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A modified Mann iterative scheme by generalized *f*-projection for a countable family of relatively quasi-nonexpansive mappings and a system of generalized mixed equilibrium problems

Siwaporn Saewan[1] and Poom Kumam[1,2*]

* Correspondence: poom.kum@kmutt.ac.th
[1]Department of Mathematics, Faculty of Science King Mongkut's University of Technology Thonburi (KMUTT) Bangmod, Bangkok 10140, Thailand
Full list of author information is available at the end of the article

Abstract

The purpose of this paper is to introduce a new hybrid projection method based on modified Mann iterative scheme by the generalized *f*-projection operator for a countable family of relatively quasi-nonexpansive mappings and the solutions of the system of generalized mixed equilibrium problems. Furthermore, we prove the strong convergence theorem for a countable family of relatively quasi-nonexpansive mappings in a uniformly convex and uniform smooth Banach space. Finally, we also apply our results to the problem of finding zeros of $\mathcal{B}$-monotone mappings and maximal monotone operators. The results presented in this paper generalize and improve some well-known results in the literature.

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1 Introduction

The theory of equilibrium problems, the development of an efficient and implementable iterative algorithm, is interesting and important. This theory combines theoretical and algorithmic advances with novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis, and numerical analysis.

Equilibrium problems theory provides us with a natural, novel, and unified framework for studying a wide class of problems arising in economics, finance, transportation, network, and structural analysis, image reconstruction, ecology, elasticity and optimization, and it has been extended and generalized in many directions. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. In particular, generalized mixed equilibrium problem and equilibrium problems are related to the problem of finding fixed points of nonlinear mappings.

Let E be a real Banach space with norm $||\cdot||$, C be a nonempty closed convex subset of E and let E^* denote the dual of E. Let $\{\theta_i\}_{i\in\Lambda} : C \times C \to \mathbb{R}$ be a bifunction, $\{\phi_i\}$

$_{i\in\Lambda}$: $C \to \mathbb{R}$ be a real-valued function, and $\{A_i\}_{i\in\Lambda} : C \to E^*$ be a monotone mapping, where Λ is an arbitrary index set. The *system of generalized mixed equilibrium problems* is to find $x \in C$ such that

$$\theta_i(x, y) + \langle A_i x, y - x\rangle + \varphi_i(y) - \varphi_i(x) \geq 0, \quad i \in \Lambda,\ \forall y \in C. \tag{1.1}$$

If Λ is a singleton, then problem (1.1) reduces to the *generalized mixed equilibrium problem* is to find $x \in C$ such that

$$\theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions to (1.2) is denoted by GMEP(θ, A, ϕ), i.e.,

$$\mathrm{GMEP}(\theta, A, \varphi) = \{x \in C : \theta(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}. \tag{1.3}$$

If $A \equiv 0$, the problem (1.2) reduces to the *mixed equilibrium problem for* θ, denoted by MEP(θ, ϕ) is to find $x \in C$ such that

$$\theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.4}$$

If $\theta \equiv 0$, the problem (1.2) reduces to the *mixed variational inequality* of Browder type, denoted by $VI(C, A, \phi)$ is to find $x \in C$ such that

$$\langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.5}$$

If $A \equiv 0$ and $\phi \equiv 0$ the problem (1.2) reduces to the *equilibrium problem for* θ, denoted by EP(θ) is to find $x \in C$ such that

$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

If $\theta \equiv 0$, the problem (1.4) reduces to the *minimize problem*, denoted by Argmin(ϕ) is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The generalized mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases. Moreover, the above formulation (1.5) was shown in [1] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, vector equilibrium problems, and Nash equilibria in noncooperative games. In other words, the GMEP(θ, A, ϕ), MEP(θ, ϕ) and EP(θ) are an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. Many authors studied and constructed some solution methods to solve the GMEP(θ, A, ϕ), MEP(θ, ϕ), EP(θ) [[1-16], and references therein].

Let C be a closed convex subset of E and recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

As we know that if C is a nonempty closed convex subset of a Hilbert space H and recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, the unique point in $P_C x \in C$ satisfying the property $||x - P_C x|| = \min_{y\in C} ||x - y||$, then we also have P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. We consider the functional defined by

$$\phi(y, x) = ||y||^2 - 2\langle y, Jx\rangle + ||x||^2, \quad \text{for } x, y \in E, \tag{1.8}$$

where J is the normalized duality mapping. In this connection, Alber [17] introduced a generalized projection Π_C from E in to C as follows:

$$\Pi_C(x) = arg \min_{y\in C} \phi(y, x), \quad \forall x \in E. \tag{1.9}$$

It is obvious from the definition of functional φ that

$$(||y|| - ||x||)^2 \leq \phi(y, x) \leq (||y|| + ||x||)^2, \quad \forall x, y \in E. \tag{1.10}$$

If E is a Hilbert space, then $\varphi(y, x) = ||y - x||^2$ and Π_C becomes the metric projection of E onto C. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x). \tag{1.11}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\varphi(y, x)$ and strict monotonicity of the mapping J [17-21]. It is well known that the metric projection operator plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, and complementarity problems, etc. [17,22]. In 1994, Alber [23] introduced and studied the generalized projections from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces. Moreover, Alber [17] presented some applications of the generalized projections to approximately solve variational inequalities and von Neumann intersection problem in Banach spaces. In 2005, Li [22] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solve the variational inequality in Banach spaces. Later, Wu and Huang [24] introduced a new generalized *f*-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by Abler [23] and proved some properties of the generalized *f*-projection operator. In 2009, Fan et al. [25] presented some basic results for the generalized *f*-projection operator and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces.

Let $\langle\cdot, \cdot\rangle$ denote the duality pairing of E^* and E. Next, we recall the concept of the generalized *f*-projection operator. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varpi) = ||\xi||^2 - 2\langle\xi, \varpi\rangle + ||\varpi||^2 + 2\rho f(\xi), \tag{1.12}$$

where $\xi \in C$, $\varpi \in E^*$, ρ is positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$is proper, convex, and lower semicontinuous. By the definitions of G, it is easy to see the following properties:

(1) $G(\xi, \varpi)$ is convex and continuous with respect to ϖ when ξ is fixed;
(2) $G(\xi, \varpi)$ is convex and lower semicontinuous with respect to ξ when ϖ is fixed.

Definition 1.1. Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. We say that $\pi_C^f : E* \to 2^C$ is *generalized f-projection operator* if

$$\pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{\xi \in C} G(\xi, \varpi)\}, \quad \forall \varpi \in E*.$$

Observe that, if $f(x) = 0$, then the generalized f-projection operator (1.12) reduces to the generalized projection operator (1.9).

