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Existence and iterative approximation for generalized equilibrium problems for a countable family of nonexpansive mappings in banach spaces

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Abstract

We first prove the existence of a solution of the generalized equilibrium problem (GEP) using the KKM mapping in a Banach space setting. Then, by virtue of this result, we construct a hybrid algorithm for finding a common element in the solution set of a GEP and the fixed point set of countable family of nonexpansive mappings in the frameworks of Banach spaces. By means of a projection technique, we also prove that the sequences generated by the hybrid algorithm converge strongly to a common element in the solution set of GEP and common fixed point set of nonexpansive mappings.

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1. Introduction

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of positive integers and real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \quad \forall x \in E,$$

where $\langle \cdot,\cdot \rangle$ denotes the generalized duality pairing. We know that if E is smooth, then J is single-valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote by J the single-valued duality mapping. Let $f : C \times C \to \mathcal{R}$ be a bifunction and $A : C \to E^*$ be a nonlinear mapping. We consider the following generalized equilibrium problem (GEP):

$$\text{Find } u \in C \text{ such that } f(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such $u \in C$ is denoted by $GEP(f)$, i.e.,

$$GEP(f) = \{u \in C : f(u, y) + \langle Au, y - u \rangle \geq 0, \quad \forall y \in C\}.$$

Whenever $E = H$ a Hilbert space, the problem (1.1) was introduced and studied by Takahashi and Takahashi [1]. Similar problems have been studied extensively recently. In the case of $A \equiv 0$, *GEP* (*f*) is denoted by *EP* (*f*). In the case of $f \equiv 0$, *EP* is also denoted by $VI(C, A)$. Problem (1.1) is very general in the sense that it includes, as spacial cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others; see, e.g., [2,3]. A mapping $T : C \to E$ is called nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. A mapping $A : C \to E^*$ is called α-inverse-strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha||Ax - Ay||^2, \quad \forall x, y \in C.$$

It is easy to see that if $A : C \to E^*$ is an α-inverse-strongly monotone mapping, then it is $1/\alpha$- Lipschitzian.

In 1953, Mann [4] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathcal{N}, \tag{1.2}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0, 1].

However, we note that Manns iteration process (1.2) has only weak convergence, in general; for instance, see [5-7].

Let C be a nonempty, closed, and convex subset of a Banach space E and $\{T_n\}$ be sequence of mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then, $\{T_n\}$ is said to satisfy the NST-condition if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} ||z_n - T_n z_n|| = 0$$

implies $\omega_w(z_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$; see [8-10].

In 2008, Takahashi et al. [11] has adapted Nakajo and Takahashi's [12] idea to modify the process (1.2) so that strong convergence has been guaranteed. They proposed the following modification for a family of nonexpansive mappings in a Hilbert space: $x_0 \in H$, $C_1 = C$, $u_1 = P_{C_1}x_0$ and

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \ ||y_n - z|| \ \leq \ ||u_n - z||\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathcal{N}, \end{cases} \tag{1.3}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathcal{N}$. They proved that if $\{T_n\}$ satisfies the NST-condition, then $\{u_n\}$ generated by (1.3) converges strongly to a common fixed point of T_n.

Recently, motivated by Nakajo and Takahashi [12] and Xu [13], Matsushita and Takahashi [14] introduced the iterative algorithm for finding fixed points of nonexpansive mappings in a uniformly convex and smooth Banach space: $x_0 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : ||z - Tz|| \leq t_n||x_n - Tx_n||\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}x, \quad n \geq 0, \end{cases} \tag{1.4}$$

where $\overline{co}D$ denotes the convex closure of the set D, $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$, and $P_{C_n \cap D_n}$ is the metric projection from E onto $C_n \cap D_n$. They proved that $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T .

