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Convergence theorem for finite family of lipschitzian demi-contractive semigroups

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Abstract

Let E be a real Banach space and K be a nonempty, closed, and convex subset of E. Let $\{\mathcal{J}_i\}_{i=1}^N$ be a finite family of Lipschitzian demi-contractive semigroups of K, with sequences of bounded measurable functions $L_i : [0, \infty) \to (0, \infty)$ and bounded functions $\lambda_i : [0, \infty) \to (0, \infty)$, respectively, where $\mathcal{J}_i := \{T_i(t) : t \geq 0\}$, $i = 1,2, ..., N$. Strong convergence theorem for common fixed point for finite family $\{\mathcal{J}_i\}_{i=1}^N$ is proved in a real Banch space. As an application, a new convergence theorem for finite family of Lipschitzian demi-contractive maps is also proved.

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1. Introduction

Let E be a real Banach space and E^* be the dual space of E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by, $x \in E$,

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x|| \, ||x^*||, ||x^*|| = ||x||\},$$

where $\langle ., . \rangle$ denotes the normalized duality pairing. For any $x \in E$, an element of Jx is denoted by $j(x)$.

Let K be a nonempty, closed and convex subset of E. Let $T : K \to K$ be a map, a point $x \in K$ is called a fixed point of T if $Tx = x$, and the set of all fixed points of T is denoted by $F(T)$. The mapping T is called L-Lipschitzian or simply Lipschitz if $\exists L > 0$, such that $||Tx - Ty|| \leq L||x - y|| \ \forall x, y \in K$ and if $L = 1$, then the map T is called *nonexpansive*.

A one parameter family $\mathcal{J} = \{T(t) : t \geq 0\}$ of self mapping of K is called a *nonexpansive semigroup* if the following conditions are satisfied,

(i) $T(0)x = x \ \forall \ x \in K$;
(ii) $T(t + s) = T(t) \circ T(s) \ \forall \ t, s \geq 0$;
(iii) for each $x \in K$, the mapping $t \to T(t)x$ is continuos;
(iv) for $x, y \in K$ and $t \geq 0$, $||T(t)x - T(t)y|| \leq ||x - y||$.

If the family $\mathcal{J} = \{T(t) : t \geq 0\}$ satisfies conditions (i) - (iii), then it is called

(a) *pseudocontractive semigroup* if for any $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y, j(x - y) \rangle \leq ||x - y||^2;$$

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(b) *strictly pseudocontractive semigroup* if there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$ and $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y, j(x - y)\rangle \leq ||x - y||^2 - \lambda(t)||(I - T(t))x - (I - T(t))y||^2$$

for all $x, y \in K$;

(c) *demi-contractive semigroup* if $F(T(t)) \neq \emptyset\ \forall t \geq 0$, there exists a bounded function $\lambda : [0, \infty) \to (0, \infty)$, and $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - q, j(x - q)\rangle \leq ||x - q||^2 - \lambda(t)||x - T(t)x||^2$$

for any $x \in K$ and $q \in F(T(t))$;

(d) *Lipschitzian semigroup* if there is a bounded measurable function $L : [0, \infty) \to (0, \infty)$ such that for $x, y \in K$ and $t \geq 0$,

$$||T(t)x - T(t)y|| \leq L(t)||x - y||.$$

It is known that every strictly pseudocontractive semigroup is Lipschitzian, and every strictly pseudocontractive semigroup with fixed point is demi-contractive semi-group.

Let E be a real Banach space and let K be a nonempty closed convex subset of E. A mapping $T : K \to K$ is *demicompact* if for every bounded sequence $\{x_n\}$ in K such that $\{xn - Tx_n\}$ converges, and there exists a subsequence, say $\{x_{n_j}\}$of $\{x_n\}$ that converges strongly to some x^* in K. T is said to be demi-contractive if $F(T) \neq \emptyset$, and there exists $\lambda > 0$ such that $\langle Tx- q, j(x - q)\rangle \leq ||x - q||^2 - \lambda||x - Tx||^2\ \forall x \in K, q \in F(T)$ and $j(x - q) \in J(x - q)$.

Let $T_1, T_2, ..., T_N$ be a family of self-mappings of K such that $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Then, the family is said to satisfy condition $\overline{C}$ if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0\ \forall r \in (0, \infty)$ such that $f(d(x, F)) \leq ||x - T_s x||$ for some s in $\{1, 2, ..., N\}$ and for all $x \in K$, where $d(x, F) = inf\{||x - q|| : q \in F\}$.