For the generalized f-projection operator, Wu and Hung [24] proved the following basic properties:

Lemma 1.2. [24]*Let E be a real reflexive Banach space with its dual E* and C a nonempty closed convex subset of E. Then the following statement holds:*

(1) $\pi_C^f \varpi$, *is a nonempty closed convex subset of C for all* $\varpi \in E^*$;
(2) *if E is smooth, then for all* $\varpi \in E^*$, $x \in \pi_C^f \varpi$ *if and only if*

$$\langle x - y, \varpi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(3) *if E is strictly convex and* $f : C \to \mathbb{R} \cup \{+\infty\}$ *is positive homogeneous (i.e.,* $f(tx) = tf(x)$ *for all* $t > 0$ *such that* $tx \in C$ *where* $x \in C$*), then* $\pi_C^f \varpi$ *is single-valued mapping.*

Recently, Fan et al. [25] show that the condition f is positive homogeneous which appeared in [[25], Lemma 2.1 (iii)] can be removed.

Lemma 1.3. [25]*Let E be a real reflexive Banach space with its dual E* and C a nonempty closed convex subset of E. If E is strictly convex, then* $\pi_C^f \varpi$ *is single valued.*

Recall that J is single value mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$ where $x \in E$. This substitution for (1.12) gives

$$G(\xi, Jx) = ||\xi||^2 - 2\langle \xi, Jx\rangle + ||x||^2 + 2\rho f(\xi). \tag{1.13}$$

Now we consider the second generalized f projection operator in Banach space [26].

Definition 1.4. Let E be a real smooth and Banach space and C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is generalized f-projection operator if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\}, \quad \forall x \in E.$$

Next, we give the following example [27] of metric projection, generalized projection operator and generalized f-projection operator do not coincide.

Example 1.5. Let $X = \mathbb{R}^3$ be provided with the norm $||(x_1, x_2, x_3)|| = \sqrt{(x_1^2 + x_2^2)} + \sqrt{x_2^2 + x_3^2}$.

This is a smooth strictly convex Banach space and $C = \{x \in \mathbb{R}^3 | x_2 = 0, x_3 = 0\}$ is a closed and convex subset of X. It is a simple computation; we get

$$P_C(1,1,1) = (1,0,0), \quad \Pi_C(1,1,1) = (2,0,0)$$

We set $\rho = 1$ is positive number and define $f : C \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} 2 + 2\sqrt{5}, & x < 0; \\ -2 - 2\sqrt{5}, & x \geq 0. \end{cases}$$

Then, f is proper, convex, and lower semicontinuous. Simple computations show that

$$\Pi_C^f(1,1,1) = (4,0,0).$$

Recall that a point p in C is said to be an *asymptotic fixed point* of T [28] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is said to be *relatively nonexpansive mapping* [29-31] if

(R1) $F(T)$ is nonempty;

(R2) $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$;

(R3) $\widehat{F}(T) = F(T)$.

A mapping T is said to be *relatively quasi-nonexpansive* (or *quasi-φ-nonexpansive*) if the conditions (R1) and (R2) are satisfied. The asymptotic behavior of a relatively nonexpansive mapping was studied in [32-34]. The class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [11,32-35] which requires the strong restriction: $F(T) = \widehat{F}(T)$. In order to explain this better, we give the following example [36] of relatively quasi-nonexpansive mappings which is not relatively nonexpansive mapping. It is clearly by the definition of relatively quasi-nonexpansive mapping T is equivalent to $F(T) \neq \emptyset$, and $G(p, JTx) \leq G(p, Jx)$ for all $x \in C$ and $p \in F(T)$.

Example 1.6. Let E be any smooth Banach space and let $x_0 \neq 0$ be any element of E.

We define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, & \text{if } x = \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0; \\ -x, & \text{if } x \neq \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0. \end{cases}$$

Then T is a relatively quasi-nonexpansive mapping but not a relatively non-expansive mapping. Actually, T above fails to have the condition ($R3$).

Next, we give some examples which are closed quasi-φ-nonexpansive [[4], Examples 2.3 and 2.4].

Example 1.7. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0 \neq \emptyset$. Then, $J_r = (J + rA)^{-1}J$ is a closed quasi-φ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Proof By Matsushita and Takahashi [[35], Theorem 4.3], we see that J_r is relatively nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$. Therefore, J_r is quasi-φ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$. On the other hand, we can obtain the closedness of J_r easily from the continuity of the mapping J and the maximal monotonicity of A; see [35] for more details. □

Example 1.8. Let C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, C is a closed quasi-φ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

In 1953, Mann [37] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.14}$$

where the initial guess element $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is real sequence in 0[1]. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [38]. In an infinite-dimensional Hilbert space, Mann iteration can conclude *only weak convergence* [39,40]. Attempts to modify the Mann iteration method (1.14) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [41] proposed the following modification of Mann iteration method as follows:

$$\begin{cases} x_1 = x \in C \text{is arbitrary}, \\ y_n = \alpha_n Jx_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \ ||y_n - z|| \ \le \ ||x_n - z||\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x, n \ge 1. \end{cases} \tag{1.15}$$

They proved that if the sequence $\{\alpha_n\}$ bounded above from one, then $\{x_n\}$ defined by (1.15) converges strongly to $P_{F(T)}x$.

In 2007, Aoyama et al. [[42], Lemma 3.1] introduced $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfy the following condition: if for each bounded subset B of C, $\sum_{n=1}^{\infty} \sup\{||T_{n+1}z - T_n z|| \ : z \in B < \infty\}$. Assume that if the mapping $T : C \to C$ defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, then $\lim_{n\to\infty} \sup\{|| Tz - T_n z|| : z \in C\} = 0$. They proved that the sequence $\{T_n\}$ converges strongly to some point of C for all $x \in C$.

In 2009, Takahashi et al. [43] studied and proved a strong convergence theorem by the new hybrid method for a family of nonexpansive mappings in Hilbert spaces as follows: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1}x_0$ and

$$\begin{cases} \quad y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ C_{n+1} = \{z \in C : \ ||y_n - z|| \ \le \ ||x_n - z||\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \ge 1, \end{cases} \tag{1.16}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \infty$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\cap_{n=1}^{\infty} F(T_n)}x_0$.

The ideas to generalize the process (1.14) from Hilbert spaces have recently been made. By using available properties on a uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [35] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C\text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{1.17}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}\, x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$. Plubtieng and Ungchittrakool [44] introduced and proved the processes for finding a common fixed point of a countable family of relatively nonexpansive mappings in a Banach space. They proved the strong convergence theorems for a common fixed point of a countable family of relatively nonexpansive mappings $\{T_n\}$ provided that $\{T_n\}$ satisfies the following condition:

- if for each bounded subset D of C, there exists a continuous increasing and convex function $h : \mathbb{R}^+ \to \mathbb{R}^+$. such that $h(0) = 0$ and $\lim k,_{l\to\infty} \sup_{z\in D} h(||T_k z - T_l z||) = 0$.

