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Fixed point-type results for a class of extended cyclic self-mappings under three general weak contractive conditions of rational type

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Abstract

This article discusses three weak ϕ-contractive conditions of rational type for a class of 2-cyclic self-mappings defined on the union of two non-empty subsets of a metric space to itself. If the space is uniformly convex and the subsets are non-empty, closed, and convex, then the iterates of points obtained through the self-mapping converge to unique best proximity points in each of the subsets.

1. Introduction

A general contractive condition has been proposed in [1,2] for mappings on a partially ordered metric space. Some results about the existence of a fixed point and then its uniqueness under supplementary conditions are proved in those articles. The rational contractive condition proposed in [3] includes as particular cases several of the previously proposed ones [1,4-12], including Banach principle [5] and Kannan fixed point theorems [4,8,9,11]. The rational contractive conditions of [1,2] are applicable only on distinct points of the considered metric spaces. In particular, the fixed point theory for Kannan mappings is extended in [4] by the use of a non-increasing function affecting the contractive condition and the best constant to ensure a fixed point is also obtained. Three fixed point theorems which extended the fixed point theory for Kannan mappings were stated and proved in [11]. More attention has been paid to the investigation of standard contractive and Meir-Keeler-type contractive 2-cyclic self-mappings T:$A \cup B \to A \cup B$ defined on subsets $A,B \subseteq X$ and, in general, p-cyclic self-mappings $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ defined on any number of subsets $A_i \subset X$, $i \in \bar{p} := \{1, 2, \ldots, p\}$, where (X,d) is a metric space (see, for instance [13-22]). More recent investigation about cyclic self-mappings is being devoted to its characterization in partially ordered spaces and also to the formal extension of the contractive condition through the use of more general strictly increasing functions of the distance between adjacent subsets. In particular, the uniqueness of the best proximity points to which all the sequences of iterates of composed self-mappings T_2: $A \cup B \to A \cup B$ converge is proven in [14] for the extension of the contractive principle for cyclic self-mappings in uniformly convex Banach spaces (then being strictly convex and reflexive [23]) if the subsets $A,B \subset X$ in the metric space (X,d), or in the Banach space $(X,|| \ ||)$, where the 2-cyclic self-mappings are defined, are both non-empty, convex and closed. The research in [14] is

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centred on the case of the cyclic self-mapping being defined on the union of two subsets of the metric space. Those results are extended in [15] for Meir-Keeler cyclic contraction maps and, in general, for the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ be a p (≥ 2) -cyclic self-mapping being defined on any number of subsets of the metric space with $\bar{p} := \{1, 2..., p\}$. Also, the concept of best proximity points of (in general) non-self-mappings $S,T{:}A \to B$ relating non-empty subsets of metric spaces in the case that such maps do not have common fixed points has recently been investigated in [24,25]. Such an approach is extended in [26] to a mapping structure being referred to as K-cyclic mapping with contractive constant $k < 1/2$. In [27], the basic properties of cyclic self-mappings under a rational-type of contractive condition weighted by point-to-point-dependent continuous functions are investigated. On the other hand, some extensions of Krasnoselskii-type theorems and general rational contractive conditions to cyclic self-mappings have recently been given in [28,29] while the study of stability through fixed point theory of Caputo linear fractional systems has been provided in [30]. Finally, promising results are being obtained concerning fixed point theory for multivalued maps (see, for instance [31-33]).

This manuscript is devoted to the investigation of several modifications of rational type of the ϕ-contractive condition of [21,22] for a class of 2-cyclic self-mappings on non-empty convex and closed subsets $A,B \subset X$. The contractive modification is of rational type and includes the nondecreasing function associated with the ϕ-contractions. The existence and uniqueness of two best proximity points, one in each of the subsets $A,B \subset X$, of 2-cyclic self-mappings T: $A\cup B \to A \cup B$ defined on the union of two non-empty, closed, and convex subsets of a uniformly convex Banach spaces, is proven. The convergence of the sequences of iterates through T: $A\cup B \to A \cup B$ to one of such best proximity points is also proven. In the case that A and B intersect, both the best proximity points coincide with the unique fixed point in the intersection of both the sets.