Existence theorems for family of nonexpansive mappings are proved in [1-5] and actually many others. Recently, Suzuki [6] proved the equivalence between the fixed point property for nonexpansive mappings and that of the nonexpansive semi-groups.

Both implicit and explicit, Mann, Ishikawa, and Halpern-type schemes were studied for approximation of common fixed points of family of nonexpansive semigroups and their generalizations in various spaces; see, for example [6-13], to list but a few.

In 1998, Shoiji and Takahashi [7] introduced and studied a Halpern-type scheme for common fixed point of a family of asymptotically nonexpansive semigroup in the framework of a real Hilbert space. Suzuki [8] proved that the implicit scheme defined by $x, x_1 \in K$,

$$x_n = \alpha_n T(t_n)x_n + (1 - \alpha_n)x$$

converges strongly to a common fixed point of the family of nonexpansive semigroup in a real Hilbert space. Xu [9] extended the result of Suzuki to a more general real uniformly convex Banach space having a weakly sequentially continuous duality mapping.

In 2005, Aleyner and Reich [10] proved the strong convergence of an explicit Halpern-type scheme defined by $x, x_1 \in K$,

$$x_{n+1} = \alpha_n T(t_n)x_n + (1 - \alpha_n)x$$

to a common fixed point of the family $\{T(t) : t \geq 0\}$ of nonexpansive semigroup in a reflexive Banach space with uniformly Gatéuax differentiable norm. Recently, Zhang et al. [11] introduced and studied a composite iterative scheme defined by $x, x_1 \in K$,

$$x_{n+1} = \alpha_n y_n + (1 - \alpha_n)x; \;\; y_n = \beta_n T(t_n)x_n + (1 - \beta_n)x_n.$$

Those authors proved strong convergence of the sequence $\{x_n\}$ to a common fixed point of the family $\{T(t) : t \geq 0\}$ of nonexpansive semigroup.

Very recently, Chang et al. [12] proved a strong convergence theorem which extended and improved the results in [10,9] and some others. They proved the following theorem.

Theorem 1.1. *Chang et al.* [12]*Let K be a nonempty, closed, and convex subset of a real Banach space E: Let* $\mathcal{J} := \{T(t) : t \geq 0\}$*be a Lipschitzian demi-contractive semigroup of K with bounded measurable function* $L : [0, \infty) \to (0, \infty)$ *and bounded function* $\lambda : [0, \infty) \to (0, \infty)$ *such that*

$$L := \sup_{t\geq 0}\{L(t)\} < \infty, \;\; \lambda := inf_{t\geq 0}\{\lambda(t)\} > 0 \text{ and } F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset.$$

Let $\{t_n\}$ *be an increasing sequence in* $[0, \infty)$ *and* $\{\alpha_n\}$ *be a sequence in (0,1) satisfying the following conditions,*

(i) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$; (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n)^2 < \infty$. *Assume that there exists a compact subset C of E such that* $\cup_{t\geq 0} T(t)(K) \subset C$ *and for any bounded set* $D \subset K$

$$\lim_{n\to\infty} \sup_{x\in D, s\in \mathbb{R}^+} ||T(s + t_n)x - T(t_n)x|| = 0.$$

Let $\{x_n\}$ *be generated by* $x_1 \in K$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(t_n)x_n. \tag{1.1}$$

Then, the sequence $\{x_n\}$ *converges strongly to some element in F.*

The purpose in this article is to prove a strong convergence theorem for common fixed point for finite families $\{\mathcal{J}_i\}_{i=1}^N$ of demi-contractive semigroups in a real Banach space. As application, we also prove convergence theorem for finite family of demi-contractive mappings. Our theorems generalize and improve several recent results. For instance, Theorem 1.1, which generalized, extended and improved several recent results, is a special case of our Theorem.

2. Preliminaries

We shall make use of the following lemmas.

Lemma 2.1. *Let E be a real normed linear space. Then, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E \text{ and } j(x + y) \in J(x + y).$$

Lemma 2.2. (Xu [14]) *Let* $\{a_n\}$ and $\{b_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + b_n)a_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$*exists. If in addition* $\{a_n\}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. (Suzuki [15]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space E and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \lim inf \beta_n \leq \lim sup \beta_n < 1$. *Suppose* $x_{n+1} = \beta_n y_n + (1 - \beta_n) x_n$ *for all integers* $n \geq 1$ *and* $\lim sup(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. *Then,* $\lim ||y_n - x_n|| = 0$.