Motivated by the results of Takahashi and Zembayashi [13], Cholumjiak and Suantai [2] proved the following strong convergence theorem by the hybrid iterative scheme for approximation of common fixed point of countable families of relatively quasi-nonexpansive mappings $\{T_i\}$ on C into itself in a uniformly convex and uniformly smooth Banach space: $x_0 \in E$, $x_1 = \Pi_{C_1} x_0, C_1 = C$

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i x_n), \\ u_{n,i} = T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}} \dots T^{F_1}_{r_{1,n}} y_{n,i} \\ C_{n+1} = \{z \in C_n : \sup_{i>1} \phi(z, Ju_{n,i}) \leq \phi(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \geq 1, \end{cases} \tag{1.18}$$

where $T^{F_i}_{r_{i,n}}$, $i = 1, 2, 3, ..., m$ defined in Lemma 2.8. Then, they proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, and $\{r_{n,i}\}$, the sequence $\{x_n\}$ converges strongly to $\Pi_{C_{n+1}} x_0$.

Recently, Li et al. [26] introduced the following hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, n \geq 1 \end{cases} \tag{1.19}$$

They obtained a strong convergence theorem for finding an element in the fixed point set of T. The results of Li et al. [26] extended and improved on the results of Matsushita and Takahashi [35].

Very recently, Shehu [45] studied and obtained the following strong convergence theorem by the hybrid iterative scheme for approximation of common fixed point of finite family of relatively quasi-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space: let $x_0 \in C$, $x_1 = \Pi_{C_1} x_0$, $C_1 = C$ and

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \dots T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 1 \end{cases} \tag{1.20}$$

where $T_n = T_n(\text{mod } N)$. He proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{C_{n+1}} x_0$ under certain appropriate conditions.

Recall that a mapping $T : C \to C$ is closed if for each $\{x_n\}$ in C, if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$. Let $\{T_n\}$ be a family of mappings of C into itself with $\mathcal{F} := \cap_{n=1}^{\infty} F(T_n) \neq \emptyset$, $\{T_n\}$ is said to satisfy the (*)-condition [46] if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} ||z_n - T_n z_n|| = 0, \quad \text{and} \quad z_n \to z \text{ imply } z \in \mathcal{F}. \tag{1.21}$$

It follows directly from the definitions above that if $T_n \equiv T$ and T is closed, then $\{T_n\}$ satisfies (*)-condition [46]. Next, we give the following example:

Example 1.9. Let $E = \mathbb{R}$ with the usual norm. We define a mapping $T_n : E \to E$ by

$$T_n(x) = \begin{cases} 0, \text{ if } & x \le \frac{1}{n}; \\ \frac{1}{n}, \text{ if } & x > \frac{1}{n}, \end{cases}$$

for all $n \geq 0$ and for each $x \in \mathbb{R}$. Hence, $\bigcap_{n=1}^{\infty} F(T_n) = F(T_n) = \{0\}$ and $\varphi(0, T_n x) = ||0 - T_n x|| \leq ||0 - x|| = \varphi(0, x)$, $\forall x \in \mathbb{R}$. Then, T is a relatively quasi-nonexpansive mapping but not a relatively nonexpansive mapping. Moreover, for each bounded sequence $z_n \in E$, we observe that $T_n z_n = \frac{1}{n} \to 0$ as $n \to \infty$, and hence $z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} T_n z_n = 0$ as $n \to \infty$; this implies that $z = 0 \in F(T_n)$. Therefore, T_n is a relatively quasi-nonexpansive mapping and satisfies the (*)-condition.

In 2010, Shehu [47] introduced a new iterative scheme by hybrid methods and proved strong convergence theorem for approximation of a common fixed point of two countable families of weak relatively nonexpansive mappings which is also a solution to a system of generalized mixed equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth using the properties of generalized *f*-projection operator.

The following questions naturally arise in connection with the above results using the (*)-condition:

Question 1: Can the Mann algorithms (1.20) of [45] still be valid for an infinite family of relatively quasi-nonexpansive mappings?

Question 2: Can an iterative scheme (1.19) to solve a system of generalized mixed equilibrium problems?

Question 3: Can the Mann algorithms (1.20) be extended to more generalized *f*-projection operator?

The purpose of this paper is to solve the above questions. We introduce a new hybrid iterative scheme of the generalized *f*-projection operator for finding a common element of the fixed point set for a countable family of relatively quasi-nonexpansive mappings and the set of solutions of the system of generalized mixed equilibrium problem in a uniformly convex and uniformly smooth Banach space by using the (*)-condition. Furthermore, we show that our new iterative scheme converges strongly to a common element of the aforementioned sets. Our results extend and improve the recent result of Li et al. [26], Matsushita and Takahashi [35], Takahashi et al. [43], Nakajo and Takahashi [41] and Shehu [45] and others.

2 Preliminaries

A Banach space E is said to be *strictly convex* if $||\frac{x+y}{2}|| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Let $U = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{||x+ty||-||x||}{t}$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. Let E be a Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty] \to [0, \infty]$ defined by $\rho_E(t) = \sup\left\{\frac{||x+y||+||x-y||}{2} - 1 : ||x|| = 1, ||y|| \leq t\right\}$. The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by $\delta_E(\varepsilon) = \inf\{1 - ||\frac{x+y}{2}|| : x, y \in E, ||x|| = ||y|| = 1, ||x - y|| \geq \varepsilon\}$. The *normalized duality mapping* $J : E \to 2^{E*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2, ||x^*|| = ||x||\}$. If E is a Hilbert space, then $J = I$, where I is the identity mapping.

It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Remark 2.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\varphi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\varphi(x, y) = 0$ then $x = y$. From (1.8), we have $||x|| = ||y||$. This implies that $\langle x, Jy\rangle = ||x||^2 = ||Jy||^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [19,21] for more details.

We also need the following lemmas for the proof of our main results:

Lemma 2.2. [20]*Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\varphi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then* $||x_n - y_n|| \to 0$.

Lemma 2.3. [48]*Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x*\rangle + \alpha, \ \forall x \in E.$$

Lemma 2.4. [26]*Let E be a reflexive smooth Banach space and C be a nonempty closed convex subset of E. The following statements hold:*

1. $\Pi_C^f x$ *is nonempty closed convex subset of C for all $x \in E$;*
2. *for all $x ⌊ E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \quad \forall y \in C;$$

3. *if E is strictly convex, then Π_C^f is a single-valued mapping.*

Lemma 2.5. [26]*Let E be a real reflexive smooth Banach space, let C be a nonempty closed convex subset of E, and let $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C.$$

Remark 2.6. Let E be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$; then Lemma 2.5 reduces to the property of the generalized projection operator considered by Alber [17].