Very recently, Kimura and Nakajo [15] investigated iterative schemes for finding common fixed points of a family of nonexpansive mappings and proved strong convergence theorems by using the Mosco convergence technique in a uniformly convex and smooth Banach space. In particular, they proposed the following algorithm: $x_1 = x \in C$ and

$$\begin{cases} C_n = \overline{co}\{z \in C : ||z - T_n z|| \leq t_n ||x_n - T_n x_n||\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad n \geq 0, \end{cases} \tag{1.5}$$

where $\{t_n\}$ is a sequence in (0,1) with $t_n \to 0$ as $n \to \infty$. They proved that if $\{T_n\}$ satisfies the NST-condition, then $\{x_n\}$ converges strongly to a common fixed point of T_n.

Motivated and inspired by Nakajo and Takahashi [12], Takahashi et al. [11], Xu [13], Masushita and Takahashi [14], and Kimura and Nakajo [15], we introduce a hybrid projection algorithm for finding a common element in the solution set of a GEP and the common fixed point set of a family of nonexpansive mappings in a Banach space setting.

2. Preliminaries

Let E be a real Banach space and let $U = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. A Banach space E is said to be strictly convex if for any $x, y \in U$,

$$x \neq y \text{ implies } —|x + y|| < 2.$$

It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$||x - y|| \geq \varepsilon \text{ implies } —|x + y|| < 2(1 - \delta).$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Define a function δ: [0, 2] $\to$ [0, 1] called the modulus of convexity of E as follows:

$$\delta(\varepsilon) = \inf\left\{1 - ||\frac{x + y}{2}|| \ : x, y \in E, ||x|| = ||y|| = 1, \ ||x - y|| \geq \varepsilon\right\}.$$

Then, E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.1}$$

exists for all $x, y \in U$. Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex and smooth Banach space E. Then, for any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$||x_0 - x|| \leq \min_{y \in C} ||y - x||.$$

The mapping $P_C : E \to C$ defined by $P_C \times = x_0$ is called the metric projection from E onto C. Let $x \in E$ and $u \in C$. Then, it is known that $u = P_C \times$ if and only if

$$\langle u - y, J(x - u)\rangle \geq 0 \tag{2.2}$$

for all $y \in C$; see [16] for more details. It is well known that if P_C is a metric projection from a real Hilbert space H onto a nonempty, closed, and convex subset C, then P_C is nonexpansive. However, in a general Banach space, this fact is not true.

In the sequel, we will need the following lemmas.

Lemma 2.1. [17]*Let E be a uniformly convex Banach space,* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 < b \le \alpha_n \le c < 1$ *for all* $n \ge 1$, *and* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in E such that* $\limsup_{n\to\infty} ||x_n|| \le d$, $\limsup_{n\to\infty} ||y_n|| \le d$ *and* $\lim_{n\to\infty} ||\alpha_n x_n + (1 - \alpha_n) y_n|| = d$. *Then,* $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.2. [18]*Let C be a bounded, closed, and convex subset of a uniformly convex Banach space E. Then, there exists a strictly increasing, convex, and continuous function* $\gamma : [0, \infty) \to [0, \infty)$ *such that* $\gamma(0) = 0$ *and*

$$\gamma\left(\left\|T\left(\sum_{i=1}^{n}\lambda_i x_i\right) - \sum_{i=1}^{n}\lambda_i T x_i\right\|\right) \le \max_{1\le j\le k\le n}(||x_j - x_k|| - ||Tx_j - Tx_k||)$$

for all $n \in \mathcal{N}$, $\{x_1, x_2,..., x_n\} \subset C$, $\{\lambda_1, \lambda_2,..., \lambda_n\} \subset [0, 1]$ *with* $\sum_{i=1}^{n} \lambda_i = 1$ *and nonexpansive mapping T of C into E.*

Following Bruck's [19] idea, we know the following result for a convex combination of nonexpansive mappings which is considered by Aoyama et al. [20] and Kimura and Nakajo [15].

