaries

Many problems in mathematics, in physical sciences and in real-world applications of various technological innovations can be modeled as a convex feasibility problem (CFP). This is the problem of finding a point in the intersection of finitely many closed convex sets in a real Hilbert spaces H. That is,

$$\text{finding an } x \in \bigcap_{m=1}^{r} C_m, \tag{1.1}$$

where $r \geq 1$ is an integer and each C_m is a nonempty closed and convex subset of H. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [1,2], computer tomography [3] and radiation therapy treatment planning [4].

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $||\cdot||$. Let C be a nonempty closed and convex subset of H and A: $C \to H$ a nonlinear mapping. Recall the following definitions:

(a) A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(b) A is said to be ρ-strongly monotone if there exists a positive real number $\rho > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \rho||x - y||^2, \quad \forall x, y \in C.$$

(c) A is said to be η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta||Ax - Ay||^2, \quad \forall x, y \in C.$$

(d) A is said to be relaxed η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\eta)||Ax - Ay||^2, \quad \forall x, y \in C.$$

(e) A is said to be relaxed (η, ρ)-cocoercive if there exist positive real numbers η, $\rho > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\eta)||Ax - Ay||^2 + \rho||x - y||^2, \quad \forall x, y \in C.$$

The main purpose of this paper is to consider the following generalized variational inequality. Given nonlinear mappings $A : C \to H$ and $B : C \to H$, find a $u \in C$ such that

$$\langle u - \tau Bu + \lambda Au, v - u\rangle \geq 0, \quad \forall v \in C, \tag{1.2}$$

where λ and τ are two positive constants. In this paper, we use $GVI(C, B, A)$ to denote the set of solutions of the generalized variational inequality (1.2).

It is easy to see that an element $u \in C$ is a solution to the variational inequality (1.2) if and only if $u \in C$ is a fixed point of the mapping $P_C(\tau B - \lambda A)$, where P_C denotes the metric projection from H onto C. Indeed, we have the following relations:

$$u = P_C(\tau B - \lambda A)u \Leftrightarrow \langle u - \tau Bu + \lambda Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

Next, we consider a special case of (1.2). If $B = I$, the identity mapping and $\tau = 1$, then the generalized variational inequality (1.1) is reduced to the following. Find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.4}$$

The variational inequality (1.4) emerging as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences was introduced by Stam-pacchia [5]. In this paper, we use $VI(C, A)$ to denote the set of solutions of the variational inequality (1.4).

Let $S : C \to C$ be a mapping. We use $F(S)$ to denote the set of fixed points of the mapping S. Recall that S is said to be nonexpansive if

$$||Sx - Sy|| \leq ||x - y||, \quad \forall x, y \in C.$$

It is well known that if C is nonempty bounded closed and convex subset of H, then the fixed point set of the nonexpansive mapping S is nonempty, see [6] more details. Recently, fixed point problems of nonexpansive mappings have been considered by many authors; see, for example, [7-16].

Recall that S is said to be demi-closed at the origin if for each sequence $\{x_n\}$ in C, $x_n \rightharpoonup x_0$ and $Sx_n \to 0$ imply $Sx_0 = 0$, where $\rightharpoonup$ and $\to$ stand for weak convergence and strong convergence.

Recently, many authors considered the variational inequality (1.4) based on iterative methods; see [17-32]. For finding solutions to a variational inequality for a cocoercive mapping, Iiduka et al. [22] proved the following theorem.

Theorem ITT. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-cocoercive operator of H into H with $VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows. $x_1 = x \in C$ and*

$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)P_C(x_n - \lambda_n Ax_n))$$

for every $n = 1, 2, \ldots$, where C is the metric projection from H onto C, $\{\alpha_n\}$ is a sequence in $[-1, 1]$, and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\alpha_n\} \in [a, b]$ for some a, b with $-1 < a < b < 1$ and $\{\lambda_n\} \in [c, d]$ for some c, d with $0 < c < d < 2(1 + a)\alpha$, then $\{x_n\}$ converges weakly to some element of $VI(C, A)$.

Subsequently, Iiduka and Takahashi [23] further studied the problem of finding solutions of the classical variational inequality (1.4) for cocoercive mappings (inverse-strongly monotone mappings) and nonexpansive mappings. They obtained a strong convergence theorem. More precisely, they proved the following theorem.