Lemma 2.7. [4]*Let E be a real uniformly smooth and strictly convex Banach space, and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and relatively quasi-nonexpansive mapping. Then $F(T)$ is a closed and convex subset of C.*

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone, i.e., $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} \theta(tz + (1 - t)x, y) \leq \theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semi-continuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$\theta(x, y) = \langle Ax, y - x\rangle, \forall x, y \in C.$$

Then, θ satisfies (A1)-(A4). The following result is in Blum and Oettli [1].

Motivated by Combettes and Hirstoaga [3] in a Hilbert space and Taka-hashi and Zembayashi [12] in a Banach space, Zhang [16] obtain the following lemma:

Lemma 2.8. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), $A : C \to E^*$ be a continuous and monotone mapping and $\phi : C \to \mathbb{R}$ be a semicontinuous and convex functional. For $r > 0$ and let $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

where $F(z, y) = \theta(x, y) + \langle Az, y - z\rangle + \phi(y) - \phi(x)$, $x, y \in C$. Furthermore, define a mapping $T_r^F : E \to C$as follows:

$$T_r^F x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(1) T_r^F*is single-valued;*

(2) T_r^F*is firmly nonexpansive, i.e., for all* $x, y \in E$, $\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leq \langle T_r^F x - T_r^F y, Jx - Jy\rangle$;

(3) $F(T_r^F) = \widehat{F}(T_r^F) = \mathrm{GMEP}(\theta, A, \varphi)$;

(4) $\mathrm{GMEP}(\theta, A, \phi)$ *is closed and convex;*

(5) $\phi(p, T_r^F z) + \phi(T_r^F z, z) \leq \phi(p, z),, \forall p \in F(T_r^F)$*and* $z \in E$.

3 Main results

In this section, by using the (*)-condition, we prove the new convergence theorems for finding a common fixed points of a countable family of relatively quasi-nonexpansive mappings, in a uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$be a countable family of relatively quasi-nonexpansive mappings of C into E satisfy the (*)-condition and $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$, where $D(f)$ is a domain of f. For each j = 1, 2, ..., m let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $A_j : C \to E^*$ be a continuous and monotone mapping, and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that $\mathfrak{F} := (\cap_{n=1}^{\infty} F(T_n)) \bigcap (\cap_{j=1}^{m} \mathrm{GMEP}(\theta_j, A_j, \varphi_j)) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^{f} x_0$and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \dots, T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le G(z, Jy_n) \le G(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \quad n \ge 1, \end{cases} \tag{3.1}$$

where J is the duality mapping on E, $\{\alpha_n\}$ is a sequence in [0, 1] and $\{r_{j,n}\}_{n=1}^{\infty} \subset [d, \infty)$for some d >0 (j = 1, 2, ..., m). If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, *then $\{x_n\}$ converges strongly to $p \in \mathfrak{F}$, where $p = \Pi_{\mathfrak{F}}^{f} x_0$.*

Proof We split the proof into five steps.

Step 1: We first show that C_n is closed and convex for each $n \in \mathbb{N}$.

Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we know $G(z, Ju_n) \le G(z, Jx_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \le \|x_n\|^2 - \|u_n\|^2.$$

So, C_{n+1} is closed and convex. This implies that $\Pi_{C_{n+1}}^{f} x_0$ is well defined.

Step 2 : We show that $\mathfrak{F} \subset C_n$ for all $n \in \mathbb{N}$.

Next, we show by induction that $\mathfrak{F} \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathfrak{F} \subset C = C_1$. Suppose that $\mathfrak{F} \subset C_n$ for some $n \in \mathbb{N}$. Let $q \in \mathfrak{F}$ and $u_n = K_n^m y_n$, when $K_n^j = T_{r_{j,n}}^{F_j} T_{r_{j-1,n}}^{F_{j-1}}, \dots, T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$, j = 1, 2, 3, ..., m, $K_n^0 = I$; since $\{T_n\}$ is relatively quasi-nonexpansive mappings, it follows by (3.2) that

$$\begin{aligned} G(q, Ju_n) &= G(q, JK_n^m y_n) \\ &\le G(q, Jy_n) \\ &= G(q, \alpha_n Jx_n + (1-\alpha_n)JT_n x_n) \\ &= \|q\|^2 - 2\langle q, \alpha_n Jx_n + (1-\alpha_n)JT_n x_n\rangle \\ &\quad + \|\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\|^2 + 2\rho f(q) \\ &\le \|q\|^2 - 2\alpha_n\langle q, Jx_n\rangle - 2(1-\alpha_n)\langle q, JT_n x_n\rangle \\ &\quad + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|JT_n x_n\|^2 + 2\rho f(q) \\ &= \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, JT_n x_n) \\ &\le \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, Jx_n) \\ &= G(q, Jx_n). \end{aligned} \tag{3.2}$$

This shows that $q \in C_{n+1}$ which implies that $\mathfrak{F} \subset C_{n+1}$ and hence, $\mathfrak{F} \subset C_n$ for all $n \in \mathbb{N}$.

Step 3 : We show that $\{x_n\}$ is a Cauchy sequence in C and $\lim_{n\to\infty} G(x_n, Jx_0)$ exist.

Since $f : E \to \mathbb{R}$ is convex and lower semicontinuous mapping, from Lemma 2.3, we know that there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, x^* \rangle + \alpha, \forall y \in E.$$

Since $x_n \in E$, it follows that

$$\begin{aligned} G(x_n, Jx_0) &= ||x_n||^2 - 2\langle x_n, Jx_0 \rangle + ||x_0||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx_0 \rangle + ||x_0||^2 + 2\rho \langle x_n, x^* \rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx_0 - \rho x^* \rangle + ||x_0||^2 + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx_0 - \rho x^*|| + ||x_0||^2 + 2\rho\alpha \\ &= (||x_n|| - ||Jx_0 - \rho x^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho x^*||^2 + 2\rho\alpha. \end{aligned} \tag{3.3}$$

Again since $x_n = \Pi^f_{C_n} x_0$ and from (3.3), we have

$$\begin{aligned} G(q, Jx_0) \geq G(x_n, Jx_0) &\geq (||x_n|| - ||Jx_0 - \rho x^*||)^2 \\ &+||x_0||^2 - ||Jx_0 - \rho x^*||^2 + 2\rho\alpha, \quad \forall q \in \mathcal{F}. \end{aligned}$$

This implies that $\{x_n\}$ is bounded and so are $\{G(x_n, Jx_0)\}$, $\{y_n\}$ and $\{u_n\}$. From the fact that $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and $x_n = \Pi^f_{C_n} x_0$, it follows by Lemma 2.5, we get

$$0 \leq (||x_{n+1} - ||x_n||)^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0). \tag{3.4}$$

This implies that $\{G(x_n, Jx_0)\}$ is nondecreasing. So, we obtain that $\lim_{n\to\infty} G(x_n, Jx_0)$ exist. For $m > n$, $x_n = \prod^f_{C_n} x_0$, $x_m = \prod^f_{C_m} x_0 \in C_m \subset C_n$ and from (3.4), we have

$$\phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0).$$

Taking $m, n \to \infty$, we have $\varphi(x_m, x_n) \to 0$. From Lemma 2.2, we get $||x_n - x_m|| \to 0$. Hence, $\{x_n\}$ is a Cauchy sequence and by the completeness of E and the closedness of C, we can assume that there exists $p \in C$ such that $x_n \to p \in C$ as $n \to \infty$.