Lemma 2.3. [15]*Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space E and* $\{S_n\}$ *be a family of nonexpansive mappings of C into itself such that* $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. *Let* $\{\beta_n^k\}$ *be a family of nonnegative numbers with indices* $n, k \in \mathcal{N}$ *with* $k \le n$ *such that*

(i) $\sum_{k=1}^{n} \beta_n^k = 1$ *for every* $n \in \mathcal{N}$;
(ii) $\lim_{n\to\infty} \beta_n^k > 0$ *for every* $k \in \mathcal{N}$

and let $T_n = \alpha_n I + (1 - \alpha_n)\sum_{k=1}^{n} \beta_n^k S_k$ *for all* $n \in \mathcal{N}$, *where* $\{\alpha_n\} \subset [a, b]$ *for some* $a, b \in (0, 1)$ *with* $a \le b$. *Then,* $\{T_n\}$ *is a family of nonexpansive mappings of C into itself with* $\bigcap_{n=1}^{\infty} F(T_n) = F$ *and satisfies the NST-condition.*

Now, let us turn to following well-known concept and result.

Definition 2.4. Let B be a subset of topological vector space X. A mapping $G : B \to 2^X$ is called a KKM mapping if $co\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} G(x_i)$ for $x_i \in B$ and $i = 1, 2,..., m$, where coA denotes the convex hull of the set A.

Lemma 2.5. [21]*Let B be a nonempty subset of a Hausdorff topological vector space* × *and let* $G : B \to 2^X$ *be a KKM mapping. If G(x) is closed for all* $\times \in B$ *and is compact for at least one* $x \in B$, *then* $\cap_{x\in B} G(x) \neq \emptyset$.

3. Existence results of gep

Motivated by Takahashi and Zembayashi [22], and Ceng and Yao [23], we next prove the following crucial lemma concerning the GEP in a strictly convex, reflexive, and smooth Banach space.

Theorem 3.1. *Let C be a nonempty, bounded, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from C × C to* $\mathcal{R}$*satisfying (A1)-(A4), where*

(A1) *$f(x, x) = 0$ for all $x \in C$;*
(A2) *f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;*
(A3) *for all $y \in C$, $f(., y)$ is weakly upper semicontinuous;*
(A4) *for all $x \in C$, $f(x,.)$ is convex.*

Let A be α-inverse strongly monotone of C into E^*. For all $r > 0$ and $\times \in E$, define the mapping $S_r : E \to 2^C$ as follows:

$$S_r(x) = \{z \in C : f(z, y) + \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, J(z - x)\rangle \geq 0, \quad \forall y \in C\}. \tag{3.1}$$

Then, the following statements hold:

(1) *for each $x \in E$, $S_r(x) \neq \emptyset$;*
(2) *S_r is single-valued;*
(3) *$\langle S_r(x) - S_r(y), J(S_rx - x)\rangle \leq \langle S_r(x) - S_r(y), J(S_ry - y)\rangle$ for all $x, y \in E$;*
(4) *$F(S_r) = GEP(f)$;*
(5) *GEP(f) is nonempty, closed, and convex.*

Proof. (1) Let x_0 be any given point in *E*. For each $y \in C$, we define the mapping $G : C \to 2^E$ by

$$G(y) = \{z \in C : f(z, y) + \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, J(z - x_0)\rangle \geq 0\} \quad \text{for all } y \in C.$$

It is easily seen that $y \in G(y)$, and hence $G(y). \neq \emptyset$

(a) First, we will show that *G* is a KKM mapping. Suppose that there exists a finite subset $\{y_1, y_2,..., y_m\}$ of *C* and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^m \alpha_i y_i \notin G(y_i)$ for all $i = 1, 2,..., m$. It follows that

$$f(\hat{x}, y_i) + \langle A\hat{x}, y_i - \hat{x}\rangle + \frac{1}{r}\langle y_i - \hat{x}, J(\hat{x} - x_0)\rangle < 0, \quad \text{for all } \ i = 1, 2, \ldots, m.$$

By (A1) and (A4), we have

$$\begin{aligned} 0 &= f(\hat{x}, \hat{x}) + \langle A\hat{x}, \hat{x} - \hat{x}\rangle + \frac{1}{r}\langle \hat{x} - \hat{x}, J(\hat{x} - x_0)\rangle \\ &\leq \sum_{i=1}^{m} \left(f(\hat{x}, y_i) + \langle A\hat{x}, y_i - \hat{x}\rangle + \frac{1}{r}\langle y_i - \hat{x}, J(\hat{x} - x_0)\rangle \right) < 0, \end{aligned}$$

which is a contradiction. Thus, *G* is a KKM mapping on *C*.