3. Main Results

Let E be a real Banach space, and K be a nonempty, closed convex subset of E. For some fixed $i \in \mathbb{N}$, let $\mathcal{J}_i := \{T_i(t) : t \geq 0\}$ be a Lipschitzian demi-contractive semigroup with bounded measurable function $L_i : [0, \infty) \to (0, \infty)$ and bounded function $\lambda_i : [0, \infty) \to (0, \infty)$ such that

$$L^i := \sup_{t\geq 0}\{L_i(t)\} < \infty,\ \lambda^i := \inf_{t\geq 0}\{\lambda_i(t)\} > 0 \text{ and } F^i := \bigcap_{t\geq 0} F(T_i(t)) \neq \emptyset.$$

Then, for $x, y \in K$, $q \in F^i$ and $t \geq 0$,

$$\langle T_i(t)x - q, j(x - q)\rangle \leq ||x - q||^2 - \lambda^i||x - T_i(t)x||^2$$

and

$$||T_i(t)x - T_i(t)y|| \leq L^i||x - y||.$$

Consider a family $\{\mathcal{J}_i\}_{i=1}^N$ of Lipschitzian demi-contractive semigroups of K and let $L := \max_{1\leq i\leq N}\{L^i\}$, $L := \max_{1\leq i\leq N}\{L^i\}$ and $\lambda := \min_{1\leq i\leq N}\{\lambda^i\}$ Clearly $L < \infty$ and $\lambda > 0$ and for $x, y \in K$, $q \in \mathcal{F}$, $t \geq 0$ and any $i \in \{1, 2, ..., N\}$,

$$\langle T_i(t)x - q, j(x - q)\rangle \leq ||x - q||^2 - \lambda||x - T_i(t)x||^2$$

and

$$||T_i(t)x - T_i(t)y|| \leq L||x - y||.$$

For a fixed $\delta \in (0, 1)$ and $t \geq 0$ define a family $S_i(t) : K \to K$ $i = 1, 2, ..., N$ by

$$S_i(t)x := (1 - \delta^2)x + \delta^2 T_i(t)x,\ \forall x \in K. \tag{3.1}$$

Then, for $x, y \in K$ and $q \in \mathcal{F}$,

$$\begin{aligned}\langle S_i(t)x - q, j(x - q)\rangle &= (1 - \delta^2)\langle x - q, j(x - q)\rangle + \delta^2\langle T_i(t)x - q, j(x - q)\rangle\\ &\leq (1 - \delta^2)||x - q||^2 + \delta^2[||x - q||^2 - \lambda||x - T_i(t)x||^2]\\ &= ||x - q||^2 - \lambda\delta^2||x - T_i(t)x||^2.\end{aligned}$$

Let $\bar{\lambda} = \lambda\delta^2 > 0$, then

$$\langle S_i(t)x - q, j(x - q)\rangle \leq ||x - q||^2 - \bar{\lambda}||x - T_i(t)x||^2. \tag{3.2}$$

Also,

$$\begin{aligned}||S_i(t)x - S_i(t)y|| &= ||(1 - \delta^2)(x - y) + \delta^2(T_i(t)x - T_i(t)y)||\\ &\leq (1 - \delta^2)||x - y|| + \delta^2 L||x - y||\\ &= [1 - \delta^2 + \delta^2 L]||x - y||\\ &\leq (1 + \delta^2 L)||x - y||.\end{aligned}$$

Let $\bar{L} = 1 + \delta^2 L$.

Then,

$$||S_i(t)x - S_i(t)y|| \leq \bar{L}||x - y||. \tag{3.3}$$

Hence, for each $i \in \{1, 2, \dots N\}$, S_i is Lipschitz with Lipschitz constant $\bar{L} > 0$.

Lemma 3.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let* $\{\mathcal{J}_i\}_{i=1}^N$ *be a finite family of Lipschitzian demi-contractive semigroups of K with sequences of bounded measurable functions* $L_i : [0, \infty) \to (0, \infty)$ *and bounded functions* $\lambda_i : [0, \infty) \to (0, \infty)$ $i = 1, 2, \dots, N$ *such that for each* $i = 1, 2, \dots, N$,

$$L^i := \sup_{t\geq 0}\{L_i(t)\} < \infty,\ \lambda^i := \inf_{t\geq 0}\{\lambda_i(t)\} > 0 \text{ and } F^i := \bigcap_{t\geq 0} F(T_i(t)) \neq \emptyset.$$

Let $\mathcal{F} := \bigcap_{1\leq i\leq N}\{\bigcap_{t\geq 0} F(T_i(t))\} \neq \emptyset$, $\{t_n\}$ *be an increasing sequence in* $[0, \infty)$ *and* $\{\alpha_n\}$ *be a sequence in (0,1) satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$, (ii) $\sum_{n=1}^{\infty}(1 - \alpha_n)^2 < \infty$.