Theorem IT. *Let C be a closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpanisve mapping and A an α-cocoercive mapping of C into H such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n)$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[a, b]$.

If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\lambda_n\} \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

In this paper, motivated by research work going on in this direction, we study the CFP in the case that each C_m is a solution set of generalized variational inequality (1.2). Strong convergence theorems of solutions are established in the framework of real Hilbert spaces.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 [33]. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and $\{\beta_n\}$ a sequence in $(0, 1)$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 1.2 [34]. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let* $S_1 : C \to C$ *and* $S_2 : C \to C$ *be nonexpansive mappings on C. Suppose that* $F(S_1) \cap F(S_2)$ *is nonempty. Define a mapping* $S : C \to C$ *by*

$$Sx = aS_1x + (1 - a)S_2x, \quad \forall x \in C,$$

where a is a constant in (0, 1). *Then S is nonexpansive with* $F(S) = F(S_1) \cap F(S_2)$.

Lemma 1.3 [35]. *Let C be a nonempty closed and convex subset of a real Hilbert space H and* $S : C \to C$ *a nonexpansive mapping. Then I - S is demi-closed at zero.*

Lemma 1.4 [36]. *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* (0, 1) *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let* $A_m : C \to H$ *be a relaxed* (η_m, ρ_m)*-cocoercive and* μ_m*-Lipschitz continuous mapping and* $B_m : C \to H$ *a relaxed* $(\widehat{\eta}_m, \widehat{\rho}_m)$*-cocoercive and* $\widehat{\mu}_m$*-Lipschitz continuous mapping for each* $1 \leq m \leq r$. *Assume that* $\bigcap_{m=1}^{r} GVI(C, B_m, A_m) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{m=1}^{r} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n), \quad n \geq 1, \quad (\Upsilon)$$

where $u \in C$ *is a fixed point,* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{(1,n)}\}$, ..., *and* $\{\delta_{(r,n)}\}$ *are sequences in* (0, 1) *satisfying the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = \sum_{m=1}^{r} \delta_{(m,n)} = 1, \forall n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(d) $\lim_{n\to\infty} \delta_{(m,n)} = \delta_m \in (0, 1), \forall 1 \leq m \leq r$,

And $\{\tau_m\}_{m=1}^{r}$, $\{\lambda_m\}_{m=1}^{r}$ *are two positive sequences such that*

(e) $\sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\mu_m^2 + 2\lambda_m\eta_m\mu_m^2} + \sqrt{1 - 2\widehat{\lambda}_m\widehat{\rho}_m + \widehat{\lambda}_m^2\widehat{\mu}_m^2 + 2\widehat{\lambda}_m\widehat{\eta}_m\widehat{\mu}_m^2} \leq 1, \quad \forall 1 \leq m \leq r.$

Then the sequence $\{x_n\}$ *generated in the iterative process* (Υ) *converges strongly to a* *common element* $\bar{x} \in \bigcap_{m=1}^{r} GVI(C, B_m, A_m)$*, which uniquely solves the following variational inequality.*

$$\langle u - \bar{x}, \bar{x} - x^* \rangle \geq 0, \quad \forall x^* \in \bigcap_{m=1}^{r} GVI(C, B_m, A_m).$$

Proof. First, we prove that the mapping $P_C(\tau_m B_m - \lambda_m A_m)$ is nonexpansive for each $1 \leq m \leq r$. For each $x, y \in C$, we have

$$\begin{aligned} &||P_C(\tau_m B_m - \lambda_m A_m)x - P_C(\tau_m B_m - \lambda_m A_m)y|| \\ \leq\ &||(\tau_m B_m - \lambda_m A_m)x - (\tau_m B_m - \lambda_m A_m)y|| \\ \leq\ &||(x - y) - \lambda_m(A_m x - A_m y)|| + ||(x - y) - \tau_m(B_m x - B_m y)||. \end{aligned} \tag{2.1}$$