(b) Next, we show that $G(y)$ is closed for all $y \in C$. Let $\{z_n\}$ be a sequence in $G(y)$ such that $z_n \to z$ as $n \to \infty$. It then follows from $z_n \in G(y)$ that,

$$f(z_n, y) + \langle Az_n, y - z_n\rangle + \frac{1}{r}\langle y - z_n, J(z_n - x)\rangle \geq 0. \tag{3.2}$$

By (A3), the continuity of J, and the lower semicontinuity of $||\cdot||^2$, we obtain from (3.2) that

$$\begin{aligned}
0 &\le \liminf_{n\to\infty}[f(z_n,y)+\langle Az_n,y-z_n\rangle+\frac{1}{r}\langle y-z_n,J(z_n-x_0)\rangle]\\
&\le \limsup_{n\to\infty}[f(z_n,y)+\langle Az_n,y-z_n\rangle+\frac{1}{r}\langle y-x_0,J(z_n-x_0)\rangle+\frac{1}{r}\langle x_0-z_n,J(z_n-x_0)\rangle]\\
&= \limsup_{n\to\infty}[f(z_n,y)+\langle Az_n,y-z_n\rangle+\frac{1}{r}\langle y-x_0,J(z_n-x_0)\rangle-\frac{1}{r}||z_n-x_0||^2]\\
&\le \limsup_{n\to\infty}f(z_n,y)+\limsup_{n\to\infty}\langle Az_n,y-z_n\rangle+\frac{1}{r}\limsup_{n\to\infty}\langle y-x_0,J(z_n-x_0)\rangle-\frac{1}{r}\liminf_{n\to\infty}||z_n-x_0||^2\\
&\le f(z,y)+\langle Az,y-z\rangle+\frac{1}{r}\langle y-x_0,J(z-x_0)\rangle-\frac{1}{r}||z-x_0||^2\\
&= f(z,y)+\langle Az,y-z\rangle+\frac{1}{r}\langle y-x_0,J(z-x_0)\rangle-\frac{1}{r}\langle z-x_0,J(z-x_0)\rangle\\
&= f(z,y)+\langle Az,y-z\rangle+\frac{1}{r}\langle y-z,J(z-x_0)\rangle.
\end{aligned}$$

This shows that $z\in G(y)$, and hence $G(y)$ is closed for all $y\in C$.

(c) We prove that $G(y)$ is weakly compact. We now equip E with the weak topology. Then, C, as closed, bounded convex subset in a reflexive space, is weakly compact. Hence, $G(y)$ is also weakly compact.

Using (a), (b), and (c) and Lemma 2.5, we have $\cap_{x\in C}G(y)\neq\emptyset$. It is easily seen that

$$S_r(x_0)=\bigcap_{y\in C}G(y)$$

Hence, $s_r(x_0)\neq\emptyset$. Since x_0 is arbitrary, we can conclude that $s_r(x)\neq\emptyset$ for all $x\in E$.

(2) We prove that S_r is single-valued. In fact, for $x\in C$ and $r>0$, let $z_1, z_2\in S_r(x)$. Then,

$$f(z_1,z_2)+\langle Az_1,z_2-z_1\rangle+\frac{1}{r}\langle z_2-z_1,J(z_1-x)\rangle\ge 0.$$

and

$$f(z_2,z_1)+\langle Az_2,z_1-z_2\rangle+\frac{1}{r}\langle z_1-z_2,J(z_2-x)\rangle\ge 0.$$

Adding the two inequalities and from the condition (A2) and monotonicity of A, we have

$$\begin{aligned}
0 &\le f(z_1,z_2)+f(z_2,z_1)+\langle Az_1,z_2-z_1\rangle+\langle Az_2,z_1-z_2\rangle+\frac{1}{r}\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle\\
&\le \langle Az_1-Az_2,z_2-z_1\rangle+\frac{1}{r}\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle\\
&\le -\alpha||Az_1-Az_2||^2+\frac{1}{r}\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle\\
&\le \frac{1}{r}\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle,
\end{aligned}\tag{3.3}$$

and hence,

$$\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle\ge 0.$$

Hence,

$$0\le\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle=\langle(z_2-x)-(z_1-x),J(z_1-x)-J(z_2-x)\rangle.$$

Since J is monotone and E is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$.