Step 4 : We will show that $p \in \mathfrak{F} := (\cap_{n=1}^{\infty} F(T_n)) \bigcap (\cap_{j=1}^{m} \mathrm{GMEP}(\theta_j, A_j, \varphi_j))$.

(a) We show that $p \in \cap_{n=1}^{\infty} F(T_n)$. Since $\varphi(x_m, x_n) \to 0$ as $m, n \to \infty$, we obtain in particular that $\varphi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. By Lemma 2.2, we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.5}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} ||Jx_{n+1} - Jx_n|| = 0. \tag{3.6}$$

From the definition of $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$G(x_{n+1}, Ju_n) \leq G(x_{n+1}, Jx_n), \quad \forall n \in \mathbb{N},$$

is equivalent to

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N}.$$

It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.7}$$

By applying Lemma 2.2, we have

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{3.8}$$

By the triangle inequality, we have

$$\begin{aligned} ||u_n - x_n|| &= ||u_n - x_{n+1} + x_{n+1} - x_n|| \\ &\leq ||u_n - x_{n+1}|| + ||x_{n+1} - x_n|| \end{aligned}$$

It follows from (3.5) and (3.8), that

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty} ||Ju_n - Jx_n|| = 0. \tag{3.10}$$

From $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we get

$$G(x_{n+1}, Jy_n) \leq G(x_{n+1}, Jx_n)$$

is equivalent to

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

Using Lemma 2.2, we have

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0. \tag{3.11}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} ||Jx_{n+1} - Jy_n|| = 0. \tag{3.12}$$

Noticing that

$$\begin{aligned} ||Jx_{n+1} - Jy_n|| &= ||Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n)JT_n x_n|| \\ &= ||(1-\alpha_n)Jx_{n+1} - (1-\alpha_n)JT_n x_n + \alpha_n Jx_{n+1} - \alpha_n Jx_n|| \\ &\geq (1-\alpha_n)||Jx_{n+1} - JT_n x_n|| - \alpha_n ||Jx_n - Jx_{n+1}||, \end{aligned} \tag{3.13}$$

we have

$$||Jx_{n+1} - JT_n x_n|| \leq \frac{1}{(1-\alpha_n)}(||Jx_{n+1} - Jy_n|| + \alpha_n ||Jx_n - Jx_{n+1}||), \tag{3.14}$$

since $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, (3.6) and (3.12), one has

$$\lim_{n\to\infty} ||Jx_{n+1} - JT_n x_n|| = 0. \tag{3.15}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} ||x_{n+1} - T_n x_n|| = 0. \tag{3.16}$$

Using the triangle inequality, we have

$$||x_n - T_n x_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - T_n x_n||.$$

From (3.5) and (3.16), we have

$$\lim_{n\to\infty} ||x_n - T_n x_n|| = 0. \tag{3.17}$$

Since $x_n \to p$ it follows from the (*)-condition that $p \in \mathfrak{F} = \cap_{n=0}^{\infty} F(T_n)$.

(b) We show that $p \in \cap_{j=1}^{m} \mathrm{GMEP}(\theta_j, A_j, \varphi_j)$.

For $q \in \mathfrak{F}$, we have

$$\begin{aligned}\phi(q, x_n) - \phi(q, u_n) &= ||x_n||^2 - ||u_n||^2 - 2\langle q, Jx_n - Ju_n\rangle \\ &\le ||x_n - u_n||(||x_n|| + ||u_n||) + 2||q||\,||Jx_n - Ju_n||.\end{aligned}$$

From $||x_n - u_n|| \to 0$ and $||Jx_n - Ju_n|| \to 0$, that

$$\phi(q, x_n) - \phi(q, u_n) \to 0 \; as \; n \to \infty. \tag{3.18}$$

Let $u_n = K_n^m y_n$; when $K_n^j = T_{r_{j,n}}^{F_j} T_{r_{j-1,n}}^{F_{j-1}}, \ldots, T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$, $j = 1, 2, 3, \ldots, m$ and $K_n^0 = I$, we obtain that

$$\begin{aligned}\phi(q, u_n) &= \phi(q, K_n^m y_n) \\ &\le \phi(q, K_n^{m-1} y_n) \\ &\le \phi(q, K_n^{m-2} y_n) \\ &\;\;\vdots \\ &\le \phi(q, K_n^j y_n).\end{aligned} \tag{3.19}$$

By Lemma 2.8(5), we have for $j = 1, 2, 3, \ldots, m$

$$\begin{aligned}\phi(K_n^j y_n, y_n) &\le \phi(q, y_n) - \phi(q, K_n^j y_n) \\ &\le \phi(q, x_n) - \phi(q, K_n^j y_n) \\ &\le \phi\,(q, x_n) - \phi(q, u_n).\end{aligned} \tag{3.20}$$

By (3.18), we have $\phi(K_n^j y_n, y_n) \to 0$ as $n \to \infty$, for $j = 1, 2, 3, \ldots, m$. By Lemma 2.2, we obtain

$$\lim_{n\to\infty} ||K_n^j y_n - y_n|| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.21}$$

Since $||x_n - y_n|| \le ||x_n - x_{n+1}|| + ||x_{n+1} - y_n||$. From (3.11) and (3.5), we get

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.22}$$

Again by using the triangle inequality, we have for $j = 1, 2, 3, \ldots, m$

$$||K_n^j y_n - p|| \quad \le \quad ||K_n^j y_n - y_n|| + ||y_n - p||.$$

Since $x_n \to p$ and $||x_n - y_n|| \to 0$, then $y_n \to p$ as $n \to \infty$. From (3.21), we get

$$\lim_{n\to\infty} ||K_n^j y_n - p|| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.23}$$

Using the triangle inequality, we obtain

$$||K_n^j y_n - K_n^{j-1} y_n|| \quad \le \quad ||K_n^j y_n - p|| + ||p - K_n^{j-1} y_n||.$$