Assume $\forall\ i \in \{1,2, \dots, N\}$ for any bounded set $D \subset K$ *the relation*

$$\lim_{n\to\infty} \sup_{x\in D, s\in\mathbb{R}^+} ||T_i(s + t_n)x - T_i(t_n)x|| = 0 \tag{3.4}$$

holds. Let $\{x_n\}$ *be a sequence generated by* $x_1 \in K$,

$$x_{n+1} = \alpha_{n+1}x_n + (1 - \alpha_{n+1})S_{n+1}(t_{n+1})x_n,\ \ n \geq 1 \tag{3.5}$$

where $T_n(t_n) = T_{n\ modN}(t_n)$.

Then,

(a) $\lim_{n\to\infty} ||x_n - q||$ *exists for all* $q \in \mathcal{F}$.

(b) $\liminf_{n\to\infty} ||x_n - T_i(t_n)x_n|| = 0$ *for all* $i \in \{1,2,3, \dots, N\}$.

Proof. For any fixed $q \in \mathcal{F}$ using (3.5), we have

$$x_{n+1} - q = (x_n - q) + (1 - \alpha_{n+1})(S_{n+1}(t_{n+1})x_n - x_n).$$

Thus,

$$\begin{aligned}
||x_{n+1} - q||^2 &= ||(x_n - q) + (1 - \alpha_{n+1})(S_{n+1}(t_{n+1})x_n - x_n)||^2 \\
&\leq ||x_n - q||^2 + 2(1 - \alpha_{n+1})\langle S_{n+1}(t_{n+1})x_n - x_n, j(x_{n+1} - q)\rangle \\
&= ||x_n - q||^2 + 2(1 - \alpha_{n+1})\Big[\langle S_{n+1}(t_{n+1})x_n - S_{n+1}(t_{n+1})x_{n+1}, j(x_{n+1} - q)\rangle \\
&\quad + \langle S_{n+1}(t_{n+1})x_{n+1} - q, j(x_{n+1} - q)\rangle - \langle x_{n+1} - q, j(x_{n+1} - q)\rangle \\
&\quad + \langle x_{n+1} - x_n, j(x_{n+1} - q)\rangle\Big] \\
&\leq ||x_n - q||^2 + 2(1 - \alpha_{n+1})(\bar{L} + 1)||x_n - x_{n+1}||x_{n+1} - q|| \\
&\quad - 2(1 - \alpha_{n+1})\bar{\lambda}||x_{n+1} - T_{n+1}(t_{n+1})x_{n+1}||^2 \\
&\leq ||x_n - q||^2 + 2(1 - \alpha_{n+1})^2(1 + \bar{L})^2||S_{n+1}(t_{n+1})x_n - x_n||\ ||x_n - q|| \\
&\quad - 2(1 - \alpha_{n+1})\bar{\lambda}||x_{n+1} - T_{n+1}(t_{n+1})x_{n+1}||^2 \\
&\leq ||x_n - q||^2 + 2(1 - \alpha_{n+1})^2(1 + \bar{L})^3||x_n - q||^2 \\
&\quad - 2(1 - \alpha_{n+1})\bar{\lambda}||x_{n+1} - T_{n+1}(t_{n+1})x_{n+1}||^2 \\
&= (1 + \sigma_{n+1})||x_n - q||^2 - 2(1 - \alpha_{n+1})\bar{\lambda}||x_{n+1} - T_{n+1}(t_{n+1})x_{n+1}||^2 \\
&\leq (1 + \sigma_{n+1})||x_n - q||^2,
\end{aligned} \tag{3.6}$$

where $\sigma_{n+1} = 2(1+\bar{L})^3(1-\alpha_{n+1})^2$.
Since $\sum_{n=1}^{\infty}(1-\sigma_{n+1})^2 < \infty$, by lemma 2.2, it follows that $\lim_{n\to\infty}||x_n - q||$ exists.
Hence, $\{x_n\}$ is bounded, which implies that $\{T_n(t_n)x_n\}$ and $\{S_n(t_n)x_n\}$ are also bounded. From (3.6)