Therefore S_r is single-valued.

(3) For $x, y \in C$, we have

$$f(S_rx, S_ry) + \langle AS_rx, S_ry - S_rx\rangle + \frac{1}{r}\langle S_ry - S_rx, J(S_rx - x)\rangle \geq 0$$

and

$$f(S_ry, S_rx) + \langle AS_ry, S_rx - S_ry\rangle + \frac{1}{r}\langle S_rx - S_ry, J(S_ry - y)\rangle \geq 0.$$

Again, adding the two inequalities, we also have

$$\langle AS_rx - AS_ry, S_ry - S_rx\rangle + \langle S_ry - S_rx, J(S_rx - x) - J(S_ry - y)\rangle \geq 0.$$

It follows from monotonicity of A that

$$\langle S_ry - S_rx, J(S_rx - x)\rangle \leq \langle S_ry - S_rx, J(S_ry - y)\rangle.$$

(4) It is easy to see that

$$\begin{aligned} z \in F(S_r) &\Leftrightarrow z = S_rz \\ &\Leftrightarrow f(z, y) + \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, J(z - z)\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow f(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow z \in GEP(f). \end{aligned}$$

Hence, $F(S_r) = GEP(f)$.

(5) Finally, we claim that $GEP(f)$ is nonempty, closed, and convex. For each $y \in C$, we define the mapping $\Theta : C \to 2^E$ by

$$\Theta(y) = \{x \in C : f(x, y) + \langle Ax, y - x\rangle \geq 0\}.$$

Since $y \in \Theta(y)$, we have $\Theta(y) \neq \emptyset$ We prove that Θ is a KKM mapping on C. Suppose that there exists a finite subset $\{z_1, z_2,..., z_m\}$ of C and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\hat{z} = \sum_{i=1}^m \alpha_i z_i \notin \Theta(z_i)$ for all $i = 1, 2,..., m$. Then,

$$f(\hat{z}, z_i) + \langle A\hat{z}, z_i - \hat{z}\rangle < 0, \quad i = 1, 2, \ldots, m.$$

From (A1) and (A4), we have

$$0 = f(\hat{z}, \hat{z}) + \langle A\hat{z}, \hat{z} - \hat{z}\rangle \leq \sum_{i=1}^m \alpha_i \left(f(\hat{z}, z_i) + \langle A\hat{z}, z_i - \hat{z}\rangle\right) < 0,$$

which is a contradiction. Thus, Θ is a KKM mapping on C.

Next, we prove that $\Theta(y)$ is closed for each $y \in C$. For any $y \in C$, let $\{x_n\}$ be any sequence in $\Theta(y)$ such that $x_n \to x_0$. We claim that $x_0 \in \Theta(y)$. Then, for each $y \in C$, we have

$$f(x_n, y) + \langle Ax_n, y - x_n\rangle \geq 0.$$

By (A3), we see that

$$f(x_0, y) + \langle Ax_0, y - x_0\rangle \geq \limsup_{n\to\infty} f(x_n, y) + \lim_{n\to\infty}\langle Ax_n, y - x_n\rangle \geq 0.$$

This shows that $x_0 \in \Theta(y)$ and $\Theta(y)$ is closed for each $y \in C$. Thus, $\bigcap_{y \in C} \Theta(y) = GEP(f)$ is also closed.

We observe that $\Theta(y)$ is weakly compact. In fact, since C is bounded, closed, and convex, we also have $\Theta(y)$ is weakly compact in the weak topology. By Lemma 2.5, we can conclude that $\bigcap_{y \in C} \Theta(y) = GEP(f) \neq \emptyset$.