From (3.23), we have

$$\lim_{n\to\infty} ||K_n^j y_n - K_n^{j-1} y_n|| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{3.24}$$

Since $\{r_{j,n}\} \subset [d, \infty)$, so

$$\lim_{n\to\infty} \frac{\|K_n^j y_n - K_n^{j-1} y_n\|}{r_{j,n}} = 0, \quad \forall j = 1, 2, 3, \dots, m. \tag{3.25}$$

From Lemma 2.8, we get for j = 1, 2, 3, ..., m

$$F_j(K_n^j y_n, y) + \frac{1}{r_{j,n}} \langle y - K_n^j y_n, JK_n^j y_n - JK_n^{j-1} y_n \rangle \geq 0, \quad \forall y \in C.$$

From the condition (A2) that

$$\frac{1}{r_{j,n}} \langle y - K_n^j y_n, JK_n^j y_n - JK_n^{j-1} y_n \rangle \geq F_j(y, K_n^j y_n), \\ \forall y \in C, \ \forall j = 1, 2, 3, \dots, m.$$

From (3.23) and (3.25), we have

$$0 \geq F_j(y, p), \quad \forall y \in C, \ \forall j = 1, 2, 3, \dots, m. \tag{3.26}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = t_y + (1 - t)p$. Then, we get that $y_t \in C$. From (3.26), it follows that

$$F_j(y_t, p) \leq 0, \quad \forall y_t \in C, \ \forall j = 1, 2, 3, \dots, m. \tag{3.27}$$

By the conditions (A1) and (A4), we have for j = 1, 2, 3, ..., m

$$\begin{aligned} 0 = &\ F_j(y_t, y_t) \\ &\leq tF_j(y_t, y) + (1-t)F_j(y_t, p) \\ &\leq tF_j(y_t, y) \\ &\leq F_j(y_t, y). \end{aligned} \tag{3.28}$$

From the condition (A3) and letting $t \to 0$, This implies that $p \in \text{GMEP}(\theta_j, A_j, \phi_j)$ for all j = 1, 2, 3, ..., m. Therefore, $p \in \cap_{j=1}^m GMEP(\theta_j, A_j, \varphi_j)$. Hence, from ($a$) and ($b$), we obtain $p \in \mathfrak{F}$.

Step 5: We show that $p = \Pi_{\mathfrak{F}}^f x_0$. Since $\mathfrak{F}$ is closed and convex set from Lemma 2.4, we have $\Pi_{\mathfrak{F}}^f x_0$ is single value, denoted by v. From $x_n = \Pi_{C_n}^f x_0$ and $v \in \mathfrak{F} \subset C_n$, we also have

$$G(x_n, Jx_0) \leq G(v, Jx_0), \quad \forall n \geq 1.$$

By definition of G and f, we know that, for each given x, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to ξ. So

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(v, Jx_0).$$

From definition of $\Pi_{\mathfrak{F}}^f x_0$ and $p \in \mathfrak{F}$, we can conclude that $v = p = \Pi_{\mathfrak{F}}^f x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof. □

Setting $T_n \equiv T$ in Theorem 3.1, then we obtain the following result:

Corollary 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let T be a relatively quasi-nonexpansive mapping of C into E and $f: E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. For each j = 1, 2, ..., m let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $A_j : C \to E^*$ be a continuous and monotone mapping and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that*

$x_1 = \Pi^f_{C_1} x_0$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi^f_{C_1} x_0$*and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n = T^{F_m}_{r_{m,n}} T^{F_{m-1}}_{r_{m-1,n}}, \ldots, T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}} y_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le G(z, Jy_n) \le G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \quad n \ge 1, \end{cases} \tag{3.29}$$

where J *is the duality mapping on* E, $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *and* $\{r_{j,n}\}_{n=1}^{\infty} \subset [d, \infty)$*for some* $d > 0$ ($j = 1, 2, \ldots, m$). *If* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, *then* $\{x_n\}$ *converges strongly to* $p \in \mathfrak{F}$, *where* $p = \Pi^f_{\mathfrak{F}} x_0$.

Remark 3.3. Corollary 3.2 extends and improves the result of Li et al. [26].

Taking $f(x) = 0$ for all $x \in E$, we have $G(\xi, Jx) = \varphi(\xi, x)$ and $\Pi^f_C x = \Pi_C x$. By Theorem 3.1, then we obtain the following Corollaries:

Corollary 3.4. *Let* C *be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space* E. *Let* $\{T_n\}_{n=1}^{\infty}$*be a countable family of relatively quasi-nonexpansive mappings of* C *to* E *satisfy the* (*) *condition. For each* $j = 1, 2, \ldots, m$ *let* θ_j *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies conditions (A1)-(A4),* $A_j : C \to E^*$ *be a continuous and monotone mapping, and* $\phi_j : C \to \mathbb{R}$ *be a lower semicontinuous and convex function. Assume that* $\mathfrak{F} := (\cap_{n=1}^{\infty} F(T_n)) \bigcap (\cap_{j=1}^{m} \mathrm{GMEP}(\theta_j, A_j, \varphi_j)) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$*and* $C_1 = C$, *we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n), \\ u_n = T^{F_m}_{r_{m,n}} T^{F_m}_{r_{m-1,n}}, \ldots, T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, Jy_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 1, \end{cases} \tag{3.30}$$

where J *is the duality mapping on* E, $\{\alpha_n\}$ *is a sequence in* $[0, 1]$ *and* $\{r_{j,n}\}_{n=1}^{\infty} \subset [d, \infty)$*for some* $d > 0$ ($j = 1, 2, \ldots, m$). *If* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, *then* $\{x_n\}$ *converges strongly to* $p \in \mathfrak{F}$, *where* $p = \Pi_{\mathfrak{F}} x_0$.

Remark 3.5. Corollary 3.4 extends and improves the result of Shehu [[45], Theorem 3.1] form finite family of relatively quasi-nonexpansive mappings to a countable family of relatively quasi-nonexpansive mappings.

4 Applications

4.1 A zero of $\mathcal{B}$-monotone mappings

Let $\mathcal{B}$ be a mapping from E to E^*. A mapping $\mathcal{B}$ is said to be

1. *monotone* if $\langle \mathcal{B}x - \mathcal{B}y, x - y\rangle \ge 0$ for all $x, y \in E$;
2. *strictly monotone* if $\mathcal{B}$ monotone and $\langle \mathcal{B}x - \mathcal{B}y, x - y\rangle = 0$ if and only if $x = y$;
3. *β-Lipschitz continuous* if there exist a constant $\beta \ge 0$ such that $\|\mathcal{B}x - \mathcal{B}y\| \le \beta\|x - y\|$ for all $x, y \in E$.