$$\begin{aligned}||x_{n+1}-q||^2 \le\ & ||x_n-q||^2 + 2(1-\alpha_{n+1})^2(1+\bar{L})^3||x_n-q||^2 \\ & -2(1-\alpha_{n+1})\bar{\lambda}||x_{n+1}-T_{n+1}(t_{n+1})x_{n+1}||^2 \\ \le\ & ||x_n-q||^2 - 2(1-\alpha_{n+1})\bar{\lambda}||x_{n+1}-T_{n+1}(t_{n+1})x_{n+1}||^2 + 2(1-\alpha_{n+1})^2 M,\end{aligned}$$

where, $M := (1+\bar{L})^3 \sup_{n\in\mathbb{N}}(||x_n-q||^2)$. Hence, for some $m \in \mathbb{N}$,

$$\begin{aligned}2\bar{\lambda}\sum_{n=1}^{m}(1-\alpha_{n+1})||x_{n+1}-T_{n+1}(t_{n+1})x_{n+1}||^2 \le & \sum_{n=1}^{m}(||x_n-q||^2 - ||x_{n+1}-q||^2) \\ & + 2M\sum_{n=1}^{m}(1-\alpha_{n+1})^2 \\ \le & ||x_1-q||^2 \\ & + 2M\sum_{n=1}^{m}(1-\alpha_{n+1})^2 < \infty.\end{aligned}$$

Since $m \in \mathbb{N}$ is arbitrary, we have

$$2\bar{\lambda}\sum_{n=1}^{\infty}(1-\alpha_{n+1})||x_{n+1}-T_{n+1}(t_{n+1})x_{n+1}||^2 < \infty$$

which implies

$$\liminf_{n\to\infty}||x_{n+1}-T_{n+1}(t_{n+1})x_{n+1}|| = 0. \tag{3.7}$$

Next, we show that,

$$\lim_{n\to\infty}||x_{n+1}-x_n|| = 0.$$

Let $\{\beta_n\}$ and $\{y_n\}$ be two sequences define by $\beta_n := \delta(1 - \delta)\alpha_{n+1} + \delta^2$ and $y_n := \frac{x_{n+1}-x_n+\beta_n x_n}{\beta_n}$. Then, using the definition of $\{\beta_n\}$ and $\{S_n\}$ we obtain that $y_n := \frac{\delta\alpha_{n+1}x_n+\delta^2(1-\alpha_{n+1})T_{n+1}(t_{n+1})x_n}{\beta_n}$. Then,

$$\begin{aligned}y_{n+1}-y_n =\ & \frac{\delta\alpha_{n+2}}{\beta_{n+1}}[x_{n+1}-x_n] + \delta\left[\frac{\alpha_{n+2}}{\beta_{n+1}} - \frac{\alpha_{n+1}}{\beta_n}\right]x_n \\ & + \frac{\delta^2(1-\alpha_{n+2})}{\beta_{n+1}}[T_{n+2}(t_{n+2})x_{n+1} - T_{n+2}(t_{n+2})x_n] \\ & + \delta^2\left[\frac{1-\alpha_{n+2}}{\beta_{n+1}} - \frac{1-\alpha_{n+1}}{\beta_n}\right]T_{n+2}(t_{n+2})x_n \\ & + \frac{\delta^2(1-\alpha_{n+1})}{\beta_n}[T_{n+2}(t_{n+2})x_n - T_{n+1}(t_{n+1})x_n].\end{aligned}$$

Therefore,

$$\begin{aligned}||y_{n+1}-y_n||-||x_{n+1}-x_n|| \le &\left(\frac{\delta\alpha_{n+2}}{\beta_{n+1}}+\frac{\delta^2 L(1-\alpha_{n+2})}{\beta_{n+1}}-1\right)||x_{n+1}-x_n||\\ &+\delta\left|\frac{\alpha_{n+2}}{\beta_{n+1}}-\frac{\alpha_{n+1}}{\beta_n}\right|||x_n||\\ &+\delta^2\left|\frac{1-\alpha_{n+2}}{\beta_{n+1}}-\frac{1-\alpha_{n+1}}{\beta_n}\right|||T_{n+2}(t_{n+2})x_n||\\ &+\frac{\delta^2(1-\alpha_{n+1})}{\beta_n}||T_{n+2}(t_{n+2})x_n-T_{n+1}(t_{n+1})x_n||.\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}(||y_{n+1}-y_n||-||x_{n+1}-x_n||)\le 0,$$