It follows from the assumption that each A_m is relaxed (η_m, ρ_m)-cocoercive and μ_m-Lipschitz continuous that

$$\begin{aligned} &||x - y - \lambda_m(A_m x - A_m y)||^2 \\ =\ &||x - y||^2 - 2\lambda_m\langle A_m x - A_m y, x - y\rangle + \lambda_m^2||A_m x - A_m y||^2 \\ \leq\ &||x - y||^2 - 2\lambda_m[(-\eta_m)||A_m x - A_m y||^2 + \rho_m||x - y||^2] + \lambda_m^2\mu_m^2||x - y||^2 \\ =\ &(1 - 2\lambda_m\rho_m + \lambda_m^2\mu_m^2)||x - y||^2 + 2\lambda_m\eta_m||A_m x - A_m y||^2 \\ =\ &(1 - 2\lambda_m\rho_m + \lambda_m^2\mu_m^2)||x - y||^2 + 2\lambda_m\eta_m\mu_m^2||A_m x - A_m y||^2 \\ =\ &\xi_m^2||x - y||^2, \end{aligned}$$

where $\xi_m = \sqrt{1 - 2\lambda_m\rho_m + \lambda_m^2\mu_m^2 + 2\lambda_m\eta_m\mu_m^2}$. This shows that

$$||x - y - \lambda_m(A_m x - A_m y)|| \leq \xi_m||x - y||. \tag{2.2}$$

In a similar way, we can obtain that

$$||x - y - \tau_m(B_m x - B_m y)|| \leq \zeta_m||x - y||, \tag{2.3}$$

where $\zeta_m = \sqrt{1 - 2\widehat{\lambda}_m\widehat{\rho}_m + \widehat{\lambda}_m^2\widehat{\mu}_m^2 + 2\widehat{\lambda}_m\widehat{\eta}_m\widehat{\mu}_m^2}$. Substituting (2.2) and (2.3) into (2.1), we from the condition (e) see that $P_C(\tau_m B_m - \lambda_m A_m)$ is nonexpansive for each $1 \leq m \leq r$. Put

$$y_n = \sum_{m=1}^{r} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n), \quad \forall n \geq 1.$$

Fixing $p \in \bigcap_{m=1}^{r} GVI(C, B_m, A_m)$, we see that

$$||y_n - p|| \leq ||x_n - p||.$$

It follows that

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_n u + \beta_n x_n + \gamma_n y_n - p|| \\ &\leq \alpha_n||u - p|| + \beta_n||x_n - p|| + \gamma_n||y_n - p|| \\ &\leq \alpha_n||u - p|| + \beta_n||x_n - p|| + \gamma_n||x_n - p|| \\ &= \alpha_n||u - p|| + (1 - \alpha_n)||x_n - p||. \end{aligned}$$

By mathematical inductions we arrive at

$$||x_n - p|| \leq \max\{||u - p||, ||x_1 - p||\}, \quad \forall n \geq 1.$$

Since the mapping $P_C(\tau_m B_m - \lambda_m A_m)$ is nonexpansive for each $1 \le m \le r$, we see that

$$\begin{aligned}
&||y_{n+1} - y_n|| \\
&= ||\sum_{m=1}^{r} \delta_{(m,(n+1))} P_C(\tau_m B_m x_{n+1} - \lambda_m A_m x_{n+1}) - \sum_{m=1}^{r} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n)|| \\
&\le ||x_{n+1} - x_n|| + M \sum_{m=1}^{r} |\delta_{(m,(n+1))} - \delta_{(m,n)}|,
\end{aligned} \tag{2.4}$$

where M is an appropriate constant such that

$$M = \max\{\sup_{n\ge 1} ||P_C(\tau_m B_m x_n - \lambda_m A_m x_n)||, \forall 1 \le m \le r\}.$$

Put $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, for all $n \ge 1$. That is,

$$x_{n+1} = (1 - \beta_n) l_n + \beta_n x_n, \quad \forall n \ge 1. \tag{2.5}$$

Now, we estimate $||l_{n+1} - l_n||$. Note that

$$\begin{aligned}
l_{n+1} - l_n &= \frac{\alpha_{n+1} u + \gamma_{n+1} y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n y_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (u - y_{n+1}) + \frac{\alpha_n}{1 - \beta_n} (y_n - u) + y_{n+1} - y_n,
\end{aligned}$$

which combines with (2.4) yields that

$$\begin{aligned}
&||l_{n+1} - l_n|| - ||x_{n+1} - x_n|| \\
&\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} ||u - y_{n+1}|| + \frac{\alpha_n}{1 - \beta_n} ||y_n - u|| + M \sum_{m=1}^{r} |\delta_{(m,(n+1))} - \delta_{(m,n)}|.
\end{aligned}$$