Let M be a set-valued mapping from E to E^* with domain $D(M) = \{z \in E : Mz \neq 0\}$ and range $R(M) = \cup\{Mz : z \in D(M)\}$. A set value mapping M is said to be

(i) *monotone* if $\langle x_1\text{-}x_2, y_1\text{-}y_2\rangle \geq 0$ for each $x_i \in D(M)$ and $y_i \in Mx_i$, $i = 1, 2$;
(ii) *r-strongly monotone* if $\langle x_1\text{-}x_2, y_1\text{-}y_2\rangle \geq r||x_1\text{-}x_2||$ for each $x_i \in D(M)$ and $y_i \in Mx_i$, $i = 1, 2$;
(iii) *maximal monotone* if M is monotone and its graph $\mathfrak{G}(M) = \{(x, y) : y \in Mx\}$ is not properly contained in the graph of any other monotone mapping;
(iv) *general $\mathcal{B}$-monotone* if M is monotone and $(\mathcal{B} + \lambda M)E = E*$ holds for every $\lambda > 0$, where $\mathcal{B}$ is a mapping from E to E^*.

We consider the problem of finding a point $x^* \in E$ satisfying $0 \in Mx^*$. We denote by $M^{-1}0$ the set of all points $x^* \in E$ such that $0 \in Mx^*$, where M is maximal monotone operator from E to E^*.

Lemma 4.1. [26]*Let E be a Banach space with the dual space E^*, $\mathcal{B} : E \to E*$be a strictly monotone mapping, and $M : E \to 2^{E*}$ be a general $\mathcal{B}$-monotone mapping. Then M is maximal monotone mapping.*

Remark 4.2. [26] Let E be a Banach space with the dual space E^*, $\mathcal{B} : E \to E*$ be a strictly monotone mapping, and $M : E \to 2^{E*}$ be a general $\mathcal{B}$-monotone mapping. Then M is a maximal monotone mapping. Therefore, $M^{-1}0 = \{z \in D(M) : 0 \in Mz\}$ is closed and convex.

Lemma 4.3. [17]*Let E be a uniformly convex and uniformly smooth Banach space, $\delta_E(\varepsilon)$ be the modulus of convexity of E, and $\rho_E(t)$ be the modulus of smoothness of E; then the inequalities*

$$8d^2\delta_E(||x - \xi||/4d) \leq \phi(x, \xi) \leq 4d^2\rho_E(4||x - \xi||/d)$$

hold for all x and ξ in E, where $d = \sqrt{(||x||^2 + ||\xi||^2)/2}$.

Lemma 4.4. [49]*Let E be a Banach space with the dual space E^*, $\mathcal{B} : E \to E*$be a strictly monotone mapping, and $M : E \to 2^{E*}$ be a general $\mathcal{B}$-monotone mapping. Then*

1. *$(\mathcal{B} + \lambda M)^{-1}$is single value;*
2. *if E is reflexive and $M : E \to 2^{E*}$ a r-strongly monotone mapping, then $(\mathcal{B} + \lambda M)^{-1}$is Lipschitz continuous with constant $\frac{1}{\lambda r}$, where $r > 0$.*

From Lemma 4.4 we note that let E be a Banach space with the dual space E^*, $\mathcal{B} : E \to E*$ a strictly monotone mapping, and $M : E \to 2^{E*}$ a general $\mathcal{B}$-monotone mapping, for every $\lambda > 0$ and $x^* \in E^*$; then there exists a unique $x \in D(M)$ such that $x = (\mathcal{B} + \lambda M)^{-1}x^*$. We can define a single-valued mapping $T_\lambda : E \to D(M)$ by $T_\lambda x = (\mathcal{B} + \lambda M)^{-1}\mathcal{B}x$. It is easy to see that $M^{-1}0 = F(T_\lambda)$ for all $\lambda > 0$. Indeed, we have

$$\begin{aligned} z \in M^{-1}0 &\Leftrightarrow 0 \in Mz \\ &\Leftrightarrow 0 \in \lambda Mz \\ &\Leftrightarrow \mathcal{B}z \in (\mathcal{B} + \lambda M)z \\ &\Leftrightarrow z = (\mathcal{B} + \lambda M)^{-1}\mathcal{B}z = T_\lambda z \\ &\Leftrightarrow z \in F(T_\lambda), \forall\lambda > 0. \end{aligned} \tag{4.1}$$

Motivated by Li et al. [26] we obtain the following result:

Theorem 4.5. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with $\delta_E(\varepsilon) \geq k\varepsilon^2$ and $\rho_E(t) \leq ct^2$ for some $c, k > 0$, and E^* be the dual space of E. Let $\mathcal{B} : E \to E*$be a strictly monotone and β-Lipschitz* *continuous mapping, and let* $M : E \to 2^{E*}$ *be a general* $\mathcal{B}$*-monotone and r-strongly monotone mapping with* $r > 0$*. Let* $\{T_{\lambda_n}\} = (\mathcal{B} + \lambda_n M)^{-1}\mathcal{B}$*satisfy the* (*)*-condition and* $f : E \to \mathbb{R}$ *be a convex lower semicontinuous mapping with* $C \subset int(D(f))$ *and suppose that for each* $n \geq 0$ *there exists* $\lambda_n > 0$ *such that* $64c\beta^2 \leq min\{\frac{1}{2}k\lambda_n^2 r^2\}$*. For each* $j = 1, 2, ..., m$ *let* θ_j *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies conditions (A1)-(A4),* $A_j : C \to E^*$ *be a continuous and monotone mapping, and* $\phi_j : C \to \mathbb{R}$ *be a lower semicontinuous and convex function. Assume that* $\mathfrak{F} := M^{-1}0 \bigcap (\cap_{j=1}^m \mathrm{GMEP}(\theta_j, A_j, \varphi_j)) \neq \emptyset$*. For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1}^f x_0$*and* $C_1 = C$*, we define the sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_{\lambda_n}x_n), \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \leq G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \geq 1, \end{cases} \tag{4.2}$$

where J *is the duality mapping on* E *and* $\{\alpha_n\}$ *is a sequence in* $[0, 1]$*, and* $\{r_{j,n}\}_{n=1}^\infty \subset [d, \infty)$*for some* $d > 0$ $(j = 1, 2, ..., m)$*. If* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$*, then* $\{x_n\}$ *converges strongly to* $p \in \mathfrak{F}$*where* $p = \Pi_{\mathfrak{F}}^f x_0$*.*