and by lemma 2.3,

$$\lim_{n\to\infty}||y_n-x_n|| = 0.$$

Thus,

$$||x_{n+1}-x_n|| = \beta_n||y_n-x_n|| \to 0 \ as\ n\to\infty.$$

This implies that,

$$||x_{n+i}-x_n||\to 0\ as\ n\to\infty,\ \forall\, i\in\{1,2,3,\dots,N\}.$$

But, for $i\in$ {1,2,3, ..., N},

$$\begin{aligned}||x_n-S_{n+i}(t_{n+i})x_n|| &\le \delta^2\Big[||x_n-x_{n+i}||+||x_{n+i}-T_{n+i}(t_{n+i})x_{n+i}||\\ &\quad+||T_{n+i}(t_{n+i})x_{n+i}-T_{n+i}(t_{n+i})x_n||\Big]\\ &\le \delta^2[(1+L)||x_{n+i}-x_n||+||x_{n+i}-T_{n+i}(t_{n+i})x_{n+i}||].\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty}||x_n-S_{n+i}(t_{n+i})x_n|| = 0.$$

Hence,

$$\liminf_{n\to\infty}||T_{n+i}(t_{n+i})x_n-x_n|| = \liminf_{n\to\infty}[\frac{1}{\delta^2}||S_{n+i}(t_{n+i})x_n-x_n||]=0.$$

From the relation,

$$\begin{aligned}||T_{n+i}(t_n)x_n-x_n|| &\le ||T_{n+i}(t_n)x_n-T_{n+i}((t_{n+i}-t_n)+t_n)x_n||\\ &\quad+||T_{n+i}(t_{n+i})x_n-x_n||\\ &\le \sup_{z\in\{x_n\},s\in\mathbb{R}^+}||T_{n+i}(t_n)z-T_{n+i}(s+t_n)z||+||T_{n+i}(t_{n+i})x_n-x_n||,\end{aligned}$$

and condition (3.4) we get

$$\liminf_{n\to\infty}||T_{n+i}(t_n)x_n-x_n|| = 0. \tag{3.8}$$

It follows from (3.8) that $\liminf_{n\to\infty}||T_l(t_n)x_n-x_n|| = 0\,\forall\, l\in\{1,2,3,\dots,N\}$. This completes the proof. □

Theorem 3.2. *Let E, K, $\mathcal{F}$, $\{\alpha_n\}$, $\{t_n\}$, $\{\mathcal{J}_i\}_{i=1}^N$and $\{x_n\}$ be as in lemma 3.1. Assume that, for at least one $i \in \{1, 2, ..., N\}$, there exists a compact subset C of E such that $\cup_{t\geq 0} T_i(t)(K) \subset C$. Then, the sequence $\{x_n\}$ converges to some element $\mathcal{F}$.*

Proof. By Lemma 3.1, we have $\liminf_{n\to\infty} ||T_l(t_n)x_n - x_n|| = 0 \,\forall\, l \in \{1, 2, 3, \ldots, N\}$.

If $\cup_{t\geq 0} T_s(t)(K) \subset C$ for some compact subet C of E and some $s \in \{1, 2, ..., N\}$, then there exists a subsequence $\{x_{n_k}\}$, of $\{x_n\}$ and $q^* \in K$, such that

$$x_{n_k} \to q^* \text{ and } ||T_s(t_{n_k})x_{n_k} - x_{n_k}|| \to 0 \text{ as } n \to \infty. \tag{3.9}$$

Observe that for $t > 0$,

$$\begin{aligned} ||T_s(t)x_{n_k} - x_{n_k}|| &\leq ||T_s(t)x_{n_k} - T_s(t)T_s(t_{n_k})x_{n_k}|| \\ &\quad + ||T_s(t)T_s(t_{n_k})x_{n_k} - T_s(t_{n_k})x_{n_k}|| + ||T_s(t_{n_k})x_{n_k} - x_{n_k}|| \\ &\leq ||T_s(t + t_{n_k})x_{n_k} - T_s(t_{n_k})x_{n_k}|| + (1 + L)||T_s(t_{n_k})x_{n_k} - x_{n_k}||. \end{aligned}$$

From the above we have $\lim_{k\to\infty} ||T_s(t)x_{n_k} - x_{n_k}|| = 0$. Using (3.9) and the fact that Ts is Lipschitzian, we get $q^* \in \cap_{t\geq 0} F(T_s(t))$.