Finally, we prove that $GEP(f)$ is convex. In fact, let $u, v \in F(S_r)$ and $z_t = tu+(1-t)v$ for $t \in (0, 1)$. From (3), we know that

$$\langle S_r u - S_r z_t, J(S_r z_t - z_t) - J(S_r u - u)\rangle \geq 0.$$

This yields that

$$\langle u - S_r z_t, J(S_r z_t - z_t)\rangle \geq 0. \tag{3.4}$$

Similarly, we also have

$$\langle v - S_r z_t, J(S_r z_t - z_t)\rangle \geq 0. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} ||z_t - S_r z_t||^2 &= \langle z_t - S_r z_t, J(z_t - S_r z_t)\rangle \\ &= t\langle u - S_r z_t, J(z_t - S_r z_t)\rangle + (1-t)\langle v - S_r z_t, J(z_t - S_r z_t)\rangle \\ &\leq 0. \end{aligned}$$

Hence, $z_t \in F(S_r) = GEP(f)$ and hence $GEP(f)$ is convex. This completes the proof.

4. Strong convergence theorem

In this section, we prove a strong convergence theorem using a hybrid projection algorithm in a uniformly convex and smooth Banach space.

Theorem 4.1. *Let E be a uniformly convex and smooth Banach space and C be a nonempty, bounded, closed, and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)-(A4), A an α-inverse strongly monotone mapping of C into E^** and *$\{T_n\}_{n=0}^{\infty}$ a sequence of nonexpansive mappings of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap GEP(f) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$* satisfies the *NST-condition. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : ||z - T_n z|| \leq t_n ||x_n - T_n x_n||\}, \quad n \geq 1, \\ u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \quad \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases} \tag{4.1}$$

where $\{t_n\}$ and *$\{r_n\}$ are sequences which satisfy the following conditions:*

(C1) $\{t_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} t_n = 0$;
(C2) $\{r_n\} \subset (0, 1)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. First, we rewrite the algorithm (4.1) as the following:

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : ||z - T_n z|| \le t_n ||x_n - T_n x_n||\}, \quad n \ge 1, \\ D_n = \{z \in D_{n-1} : \langle S_{r_n} x_n - z, J(x_n - S_{r_n} x_n)\rangle \ge 0\}, n \ge 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, n \ge 0, \end{cases} \tag{4.2}$$

where S_r is the mapping defined by (3.1) for all $r > 0$. We first show that the sequence $\{x_n\}$ is well defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $\Omega \subset C_n$ for all $n \ge 0$. Next, we prove that $\Omega \subset C_n \cap D_n$. Since $D_0 = C$, we also have $\Omega \subset C_0 \cap D_0$. Suppose that $\Omega \subset C_{k-1} \cap D_{k-1}$ for $k \ge 2$. It follows from Lemma (3) that

$$\langle S_{r_k} x_k - S_{r_k} u, J(S_{r_k} u - u) - J(S_{r_k} x_k - x_k)\rangle \ge 0,$$

for all $u \in \Omega$. This implies that

$$\langle S_{r_k} x_k - u, J(x_k - S_{r_k} x_k)\rangle \ge 0,$$

for all $u \in \Omega$. Hence, $\Omega \subset D_k$. By the mathematical induction, we get that $\Omega \subset C_n \cap D_n$ for each $n \ge 0$ and hence $\{x_n\}$ is well defined. Let $w = P_F x_0$. Since $\Omega \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$||x_{n+1} - x_0|| \le ||w - x_0||, \quad n \ge 0. \tag{4.3}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$||x_{n+1} - x_0|| \le ||x_{n+2} - x_0||.$$

Since $\{x_n - x_0\}$ is bounded, we have $\lim_{n\to\infty} ||x_n - x_0|| = d$ for some a constant d. Moreover, by the convexity of D_n, we also have $\frac{1}{2}(x_{n+1} + x_{n+2}) \in D_n$ and hence

$$||x_0 - x_{n+1}|| \le \left\| x_0 - \frac{x_{n+1} + x_{n+2}}{2} \right\| \le \frac{1}{2}\left(||x_0 - x_{n+1}|| + ||x_0 - x_{n+2}||\right).$$

This implies that

$$\lim_{n\to\infty} \left\| \frac{1}{2}(x_0 - x_{n+1}) + \frac{1}{2}(x_0 - x_{n+2}) \right\| = \lim_{n\to\infty} \left\| x_0 - \frac{x_{n+1} + x_{n+2}}{2} \right\| = d.$$

By Lemma 2.1, we have $\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0$.