2. Basic properties of some modified constraints of 2-cyclic ϕ-contractions

Let (X,d) be a metric space and consider two non-empty subsets A and B of X. Let T: $A\cup B \to A \cup B$ be a 2-cyclic self-mapping, i.e., $T(A) \subseteq B$ and $T(B) \subseteq A$. Suppose, in addition, that T: $A\cup B \to A \cup B$ is a 2-cyclic modified weak ϕ-contraction (see [21,22]) for some non-decreasing function $\phi{:}\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ subject to the rational modified ϕ-contractive constraint:

$$d(Tx, Ty) \leq \alpha \left[\frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} - \varphi\left(\frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} \right) \right] + \beta\left(d(x, y) - \varphi\left(d(x, y) \right) \right) + \varphi(D); \quad \forall x, y\, (\neq x) \in A \cup B \tag{2.1}$$

where

$$D := dist(A, B) := \inf \left\{ d(x, y) \ : x \in A, y \in B \right\} \tag{2.2}$$

$$D \leq \limsup_{n\to\infty} d\left(T^{n+1}x, T^n x\right) \leq \lim_{n\to\infty} \left(k^n d(x, Tx) + \frac{(1-k^n)\ (1-k)}{1-k} \varphi(D) \right) = \varphi(D);\ \forall x \in A\cup B \tag{2.3}$$

Note that (2.1) is, in particular, a so-called 2-cyclic ϕ-contraction if $\alpha = 0$ and $\phi(t) = (1-\alpha)t$ for some real constant $\alpha \in [0,1)$ since $\phi{:}\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is strictly increasing [1]. We refer to "modified weak ϕ-contraction" for (2.1) in the particular case $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta < 1$, and $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ being non-decreasing as counterpart to the term ϕ-contraction (or via an abuse of terminology "modified strong ϕ-contraction") for the case of $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ in (2.1) being strictly increasing. There are important background results on the properties of weak contractive mappings (see, for instance, [1,2,34] and references therein). The so-called "ϕ-contraction", [1,2], involves the particular contractive condition obtained from (2.1) with $\alpha = 0$, $\beta = 1$, and $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ being strictly increasing, that is,$d\left(Tx, Ty\right) \leq d\left(x, y\right) - \varphi\left(d\left(x, y\right)\right) + \varphi\left(D\right);\ \forall x \in A \cup B,$

In the following, we refer to 2-cyclic self-maps $T:A \cup B \to A \cup B$ simply as cyclic self-maps. The following result holds:

Lemma 2.1. Assume that $T:A \cup B \to A \cup B$ is a modified weak ϕ-contraction, that is, a cyclic self-map satisfying the contractive condition (2.1) subject to the constraints $min\ (\alpha, \beta) \geq 0$ and $\alpha + \beta < 1$ with $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ being non-decreasing. Then, the following properties hold:

(i) Assume that $\phi(D) \geq D$

$$D \leq d\left(T^{n+1}x, T^n x\right) \leq kd\left(Tx, x\right) + (1-k)\ \varphi\left(D\right);\ \forall n \in \boldsymbol{N}_0 := \boldsymbol{N} \cup \{0\},\ \forall x \in A \cup B \tag{2.4}$$

$$D \leq \liminf_{n\to\infty}\ d\left(T^{n+m+1}x, T^{n+m}x\right) \leq \limsup_{n\to\infty}\ d\left(T^{n+m+1}x, T^{n+m}x\right) \leq \varphi\left(D\right);\ \forall x \in A \cup B\ \forall m \in \boldsymbol{N}_0 \tag{2.5}$$

and $\limsup_{n\to\infty}\ d\left(T^{n+m+1}x, T^{n+m}x\right) \leq \varphi\left(D\right)$ if $D \neq 0$ If $\phi(D) = D = 0$ then $\exists \lim_{n\to\infty}\ d\left(T^{n+m+1}x, T^{n+m}x\right) = 0$; $\forall x \in A \cup B$, $\forall m \in \boldsymbol{N}_0$.

(ii) Assume that $d(x,Tx) \leq m(x)$ for any given $x \in A \cup B$. Then

$$d\left(T^n x, x\right) \leq \frac{km\left(x\right)}{1-k} + \frac{1-k}{k}\varphi\left(D\right);\ \forall x \in A \cup B,\ \forall n \in \boldsymbol{N} \tag{2.6}$$

If $d(x,Tx)$ is finite and, in particular, if x and Tx in $A \cup B$ are finite then the sequences $\{T^n x\}_{n\in\boldsymbol{N}_0}$ and $\left\{T^{n+1}x\right\}_{n\in\boldsymbol{N}_0}$ are bounded sequences where $T^n \in A$ and $T^{n+1}x \in B$ if $x \in A$ and n is even, $T^n \in B$ and $T^{n+1} \in B$ if $x \in B$ and n is even.