Now, for any $l \in \{1,2, ...,N\}$, since $\liminf_{k\to\infty} ||T_l(t_{n_k})x_{n_k} - x_{n_k}|| = 0$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\lim_{j\to\infty} ||T_l(t_{n_{k_j}})x_{n_{k_j}} - x_{n_{k_j}}|| = \liminf_{k\to\infty} ||T_l(t_{n_k})x_{n_k} - x_{n_k}|| = 0$. Then, similarly for $t \geq 0$, we obtain

$$\begin{aligned} ||T_l(t)x_{n_{k_j}} - x_{n_{k_j}}|| &\leq ||T_l(t)x_{n_{k_j}} - T_l(t)T_l(t_{n_{k_j}})x_{n_{k_j}}|| \\ &+ ||T_l(t)T_l(t_{n_{k_j}})x_{n_{k_j}} - T_l(t_{n_{k_j}})x_{n_{k_j}}|| + ||T_l(t_{n_{k_j}})x_{n_{k_j}} - x_{n_{k_j}}|| \\ &\leq ||T_l(t + t_{n_{k_j}})x_{n_{k_j}} - T_l(t_{n_{k_j}})x_{n_{k_j}}|| + (1 + L)||T_l(t_{n_{k_j}})x_{n_{k_j}} - x_{n_{k_j}}||. \end{aligned}$$

This implies that $\lim_{j\to\infty} ||T_l(t)x_{n_{k_j}} - x_{n_{k_j}}|| = 0$ and hence $q^* \in \cap_{t\geq 0} F(T_l(t))$. Since $l \in \{1, 2, \ldots N\}$ is arbitrarily chosen, we have $q^* \in \mathcal{F}$. As the limit $\lim_{n\to\infty} ||x_n - q^*||$ exists, the conclusion of the theorem follows immediately and this completes the proof. □

Remark 3.3. Observe that considering a single one-parameter family of demi-contractive semigroup in Theorem 3.2, we obtain the conclusion of Theorem 1.1.

Let $T_1, T_2, ..., T_N$ be a finite family of Lipschitzian demi-contractive self-mapping of K with positive constants $\lambda_1, \lambda_2, ..., \lambda_N$ and Lipschitz constants $L_1, L_2, ..., L_N$, respectively. Let $F := \bigcap_{1\leq i\leq N} F(T_i) \neq \emptyset$.

For a fixed $\delta \in (0, 1)$, define $S_n : K \to K$ by

$$S_n x := (1 - \delta^2)x + \delta^2 T_n x, \quad \forall\, x \in K. \tag{3.10}$$

Then, it follows that for $x, y \in K$ and $i \in F$,

$$\begin{aligned} \langle S_n x - q, j(x - q)\rangle &\leq ||x - q||^2 - \bar{\lambda}||x - T_n x||^2 \textit{ and} \\ ||S_n x - S_n y|| &\leq \bar{L}||x - y||, \end{aligned}$$

where $\bar{\lambda} = \lambda\delta^2 > 0$, $\bar{L} = 1 + \delta^2 L$, $\lambda := \min_{1\leq i\leq N}\{\lambda_i\}$ and $L := \max_{1\leq i\leq N}\{L_i\}$.

The following Theorem is a consequence of Lemma 3.1.

Theorem 3.4. *Let E, K and $\{\alpha_n\}$ be as in Lemma 3.1. Let $T_1, T_2, ..., T_N : K \to K$ be Lipschitzian demi-contractive mappings with T_s demicompact for at least one $s \in \{1, 2,$*

..., N}. Let $\{x_n]$ be a sequence generated by $x_1 \in K$

$$x_{n+1} = \alpha_{n+1}x_n + (1 - \alpha_{n+1})S_{n+1}x_n, \tag{3.11}$$

where $T_n = T_{n \ modN}$. Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$.

Proof. Following the line of proof of lemma 3.1 we immediately obtain $\lim_{n\to\infty} ||x_n - q||qk$ exists for any $q \in F$ and $\liminf_{n\to\infty} ||T_i x_n - x_n|| = 0$, $\forall i \in \{1,2, \dots N\}$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} ||T_i x_{n_k} - x_{n_k}|| = \liminf_{n\to\infty} ||T_i x_n - x_n|| = 0.$$

Therefore $\lim_{k\to\infty} ||T_s x_{n_k} - x_{n_k}|| = 0$ and, by demicompactness of T_s, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $q^* \in K$, such that $x_{n_{k_j}} \to q^*$ as $j \to \infty$.