Next, we show that $v \in \bigcap_{n=0}^{\infty} F(T_n)$. Since $x_{n+1} \in C_n$ and $t_n > 0$, there exists $m \in \mathcal{N}$, $\{\lambda_0, \lambda_1,..., \lambda_m\} \subset [0, 1]$ and $\{y_0, y_1,..., y_m\} \subset C$ such that

$$\sum_{i=1}^{m} \lambda_i = 1, \quad \left\| x_{n+1} - \sum_{i=0}^{m} \lambda_i y_i \right\| < t_n, \text{ and } —|y_i - T_n y_i|| \le t_n ||x_n - T_n x_n||$$

for each $i = 0, 1,..., m$. Since C is bounded, by Lemma 2.2, we have

$$\begin{aligned}||x_n - T_n x_n|| &\leq ||x_n - x_{n+1}|| + \left\| x_{n+1} - \sum_{i=0}^{m} \lambda_i y_i \right\| + \left\| \sum_{i=0}^{m} \lambda_i y_i - \sum_{i=0}^{m} \lambda_i T_n y_i \right\| \\ &\quad + \left\| \sum_{i=0}^{m} \lambda_i T_n y_i - T_n \left(\sum_{i=0}^{m} \lambda_i y_i \right) \right\| + \left\| T_n \left(\sum_{i=0}^{m} \lambda_i y_i \right) - T_n x_n \right\| \\ &\leq 2||x_n - x_{n+1}|| + (2 + 2M)t_n \\ &\quad + \gamma^{-1} \left(\max_{0 \leq i \leq j \leq m} (||y_i - y_j|| - ||T_n y_i - T_n y_j||) \right) \\ &\leq 2||x_n - x_{n+1}|| + (2 + 2M)t_n \\ &\quad + \gamma^{-1} \left(\max_{0 \leq i \leq j \leq m} (||y_i - T_n y_i|| - ||y_j - T_n y_j||) \right) \\ &\leq 2||x_n - x_{n+1}|| + (2 + 2M)t_n + \gamma^{-1}(4Mt_n),\end{aligned}$$

where $M = \sup_{n \geq 0} ||x_n - w||$. It follows from (C1) that $\lim_{n \to \infty} ||x_n - T_n x_n|| = 0$. Since $\{T_n\}$ satisfies the NST-condition, we have $v \in \bigcap_{n=0}^{\infty} F(T_n)$.

Next, we show that $v \in GEP(f)$. By the construction of D_n, we see from (2.2) that $S_{r_n} x_n = P_{D_n} x_n$. Since $x_{n+1} \in D_n$, we obtain

$$||x_n - S_{r_n} x_n|| \ \leq \ ||x_n - x_{n+1}|| \to 0,$$

as $n \to \infty$. From (C2), we also have

$$\frac{1}{r_n} \left\| J(x_n - S_{r_n} x_n) \right\| = \frac{1}{r_n} ||x_n - S_{r_n} x_n|| \ \to 0, \tag{4.4}$$

as $n \to \infty$. Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_i}\}$ which weakly converges to some $v \in E$.

By (4.4), we also have $S_{r_{n_i}} \rightharpoonup v$. By the definition of $S_{r_{n_j}}$, for each $y \in C$, we obtain

$$f(S_{r_{n_i}} x_{n_i}, y) + \langle AS_{r_{n_i}} x_{n_i}, y - S_{r_{n_i}} x_{n_i} \rangle + \frac{1}{r_{n_i}} \langle y - S_{r_{n_i}} x_{n_i}, J(S_{r_{n_i}} x_{n_i} - x_{n_i}) \rangle \geq 0.$$

By (A3) and (4.4), we have

$$f(v, y) + \langle Av, y - v \rangle \geq 0, \quad \forall y \in C.$$

This shows that $v \in GEP(f)$ and hence $v \in \Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap GEP(f)$.