Proof: Take $y = Tx$ so that $Ty = T^2x$. Since $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing $\phi(x) \geq \phi(D)$ for $x \geq D$, one gets for any $x \in A$ and any $Tx \in B$ or for any $x \in B$ and any $Tx \in A$:

$$\begin{aligned} &(1-\alpha)\, d\left(T^2x, Tx\right) \leq -\alpha\varphi\left(d\left(Tx, T^2x\right)\right) + \beta\left[d\left(x, Tx\right) - \varphi\left(d\left(x, Tx\right)\right)\right] + \varphi\left(D\right) \\ &= \beta d\left(x, Tx\right) + \varphi\left(D\right) - \alpha\varphi\left(d\left(Tx, T^2x\right)\right) - \beta\varphi\left(d\left(x, Tx\right)\right);\ \forall x \in A \cup B \\ &\Leftrightarrow d\left(T^2x, Tx\right) \leq kd\left(x, Tx\right) + \frac{1-\alpha-\beta}{1-\alpha}\varphi\left(D\right) = kd\left(x, Tx\right) + (1-k)\,\varphi\left(D\right);\ \forall x \in A \cup B \end{aligned} \tag{2.7}$$

if $Tx \neq x$ where $k := \dfrac{\beta}{1-\alpha} < 1$, since $T:A \cup B \to A \cup B$ is cyclic, $d(x,Tx) \geq D$ and $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is increasing. Then

$$d\left(T^{n+1}x, T^n x\right) \leq k^n d\left(x, Tx\right) + \left(1-k^n\right)\varphi\left(D\right);\ \forall x \in A \cup B;\ \forall n \in \boldsymbol{N} \tag{2.8}$$

$\Phi(D) \geq D \neq 0$ since $min\ (\alpha, \beta) \geq 0$ and $\alpha + \beta < 1$ Proceeding recursively from (2.8), one gets for any $m \in \boldsymbol{N}$:

$$D \leq d\left(T^{n+1}x, T^n x\right) \leq k^n d\left(Tx, x\right) + \varphi\left(D\right)(1-k)\left(\sum_{i=0}^{n-1} k^i\right) \leq kd\left(Tx, x\right) + \varphi\left(D\right)\left(1-k^n\right) \tag{2.9a}$$

$$\leq kd(Tx,x)+(1-k)\varphi(D)\leq kd(Tx,x)+\varphi(D)<d(Tx,x)+\varphi(D);\ \forall x\in A\cup B \tag{2.9b}$$

$$\begin{aligned} D &\leq \limsup_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) \leq \lim_{n\to\infty}\left(k^{n+m}d(Tx,x)+\varphi(D)(1-k)\left(\sum_{i=0}^{n+m-1}k^i\right)\right)\\ &\leq \varphi(D)(1-k)\lim_{n\to\infty}\left(\frac{1-k^{n+m}}{1-k}\right)=\varphi(D);\ \forall x\in A\cup B \end{aligned} \tag{2.10}$$

$\Phi(D) \geq D \neq 0$ and if $\Phi(D) = D = 0$ then the $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = 0$; $\forall x \in A \cup B$. Hence, Property (i) follows from (2.9) and (2.10) since $\phi(D) \geq D$ and $d(x,Tx) \geq D$; $\forall x \in A \cup B$, since $T{:}A\cup B \to A \cup B$ is a 2-cyclic self-mapping and $\phi{:}\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing. Now, it follows from triangle inequality for distances and (2.9a) that:

$$\begin{aligned} d(T^n x,x) &\leq \sum_{i=1}^{n-1} d\left(T^{i+1}x,T^ix\right) \leq \left(\sum_{i=1}^{n-1}k^i\right)d(x,Tx)+\varphi(D)\left(\sum_{i=1}^{n-1}\left(1-k^i\right)\right)\\ &\leq \frac{k\left(1-k^{n-1}\right)}{1-k}d(x,Tx)+\varphi(D)\left(\sum_{i=1}^{n-1}(1-k)\right)^i \leq \frac{k\left(1-k^{n-1}\right)}{1-k}d(x,Tx)+\frac{(1-k)\left(1-(1-k)^{n-1}\right)}{k}\varphi(D)\\ &\leq \frac{k}{1-k}d(x,Tx)+\frac{1-k}{k}\varphi(D)<\infty,\ \forall x\in A\cup B,\ \forall n\in \boldsymbol{N} \end{aligned} \tag{2.11}$$

which leads directly to Property (ii) with $\{T^n x\}_{n\in \boldsymbol{N}_0}$ and $\left\{T^{n+1}x\right\}_{n\in \boldsymbol{N}_0}$ being bounded sequences for any finite $x \in A \cup B$. □

Concerning the case that A and B intersect, we have the following existence and uniqueness result of fixed points:

Theorem 2.2. If $\phi(D) = D = 0$ (i.e., $A^0 \cap B^0 \neq \varnothing$) then $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = 0$ and $d(T^n x, x) \leq \dfrac{kd(x,Tx)}{1-k}$; $\forall x \in A\cup B$. Furthermore, if (X,d) is complete and A and B are non-empty closed and convex then there is a unique fixed point $z \in A\cap B$ of $T{:}A\cup B \to A\cup B$ to which all the sequences $\{T^n x\}_{n\in \boldsymbol{N}_0}$, which are Cauchy sequences, converge; $\forall x \in A\cup B$.