Since,

$$\begin{aligned} 0 = \lim_{j\to\infty} ||T_i x_{n_{k_j}} - x_{n_{k_j}}|| &= ||T_i \lim_{j\to\infty} x_{n_{k_j}} - \lim_{j\to\infty} x_{n_{k_j}}|| \\ &= ||T_i q^* - q^*||, \end{aligned}$$

we obtain $q^* \in F$. But, $\lim_{n\to\infty} ||x_n - q^*||$ exists, thus $x_n \to q^* \in F$ and this completes the proof. □

The following corollaries follow from Theorem 3.4

Corollary 3.5. *Let E, K and $\{\alpha_n\}$ be as in Theorem 3.4. Let $T_1, T_2, \dots, T_N : K \to K$ be Lipschitzian demi-contractive mappings. Suppose there exists a compact subset C in E such that $\bigcup_{i=1}^N T_i(K) \subset C$. Let $\{x_n\}$ be defined by* (3.11). *Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$.*

Corollary 3.6. *Let E; K and $\{\alpha_n\}$ be as in Theorem 3.4. Let $T_1, T_2, \dots, T_N : K \to K$ be Lipschitzian demi-contractive mappings satisfying condition $\overline{C}$. Let $\{x_n\}$ be defined by* (3.11). *Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$.*

Proof. Following the line of proof of lemma 3.1, we obtain $\liminf_{n\to\infty} ||x_n - T_i x_n|| = 0$ for all $i \in \{1, 2, 3, \dots, N\}$ and $||x_{n+1} - q||^2 \le (1 + \sigma_{n+1}) ||x_n - q||^2$, where $\sigma_{n+1} = 2(1 + \bar{L})^3(1 - \alpha_{n+1})^2$. Since $\sum_{n=1}^\infty (1 - \sigma_{n+1})^2 < \infty$, by lemma 2.2 $\lim_{n\to\infty} ||x_n - p||$ exists and consequently $\lim_{n\to\infty} d(x_n, F)$ exists. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty} ||x_{n_k} - T_i x_{n_k}|| = \liminf_{n\to\infty} ||x_n - T_i x_n|| = 0$. Then, by using condition $\overline{C}$, there exists $s \in \{1, 2, \dots, N\}$ such that $0 = \lim_{k\to\infty} ||x_{n_k} - T_s x_{n_k}|| \ge \lim_{k\to\infty} f(d(x_{n_k}, F))$ and, using the property of *f*, we get that $\lim_{k\to\infty} d(x_{n_k}, F) = 0$, and since the limit $\lim_{n\to\infty} d(x_n, F)$ exists we have that $\lim_{n\to\infty} d(x_n, F) = 0$. We next show that $\{x_n\}$ is Cauchy. Let $\varepsilon > 0$ be given, then there exists $p^* \in F$ and $n^* \in \mathbb{N}$ such that $\forall n \ge n^*$, $||x_n - p^*|| < \frac{\varepsilon}{2}$. Hence, for $n \ge n^*$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} ||x_{n+k} - x_n|| &\le ||x_{n+k} - p^*|| + ||x_n - p^*|| \\ &< \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is Cauchy and so $x_n \to q^* \in$ K. We now show that q^* is in *F*. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists $m_0 \in \mathbb{N}$ large enough and $p^* \in F$ such that for all $n \ge m_0$, and $||x_n - p^*|| < \frac{\varepsilon}{6(1+L)}$. Hence,

$$\begin{aligned}
||q^* - T_l q^*|| &\le ||x_n - q^*|| + ||x_n - p^*|| + ||p^* - T_l q^*|| \\
&\le \frac{\varepsilon}{6(1+L)} + \frac{\varepsilon}{6(1+L)} + L||p^* - q^*|| \\
&< \frac{\varepsilon}{6(1+L)} + \frac{\varepsilon}{6(1+L)} + \frac{3L\varepsilon}{6(1+L)} \\
&< \varepsilon.
\end{aligned}$$

Thus, $q^* \in F(T_l)$ and since $l \in \{1, 2, ..., N\}$ is arbitrary, we have $q^* \in F$. This completes the proof. □

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Authors' contributions
BA conceived the study, GCU carried out the computations for Theorem 3.4. BA Modified Theorem 3.4 to obtain Theorem 3.2. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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