Note that $w = P_{\Omega} x_0$. Finally, we show that $x_n \to w$ as $n \to \infty$. By the weakly lower semicontinuity of the norm, it follows from (4.3) that

$$||x_0 - w|| \ \leq \ ||x_0 - v|| \ \leq \liminf_{i \to \infty} ||x_0 - x_{n_i}|| \ \leq \limsup_{i \to \infty} ||x_0 - x_{n_i}|| \ \leq \ ||x_0 - w||.$$

This shows that

$$\lim_{i \to \infty} ||x_0 - x_{n_i}|| \ = \ ||x_0 - w|| \ = \ ||x_0 - v||$$

and $v = w$. Since E is uniformly convex, we obtain that $x_0 - x_{n_i} \to x_0 - w$. It follows that $x_{n_i} \to w$. Hence, we have $x_n \to w$ as $n \to w$. This completes the proof.

5. Deduced theorems

If we take $f \equiv 0$ and $A \equiv 0$ in Theorem 4.1, then we obtain the following result.

Theorem 5.1. *Let E be a uniformly convex and smooth Banach space, C a nonempty, bounded, closed, and convex subset of E and* $\{T_n\}_{n=0}^{\infty}$ *a sequence of nonexpansive mappings of C into itself such that* $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$ *and suppose that* $\{T_n\}_{n=0}^{\infty}$ *satisfies the NST-condition. Let* $\{x_n\}$ *be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : \ ||z - T_n z|| \ \le t_n ||x_n - T_n x_n||\}, n \ge 1, \\ x_{n+1} = P_{C_n} x_0, n \ge 0. \end{cases} \tag{5.1}$$

If $\{t_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} t_n = 0$, *then* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Remark 5.2. By Lemma 2.3, if we define $T_n = \alpha_n I + (1 - \alpha_n) \sum_{k=1}^{n} \beta_n^k S_k$ for all $n \ge 0$ in Theorems 3.1 and 5.1, then the theorems also hold.

If we take $T_n \equiv I$, the identity mapping on C, for all $n \ge 0$ in Theorem 4.1, then we obtain the following result.

Theorem 5.3. *Let E be a uniformly convex and smooth Banach space, C a nonempty, bounded, closed, and convex subset of E. Let f be a bifunction from* $C \times C$ to $\mathcal{R}$ *satisfying (A1)-(A4) and A an* α*-inverse strongly monotone mapping of C into* E^*. *Let* $\{x_n\}$ *be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \ge 0, \forall y \in C, n \ge 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \quad n \ge 1, \\ x_{n+1} = P_{D_n} x_0, \quad n \ge 0. \end{cases} \tag{5.2}$$

If $\{r_n\} \subset (0, 1)$ *and* $\liminf_{n\to\infty} r_n > 0$, *then* $\{x_n\}$ *converges strongly to* $P_{GEP\,(f)} x_0$.

If we take $A \equiv 0$ in Theorem 4.1, then we obtain the following result concerning an equilibrium problem in a Banach space setting.

Theorem 5.4. *Let E be a uniformly convex and smooth Banach space and C be a nonempty, bounded, closed, and convex subset of E. Let f be a bifunction from* $C \times C$ to $\mathcal{R}$ *satisfying (A1)-(A4) and let* $\{T_n\}_{n=0}^{\infty}$ *be a sequence of nonexpansive mappings of C into itself such that* $\Omega := \bigcap_{n=0}^{\infty} F(T_n) \cap EP(f) \neq \emptyset$ *and suppose that* $\{T_n\}_{n=0}^{\infty}$ *satisfies the NST-condition. Let* $\{x_n\}$ *be the sequence in C generated by*

$$\begin{cases} x_0 \in C, D_0 = C, \\ C_n = \overline{co}\{z \in C : ||z - T_n z|| \ \le t_n ||x_n - T_n x_n||\}, \quad n \ge 1, \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \ge 0, \quad \forall y \in C, \quad n \ge 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \quad n \ge 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \ge 0, \end{cases} \tag{5.3}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are sequences which satisfy the conditions:*

(C1) $\{t_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty} t_n = 0$;
(C2) $\{r_n\} \subset (0, 1)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Abbreviations
GEP: generalized equilibrium problem.

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