Proof: It follows from Lemma 2.1(i)-(ii) for $\phi(D) = D = 0$ It also follows that $\lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = \lim_{n\to\infty}(k^n)\, d\left(T^{m+2}x, T^{m+1}x\right) = 0$; $\forall x \in A\cup B$, $\forall m \in \boldsymbol{N}_0$ what implies $\lim_{n,m\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = 0$ so that $\{T^n x\}_{n\in \boldsymbol{N}_0}$ is a Cauchy sequence, $\forall x \in A\cup B$, then being bounded and also convergent in $A\cap B$ as $n \to \infty$ since (X,d) is complete and A and B are non-empty, closed, and convex. Thus, $\lim_{n\to\infty} T^n x = z \in A\cap B$ and $z = \lim_{n\to\infty} T^{n+1}x = T\left(\lim_{n\to\infty} T^{n+1}x\right) = Tz$, since the iterate composed self-mapping $T^n{:}A\cup B \to A\cup B$, $\forall n \in \boldsymbol{N}_0$ is continuous for any initial point $x \in A\cup B$ (since it is contractive, then Lipschitz continuous in view of (2.9a) with associate Lipschitz constant $0 \leq k < 1$ for $\phi(D) = D = 0$). Thus, $z \in A\cap B$ is a fixed point of $T{:}A\cup B \to A\cup B$. Its uniqueness is proven by contradiction. Assume that there are two distinct fixed points z and y of $T{:}A\cup B \to A\cup B$ in $A\cap B$. Then, one gets from (2.1) that either $0 < d(Tz,Ty) \leq \beta(d(z,y)-\phi(d(z,y))) \leq \beta d(z,y) < d(z,y)$ or $d(Tz,Ty) = d(z,y) = 0$ what contradicts $d(z,y) > 0$ since $z \neq y$. Then, $d(Tz,Ty) \leq \beta(d(x,y) \leq \beta d(x,y) < d(z,y)$ what leads to the contradiction $\lim_{n\to\infty} d\left(T^n z, T^n y\right) = 0 = d\left(z,y\right) > 0$. Thus, $z = y$. Hence, the theorem. □

Now, the contractive condition (2.1) is modified as follows:

$$d(Tx,Ty) \leq \alpha_0\left[\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)} - \varphi\left(\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)}\right)\right] + \beta_0\left(d(x,y) - \varphi\left(d(x,y)\right)\right) + \varphi(D) \tag{2.12}$$

for $x,y(\neq x) \in X$, where $min\,(\alpha_0, \beta_0) \geq 0$, $min\,(\alpha_0, \beta_0) > 0$, and $\alpha_0 + \beta_0 \leq 1$. Note that in the former contractive condition (2.1), $\alpha + \beta < 1$. Thus, for any non-negative real constants $\alpha \leq \alpha_0$ and $\beta \leq \beta_0$, (2.12) can be rewritten as

$$\begin{aligned} d\left(Tx, Ty\right) \leq \alpha \left[\frac{d\left(x, Tx\right) d\left(y, Ty\right)}{d\left(x, y\right)} - \varphi\left(\frac{d\left(x, Tx\right) d\left(y, Ty\right)}{d\left(x, y\right)}\right)\right] + \beta\left(d\left(x, y\right) - \varphi\left(d\left(x, y\right)\right)\right) + \varphi\left(D\right) \\ + \left(\alpha_0 - \alpha\right)\left[\frac{d\left(x, Tx\right) d\left(y, Ty\right)}{d\left(x, y\right)} - \varphi\left(\frac{d\left(x, Tx\right) d\left(y, Ty\right)}{d\left(x, y\right)}\right)\right] + \left(\beta_0 - \beta\right)\left(d\left(x, y\right) - \varphi\left(d\left(x, y\right)\right)\right); \ \forall x, y \in A \cup B. \end{aligned} \tag{2.13}$$

The following two results extend Lemma 2.1 and Theorem 2.2 by using constants α_0 and β_0 in (2.1) whose sum can equalize unity $\alpha_0+\beta_0 = 1$.