It follows from the conditions (b), (c) and (d) that

$$\limsup_{n\to\infty} (||l_{n+1} - l_n|| - ||x_{n+1} - x_{n+1}||) \le 0.$$

It follows from Lemma 1.1 that $\lim_{n\to\infty} ||l_n - x_n|| = 0$. In view of (2.5), we see that x_n +1 $x_n = (1 - \beta_n)(l_n - x_n)$. It follows that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{2.6}$$

On the other hand, from the iterative algorithm (Υ), we see that x_n+1 - $x_n = \alpha_n(u - x_n) + \gamma_n(y_n - x_n)$. It follows from (2.6) and the conditions (b), (c) that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{2.7}$$

Next, we show that $\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x}\rangle \le 0$. To show it, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x}\rangle = \lim_{i\to\infty} \langle u - \bar{x}, x_{n_i} - \bar{x}\rangle. \tag{2.8}$$

Since $\{x_{n_i}\}$ is bounded, we obtain that there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to q. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$. Next, we show that $q \in \bigcap_{m=1}^{r} GVI(C, B_m, A_m)$. Define a mapping $R : C \to C$ by

$$Rx = \sum_{m=1}^{r} \delta_m P_C(\tau_m B_m - \lambda_m A_m)x, \quad \forall x \in C,$$

where $\delta_m = \lim_{n\to\infty} \delta_{(m,n)}$. From Lemma 1.2, we see that R is nonexpansive with

$$F(R) = \bigcap_{m=1}^{r} F(P_C(\tau_m B_m - \lambda_m A_m)) = \bigcap_{m=1}^{r} GVI(C, B_m, A_m).$$

Now, we show that $Rx_n - x_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}
&||Rx_n - x_n|| \\
&= ||\sum_{m=1}^{r} \delta_m P_C(\tau_m B_m - \lambda_m A_m)x_n - \sum_{m=1}^{r} \delta_{(m,n)} P_C(\tau_m B_m x_n - \lambda_m A_m x_n)|| + ||y_n - x_n|| \\
&\le M \sum_{m=1}^{r} |\delta_{(m,n)} - \delta_m| + ||y_n - x_n||.
\end{aligned}$$

From the condition (d) and (2.7), we obtain that $\lim_{n\to\infty} ||Rx_n - x_n|| = 0$. From Lemma 1.3, we see that

$$q \in F(R) = \bigcap_{m=1}^{r} F(P_C(\tau_m B_m - \lambda_m A_m)) = \bigcap_{m=1}^{r} GVI(C, B_m, A_m).$$

In view of (2.8), we arrive at

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_n - \bar{x}\rangle = \langle u - \bar{x}, q - \bar{x}\rangle \le 0. \tag{2.9}$$

Finally, we show that $x_n \to \bar{x}$ as n - ∞. Note that

$$\begin{aligned}
&||x_{n+1} - \bar{x}||^2 \\
&= \langle \alpha_n u + \beta_n x_n + \gamma_n y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&= \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n \langle x_n - \bar{x}, x_{n+1} - \bar{x}\rangle + \gamma_n \langle y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\
&\le \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n ||x_n - \bar{x}|| ||x_{n+1} - \bar{x}|| + \gamma_n ||y_n - \bar{x}|| \, ||x_{n+1} - \bar{x}|| \\
&\le \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + (1 - \alpha_n) ||x_n - \bar{x}|| \, ||x_{n+1} - \bar{x}|| \\
&\le \frac{1 - \alpha_n}{2} (||x_n - \bar{x}||^2 + ||x_{n+1} - \bar{x}||^2) + \alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle,
\end{aligned}$$

which implies that

$$||x_{n+1} - \bar{x}||^2 \le (1 - \alpha_n) ||x_n - \bar{x}||^2 + 2\alpha_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle. \tag{2.10}$$

From the condition (c), (2.9) and applying Lemma 1.4 to (2.10), we obtain that

$$\lim_{n\to\infty} ||x_n - \bar{x}|| = 0.$$

This completes the proof.