Proof We show that $\{T_{\lambda_n}\}$ is a family of relatively quasi-nonexpansive mappings with common fixed point $\cap_{n=1}^\infty F(T_{\lambda_n}) = M^{-1}0$. We only show that $\phi(p, T_{\lambda_n}q) \leq \phi(p, q)$, $\forall q \in E$, $p \in F(T_{\lambda_n})$, $n \geq 1$. From Lemma 4.3, and $\mathcal{B}$ is a β-Lipschitz continuous mapping, we have

$$\begin{aligned} \phi(p, T_{\lambda_n}q) &= \phi(T_{\lambda_n}p, T_{\lambda_n}q) \\ &\leq 4d^2\rho_E\left(\frac{4||T_{\lambda_n}p - T_{\lambda_n}q||}{d}\right) \\ &\leq 64c||T_{\lambda_n}p - T_{\lambda_n}q||^2 \\ &= 64c||(\mathcal{B} + \lambda_n M)^{-1}\mathcal{B}p - (\mathcal{B} + \lambda_n M)^{-1}\mathcal{B}q||^2 \\ &\leq \frac{64c}{\lambda_n^2 r^2}||\mathcal{B}p - \mathcal{B}q||^2 \\ &\leq \frac{64c\beta^2}{\lambda_n^2 r^2}||p - q||^2 \end{aligned} \tag{4.3}$$

and we also have

$$\phi(p, q) \geq 8d^2\delta_E\left(\frac{||p-q||}{4d}\right) \geq \frac{1}{2}k||p - q||^2. \tag{4.4}$$

Since

$$64c\beta^2 \leq \frac{1}{2}k\lambda_n^2 r^2,$$

it follows from (4.3) and (4.4) that $\phi(p, T_{\lambda_n}q) \leq \phi(p, q)$ for all $q \in E$, $p \in F(T_{\lambda_n})$, $n \geq 1$. Therefore, $\{T_{\lambda_n}\}$ is a family of relatively quasi-nonexpansive mapping. It follows from Theorem 3.1, so the desired conclusion follows. □

4.2 A zero point of maximal monotone operators

In this section, we apply our results to find zeros of maximal monotone operator. Such a problem contains numerous problems in optimization, economics, and physics. The following result is also well known.

Lemma 4.6. [50]*Let* E *be a reflexive strictly convex and smooth Banach space and let* M *be a monotone operator from* E *to* E^**. Then* M *is maximal if and only if* $R(J + \lambda M) = E^*$ *for all* $\lambda > 0$*.*

Let E be a reflexive strictly convex and smooth Banach space, $\mathcal{B} = J$ and let M be a maximal monotone operator from E to E^*. Using Lemma 4.6 and strict convexity of E, we obtain that for every $\lambda > 0$ and $x \in E$, there exists a unique x_λ such that $Jx \in (Jx_\lambda + \lambda Mx_\lambda)$. Then we can defined a single-valued mapping $J_\lambda : E \to D(M)$ by $J_\lambda = (J + \lambda M)^{-1}J$ and J_λ is called the *resolvent* of M. We know that $M^{-1}0 = F(J_\lambda)$ [21,51].

Theorem 4.7. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E with the dual space E^*. Let $M \subset E \times E^*$ be a maximal monotone mapping and $D(M) \subset C \subset J^{-1}(\cap_{\lambda_n>0} R(J + \lambda_n M)$. Let $\{J_{\lambda_n}\} = (J + \lambda_n M)^{-1}J$satisfy the (*)-condition where $\lambda_n > 0$ be the resolvement of M and $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. For each $j = 1, 2, \ldots, m$ let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $Aj : C \to E^*$ be a continuous and monotone mapping, and $\phi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that $\mathfrak{F} = M^{-1}0 \bigcap (\cap_{j=1}^m \mathrm{GMEP}(\theta_j, A_j, \varphi_j)) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}^f x_0$and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_{\lambda_n}x_n), \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}}, \ldots, T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le G(z, Jy_n) \le G(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \ge 1, \end{cases} \tag{4.5}$$

where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in $[0, 1]$ *and* $\{r_{j,n}\}_{n=1}^\infty \subset [d, \infty)$*for some $d > 0$ ($j = 1, 2, \ldots, m$). If* $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, *then $\{x_n\}$ converges strongly to $p \in \mathfrak{F}$, where $p = \Pi_{\mathfrak{F}}^f x_0$.*

Proof First, we have $\cap_{n=1}^\infty F(J_{\lambda_n}) = M^{-1}0 \neq \emptyset$. Second, from the monotonicity of M, let $p \in \cap_{n=1}^\infty F(J_{\lambda_n})$ and $q \in E$; we have

$$\begin{aligned} \phi(p, J_{\lambda_n}q) &= ||p||^2 - 2\langle p, JJ_{\lambda_n}q\rangle + ||J_{\lambda_n}q||^2 \\ &= ||p||^2 + 2\langle p, Jq - JJ_{\lambda_n}q - Jq\rangle + ||J_{\lambda_n}q||^2 \\ &= ||p||^2 + 2\langle p, Jq - JJ_{\lambda_n}q\rangle - 2\langle p, Jq\rangle + ||J_{\lambda_n}q||^2 \\ &= ||p||^2 - 2\langle J_{\lambda_n}q - p - J_{\lambda_n}q, Jq - JJ_{\lambda_n}q\rangle - 2\langle p, Jq\rangle + ||J_{\lambda_n}q||^2 \\ &= ||p||^2 - 2\langle J_{\lambda_n}q - p, Jq - JJ_{\lambda_n}q\rangle + 2\langle J_{\lambda_n}q, Jq - JJ_{\lambda_n}q\rangle - 2\langle p, Jq\rangle + ||J_{\lambda_n}q||^2 \\ &\le ||p||^2 + 2\langle J_{\lambda_n}q, Jq - JJ_{\lambda_n}q\rangle - 2\langle p, Jq\rangle + ||J_{\lambda_n}q||^2 \\ &= ||p||^2 - 2\langle p, Jq\rangle + ||q||^2 - ||J_{\lambda_n}q||^2 + 2\langle J_{\lambda_n}q, Jq\rangle - ||q||^2 \\ &= \phi(p, q) - \phi(J_{\lambda_n}q, q) \\ &\le \phi(p, q) \end{aligned}$$

for all $n \ge 1$. Therefore, $\{J_{\lambda_n}\}$ is a family of relatively quasi-nonexpansive mapping for all $\lambda_n > 0$ with the common fixed point set $\cap_{n=1}^\infty F(J_{\lambda_n}) = M^{-1}0$. Hence, it follows from Theorem 3.1, the desired conclusion follows: □

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Author details

[1]Department of Mathematics, Faculty of Science King Mongkut's University of Technology Thonburi (KMUTT) Bangmod, Bangkok 10140, Thailand [2]Centre of Excellence in Mathematics, CHE Si Ayutthaya Rd., Bangkok 10400, Thailand

Authors' contributions

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Competing interests

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