Lemma 2.3. Assume that $T{:}A\cup B \to A\cup B$ is a cyclic self-map satisfying the contractive condition (2.13) with $min(\alpha_0, \beta_0) \geq 0$, $\alpha_0 + \beta_0 \leq 1$, and $\phi{:}\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing. Assume also that

$$\varphi\left(d\left(Tx, x\right)\right) \geq d\left(Tx, x\right) - \frac{1-\alpha}{1-\alpha-\beta} M_0; \ \forall x \in A \cup B \tag{2.14}$$

For some non-negative real constants $M_0 \leq \dfrac{1-\alpha-\beta}{1-\alpha} D$, $\alpha \leq \alpha_0$ and $\beta \leq \beta_0$ with $\alpha + \beta < 1$. Then, the following properties hold:

(i) $$D \leq \limsup_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) \leq \varphi\left(D\right) + \left(\alpha_0 + \beta_0 - \alpha - \beta\right) D; \ \forall x \in A \cup B, \ \forall m \in \boldsymbol{N}_0 \tag{2.15}$$

for any arbitrarily small $\varepsilon \in \boldsymbol{R}_+$.

(ii) If $\phi(D) = (1+\alpha+\beta-\alpha_0-\beta_0)D$ then $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = D$; $\forall x \in A\cup B$, $\forall m \in \boldsymbol{N}_0$.

(iii) If $d(x,Tx)$ is finite and, in particular, if x and Tx are finite then the sequence $\{T^n x\}_{n\in \boldsymbol{N}_0}$ and $\left\{T^{n+1}x\right\}_{n\in \boldsymbol{N}_0}$ are bounded sequences, where $T^n x \in A$ and $T^{n+1}x \in B$ if $x \in A$ and n is even and $T^n x \in B$ and $T^{n+1}x \in B$ if $x \in A$ and n is even.

Proof: Since $\phi{:}\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing then $\phi(x) \geq \phi(D)$ for $x(\in \boldsymbol{R}_{0+}) \geq D$. Note also that $M_0 \leq \dfrac{1-\alpha-\beta}{1-\alpha} D$ implies the necessary condition $\phi(d(Tx,x)) \geq 0$ and (2.14) implies that $0 \leq \phi(D) \leq D$. Note also for $y = Tx$ and $Ty = Tx^2$ and (2.14), since $\phi(x) > \phi(D)$ for $x > D$, that for $x \in A\cup B$, one gets from (2.14):

$$\varphi\left(d\left(T^2x, Tx\right)\right) \geq d\left(T^2x, Tx\right) - \frac{1-\alpha}{1-\alpha-\beta} M_0; \ \forall x \in A \cup B \tag{2.16}$$

leading from (2.14) to

$$\begin{aligned} &\left(\alpha_0 - \alpha\right)\left[d\left(T^2x, Tx\right) - \phi\left(d\left(T^2x, Tx\right)\right)\right] + \left(\beta_0 - \beta\right)\left(d\left(Tx, x\right) - \phi\left(d\left(Tx, x\right)\right)\right) \\ &\leq M := \left(\alpha_0 + \beta_0 - \alpha - \beta\right) \frac{1-\alpha}{1-\alpha-\beta} M_0 \end{aligned} \tag{2.17}$$

and $M \leq (\alpha_0 + \beta_0-\alpha-\beta)D$ since $M_0 \leq \dfrac{1-\alpha-\beta}{1-\alpha} D$. One gets from (2.13) and (2.17) the following modifications of (2.9) and (2.10) by taking $y = Tx$, $Ty = T^2x$, and successive iterates by composition of the self-mapping $T{:}A \cup B \to A \cup B$:

$$
\begin{aligned}
&D \le d\left(T^{n+1}x, T^n x\right) \le k^n d\left(Tx, x\right) + \left(\varphi\left(D\right) + M\right)\left(1-k\right)\left(\sum_{i=0}^{n-1} k^i\right) \le k^n d\left(Tx, x\right) + \left(1-k^n\right)\left(\varphi\left(D\right) + M\right) \\
&\le kd\left(Tx, x\right) + \varphi\left(D\right) + M;\ \forall x \in A \cup B,\ \forall n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}
\end{aligned} \tag{2.18}
$$

$$
D \le \limsup_{n\to\infty} d\left(T^{n+1}x, T^n x\right) \le \varphi\left(D\right) + M \le \varphi\left(D\right) + \left(\alpha_0 + \beta_0 - \alpha - \beta\right) D; \qquad \forall x \in A \cup B,\ \forall m \in \mathbf{N}_0 \tag{2.19}
$$

$$
\begin{aligned}
&D \le \limsup_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) \\
&\le \lim_{n\to\infty}\left(k^{n+m} d\left(Tx, x\right) + \left(\varphi\left(D\right) + \left(\alpha_0 + \beta_0 - \alpha - \beta\right) D\right)\left(1-k\right)\left(\sum_{i=0}^{n+m-1} k^i\right)\right) \\
&\le \varphi\left(D\right) + \left(\alpha_0 + \beta_0 - \alpha - \beta\right) D;\ \forall x \in A \cup B,\ \forall m \in \mathbf{N}_0
\end{aligned} \tag{2.20}
$$