If $B_m \equiv I$, the identity mapping and $\tau_m \equiv 1$, then Theorem 2.1 is reduced to the following result on the classical variational inequality (1.4).

Corollary 2.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A_m : C \to H$ be a relaxed (η_m, ρ_m)-cocoercive and μ_m-Lipschitz continuous mapping for each $1 \le m \le r$. Assume that $\bigcap_{m=1}^{r} VI(C, A_m) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{m=1}^{r} \delta_{(m,n)} P_C(x_n - \lambda_m A_m x_n), \quad n \geq 1,$$

where $u \in C$ *is a fixed point*, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_{(1,n)}\}$, ..., *and* $\{\delta_{(r,n)}\}$ *are sequences in* (0, 1) *satisfying the following restrictions.*

(a) $\alpha_n + \beta_n + \gamma_n = \sum_{m=1}^{r} \delta_{(m,n)} = 1, \forall n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha = \infty$;
(d) $\lim_{n\to\infty} \delta_{(m,n)} = \delta_m \in (0, 1), \forall 1 \leq m \leq r$, *and* $\{\lambda_m\}_{m=1}^{r}$*is a positive sequence such that*
(e) $\lambda_m \leq \frac{2\rho_m - 2\eta_m\mu_m^2}{\mu_m^2}, \forall 1 \leq m \leq r$.

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\bar{x} \in \bigcap_{m=1}^{r} VI(C, A_m)$, *which uniquely solves the following variational inequality*

$$\langle u - \bar{x}, \bar{x} - x^* \rangle \geq 0, \quad \forall x^* \in \bigcap_{m=1}^{r} VI(C, A_m).$$

If $r = 1$, then Theorem 2.1 is reduced to the following.

Corollary 2.3. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a relaxed* (η, ρ)*-cocoercive and* μ*-Lipschitz continuous mapping and* $B : C \to H$ *a relaxed* $(\widehat{\eta}, \widehat{\rho})$*-cocoercive and* $\widehat{\mu}$*-Lipschitz continuous mapping. Assume that* $GV I(C, B, A)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(\tau B x_n - \lambda A x_n), \quad n \geq 1,$$

where $u \in C$ *is a fixed point*, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* (0, 1) *satisfying the following restrictions.*

(a) $\alpha_n + \beta_n + \gamma_n = 1, \forall_n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$
(d) $\sqrt{1 - 2\lambda\rho + \lambda^2\mu^2 + 2\lambda\eta\mu^2} + \sqrt{1 - 2\widehat{\lambda}\widehat{\rho} + \widehat{\lambda}^2\widehat{\mu}^2 + 2\widehat{\lambda}\widehat{\eta}\widehat{\mu}^2} \leq 1$.

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\bar{x} \in GVI(C, B, A)$, *which uniquely solves the following variational inequality*

$$\langle u - \bar{x}, \bar{x} - x^* \rangle \geq 0, \quad \forall x^* \in GVI(C, B, A).$$

For the variational inequality (1.4), we can obtain from Corollary 2.3 the following immediately.

Corollary 2.4. *Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a relaxed* (η, ρ)*-cocoercive and* μ*-Lipschitz continuous mapping. Assume that* $V I(C, A)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(x_n - \lambda A x_n), \quad n \geq 1,$$

where $u \in C$ is a fixed point, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *satisfying the following restrictions.*

(a) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(d) $\lambda \leq \frac{2\rho - 2\eta\mu^2}{\mu^2}$.

Then the sequence $\{x_n\}$ *converges strongly to a common element* $\bar{x} \in VI(C, A)$, *which uniquely solves the following variational inequality*

$$\langle u - \bar{x}, \bar{x} - x^* \rangle \geq 0, \quad \forall x^* \in VI(C, A).$$

Remark 2.5. In this paper, the generalized variational inequality (1.2), which includes the classical variational inequality (1.4) as a special case, is considered based on iterative methods. Strong convergence theorems are established under mild restrictions imposed on the parameters. It is of interest to extend the main results presented in this paper to the framework of Banach spaces.

Abbreviation

CFP: convex feasibility problem.

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Authors' contributions
LY designed and performed all the steps of proof in this research and also wrote the paper. **ML** participated in the design of the study. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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