and Property (i) has been proven. Property (ii) follows from (2.20) directly by replacing $\phi(D) = (1+\alpha+\beta-\alpha_0-\beta_0)D$ in (2.15). To prove Property (iii), note from (2.18) that

$$
\begin{aligned}
d\left(T^n x, x\right) &\le \sum_{i=1}^{n-1} d\left(T^{i+1}x, T^i x\right) \le \left(\sum_{i=1}^{n-1} k^i\right) d\left(x, Tx\right) + \left(\varphi\left(D\right) + M\right)\left(\sum_{i=1}^{n-1}\left(1-k^i\right)\right) \\
&\le \frac{k\left(1-k^{n-1}\right)}{1-k} d\left(x, Tx\right) + \left(\varphi\left(D\right) + M\right)\left(\sum_{i=1}^{n-1}\left(1-k\right)\right)^i \\
&\le \frac{k\left(1-k^{n-1}\right)}{1-k} d\left(x, Tx\right) + \frac{\left(1-k\right)\left(1-\left(1-k\right)^{n-1}\right)}{k}\left(\varphi\left(D\right) + M\right) \\
&\le \frac{k}{1-k} d\left(x, Tx\right) + \frac{1-k}{k}\left(\varphi\left(D\right) + M\right) < \infty;\ \forall x \in A \cup B,\ \forall n \in \mathbf{N}.
\end{aligned}
$$

Hence, $\{T^n x\}_{n\in \mathbf{N}_0}$ and $\left\{T^{n+1}x\right\}_{n\in \mathbf{N}_0}$ are bounded for any finite $x \in A\cup B$. Property (iii) has been proven. Hence, the lemma. □

Theorem 2.4. If $\phi(D) = D = 0$ then $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = 0;\ \forall x \in A\cup B$. Furthermore, if (X,d) is complete and both A and B are non-empty, closed, and convex then there is a unique fixed point $z \in A\cap B$ of $T{:}A \cup B \to A \cup B$ to which all the sequences $\{T^n x\}_{n\in \mathbf{N}_0}$, which are Cauchy sequences, converge; $\forall x\in A\cup B$.

Proof guideline: It is identical to that of Theorem 2.2 by using $\phi(D) = D = M_0\ M = 0$ and the fact that from (2.17) $\alpha_0 = \alpha$ and $\beta_0 = \beta$ with $0 \le \alpha + \beta < 1$ if there is a pair $(x,Tx) \in A \times B\cup B \times A$ such that $d(Tx,x) = \phi(d(Tx,x))$; $d(T^2x,Tx) = \phi(d(T^2x,Tx))$; $\forall x \in A\cup B$. Hence, the theorem. □

Remark 2.5. Note that Lemma 2.2 (ii) for $\phi(D) \le D$ ($\phi(D) < D$ if $\alpha + \beta \le \alpha_0 + \beta_0 \le 1$) leads to an identical result as Lemma 2.1 (i) for $\phi(D) = D$ and $\alpha + \beta < 1$ consisting in proving that $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = D$. This result is similar to a parallel obtained for standard 2-cyclic contractions [2,5,8]. □

Remark 2.6. Note from (2.7) that Lemma 2.1 is subject to the necessary condition $D \le \phi(D)$ since $d(T^2x,Tx) \ge D$ and; $\forall x \in A\cup B$. On the other hand, note from Lemma 2.2, Equation (2.14) that $\varphi\left(D\right) \ge D - \frac{1-\alpha}{1-\alpha-\beta}M_0$, and one also gets from (2.18) for $n = 1$ the dominant lower-bound $\varphi\left(D\right) \ge D - M \ge D - \frac{1-\alpha}{1-\alpha-\beta}M_0\left(\alpha_0 + \beta_0 - \alpha - \beta\right)$, that is, $D \le \varphi\left(D\right) + \frac{1-\alpha}{1-\alpha-\beta}M_0\left(\alpha_0 + \beta_0 - \alpha - \beta\right)$ which coincides with the parallel constraint obtained from Lemma 2.1 if $\alpha_0 + \beta_0 = \alpha + \beta$. □

Remark 2.7. Note that Lemmas 2.2 and 2.3 apply for non-decreasing functions $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$. The case of $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ being monotone increasing, then unbounded, is also included as it is the case of $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ being bounded non-decreasing. □

Now, modify the modified cyclic ϕ-contractive constraint (2.1) as follows:

$$d\left(Tx, Ty\right) \le \alpha \frac{d\left(x, Tx\right) d\left(y, Ty\right)}{d\left(x, y\right)} + \beta\left(d\left(x, y\right) - \varphi\left(d\left(x, y\right)\right)\right) + (1-\alpha)\varphi\left(\frac{d\left(x, Tx\right) d\left(y, Ty\right)}{d\left(x, y\right)}\right) + \varphi\left(D\right); \forall x \in A \cup B \tag{2.21}$$

Thus, the following parallel result to Lemmas 2.1 and 2.2 result holds under a more restrictive modified weak ϕ-contraction Assume that $T:A \cup B \to A \cup B$ is modified weak ϕ-contraction subject to $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ subject to the constraint $\limsup_{x\to+\infty}\left(x - \varphi\left(x\right)\right) > \frac{\varphi\left(D\right)}{1-\alpha-\beta}$ and having a finite limit:

Lemma 2.8. Assume that $T:A \cup B \to A \cup B$ is a cyclic self-map satisfying the contractive condition (2.21) with $\min\left(\alpha,\beta\right) \ge 0$, $\alpha + \beta < 1$, and $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing having a finite limit $\lim_{x\to\infty}\varphi\left(x\right) = \bar{\varphi}$ and subject to $\phi(0) = 0$ Assume also that $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ satisfies $\limsup_{x\to+\infty}\left(x - \varphi\left(x\right)\right) > \frac{\varphi\left(D\right)}{1-\alpha-\beta}$. Then, the following properties hold:

(i) The following relations are fulfilled:

$$\frac{1-\alpha-\beta}{2-\alpha-\beta}\varphi\left(D\right) \le D \le d\left(T^{n+1}x, T^n x\right) \le \frac{\varphi\left(D\right)}{1-\alpha-\beta} + \bar{\varphi} \le \frac{2-\alpha-\beta}{1-\alpha-\beta}\bar{\varphi} < \infty;\ \forall n \in \boldsymbol{N},\ \forall x \in A \cup B \tag{2.22}$$

$$\frac{1-\alpha-\beta}{2-\alpha-\beta}\varphi\left(D\right) \le D \le \limsup_{n\to\infty} d\left(T^{n+1}x, T^n x\right) \le \frac{\varphi\left(D\right)}{1-\alpha-\beta} + \bar{\varphi} \le \frac{2-\alpha-\beta}{1-\alpha-\beta}\bar{\varphi} < \infty;\ \forall x \in A \cup B \tag{2.23}$$

(ii) If, furthermore, $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is, in addition, sub-additive and $d(x,Tx)$ is finite (in particular, if x and Tx are finite) then the sequences $\left\{T^n x\right\}_{n\in \boldsymbol{N}_0}$ and $\left\{T^{n+1}x\right\}_{n\in \boldsymbol{N}_0}$ are both bounded, where $T^n x \in A$ and $T^{n+1}x \in B$ if $x \in A$ and n is even and $T^n x \in B$ and $T^{n+1}x \in A$ if $x \in B$ and n is even. If $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is identically zero then $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = 0;\ \forall x \in A \cup B$.

Proof: One gets directly from (2.21):

$$(1-\alpha)\left(d\left(T^2x, Tx\right) - \varphi\left(d\left(T^2x, Tx\right)\right)\right) \le \beta d\left(T^2x, Tx\right) - \varphi\left(d\left(T^2x, Tx\right)\right) + \varphi\left(D\right);\ \forall x \in A \cup B \tag{2.24}$$

or, equivalently, one gets for $k = \frac{\beta}{1-\alpha} < 1$ that

$$d\left(T^2x, Tx\right) - \varphi\left(d\left(T^2x, Tx\right)\right) \le kd\left(T^2x, Tx\right) - \varphi\left(d\left(T^2x, Tx\right)\right) + \frac{\varphi\left(D\right)}{1-\alpha};\ \forall x \in A \cup B \tag{2.25}$$

leading to

$$0 \le D - \varphi\left(D\right) \le \liminf_{n\to\infty}\left(d\left(T^{n+1}x, T^n x\right) - \varphi\left(d\left(T^{n+1}x, T^n x\right)\right)\right) \le \limsup_{n\to\infty}\left(d\left(T^{n+1}x, T^n x\right) - \varphi\left(d\left(T^{n+1}x, T^n x\right)\right)\right) \le \frac{\varphi\left(D\right)}{(1-\alpha)(1-k)} = \frac{\varphi\left(D\right)}{1-\alpha-\beta};\ \forall x \in A \cup B \tag{2.26}$$

what implies the necessary condition $\varphi(D) \geq \frac{1-\alpha-\beta}{2-\alpha-\beta}D$ leading to $\frac{D}{\varphi(D)} = \frac{2-\alpha-\beta}{1-\alpha-\beta} > 1$ if $D \neq 0$ and then $\liminf_{n\to\infty}\left(d\left(T^{n+1}x, T^n x\right) - \varphi\left(d\left(T^{n+1}x, T^n x\right)\right)\right) \geq D - \varphi(D) \geq 0$; $\forall x \in A \cup B$. Also, since $\limsup_{x\to+\infty}(x - \varphi(x)) > \frac{\varphi(D)}{1-\alpha-\beta}$; $\forall x \in \boldsymbol{R}_+$, by construction, then $d(T^{n+1}x, T^n x)$ is bounded; $\forall n \in \boldsymbol{N}$ since, otherwise, a contradiction to (2.24) holds. Since $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing and has a finite limit $\bar{\varphi} \geq \varphi(x) \geq 0$; $\forall x\ \boldsymbol{R}_{0+}$ ($\bar{\varphi} = 0$ if and only if $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is identically zero), thus $\bar{\varphi} \geq \varphi(D) \geq 0$. Then, (2.22)-(2.23) hold and Property (i) has been proven. On the other hand, one gets from (2.25), since $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is sub-additive and nondecreasing and has a finite limit, that:

$$\begin{aligned}
d\left(T^n x, x\right) - \varphi\left(d\left(T^n x, x\right)\right) &\leq \left(\sum_{i=1}^{n-1} d\left(T^{i+1}x, T^i x\right) - \varphi\left(d\left(T^{i+1}x, T^i x\right)\right)\right) \\
&\leq \left(\sum_{i=1}^{n-1} k^i\right)\left(d(x, Tx) - \varphi(d(x, Tx))\right) + \frac{\varphi(D)}{1-\alpha}\left(\sum_{i=1}^{n-1}\left(1-k^i\right)\right) \\
&\leq \frac{k\left(1-k^{n-1}\right)}{1-k}\left(d(x, Tx) - \varphi(d(x, Tx))\right) + \frac{\varphi(D)}{1-\alpha}\left(\sum_{i=1}^{n-1}(1-k)\right)^i \\
&\leq \frac{k\left(1-k^{n-1}\right)}{1-k} d(x, Tx) + \frac{(1-k)\left(1-(1-k)^{n-1}\right)}{k}\frac{\varphi(D)}{1-\alpha} \\
&\leq \frac{k}{1-k}\left(d(x, Tx) - \varphi(d(x, Tx))\right) + \frac{1-k}{k}\frac{\varphi(D)}{1-\alpha} < \infty;\ \forall x \in A \cup B,\ \forall n \in \boldsymbol{N}
\end{aligned} \tag{2.27}$$

$$\limsup_{n\to\infty} d\left(T^n x, x\right) \leq \frac{k}{1-k}\left(d(x, Tx) - \varphi(d(x, Tx))\right) + \frac{1-k}{k}\frac{\varphi(D)}{1-\alpha} + \bar{\varphi} < \infty;\ \forall x \in A \cup B \tag{2.28}$$

Then the sequences $\{T^n x\}_{n\in \boldsymbol{N}_0}$ and $\left\{T^{n+1}x\right\}_{n\in \boldsymbol{N}_0}$ are both bounded for any $x \in A \cup B$. Hence, the first part of Property (ii). If $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is identically zero then $\bar{\varphi} \equiv \varphi(x, Tx) = 0$; $\forall x \in A \cup B$ so that $\exists \lim_{n\to\infty} d\left(T^{n+m+1}x, T^{n+m}x\right) = 0$ from (2.23). Hence, the lemma. □

The existence and uniqueness of a fixed point in $A \cap B$ if A and B are non-empty, closed, and convex and (X,d) is complete follows in the subsequent result as its counterpart in Theorem 2.2 modified cyclic ϕ-contractive constraint (2.21):

Theorem 2.9. if (X,d) is complete and A and B intersect and are non-empty, closed, and convex then there is a unique fixed point $z \in A \cap B$ of $T:A \cup B \to A \cup B$ to which all the sequences $\{T^n x\}_{n\in \boldsymbol{N}_0}$, which are Cauchy sequences, converge; $\forall x \in A \cup B$. □

Remark 2.7. Note that the nondecreasing function $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ of the contractive condition (2.21) is not monotone increasing under Lemma 2.5 since it possesses a finite limit and it is then bounded. □

Remark 2.8. The case of $T:A \cup B \to A \cup B$ being a ϕ-contraction, namely, $d(Tx,Ty) \leq d(x,y)-\phi(d(x,y)) + \phi(D)$ with strictly increasing $\phi:\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$; $\forall x \in A \cup B$, [1,2] implies, since $\phi(x) = 0$ if and only if $x = 0$, implies the relation

$$d\left(Tx, Ty\right) \leq \beta_1 d\left(x, y\right) + \varphi(D) < d\left(x, y\right) + \varphi(D);\ \forall x, y\,(\neq x) \in A \cup B \tag{2.29}$$

for some real constant $0 \leq \beta_1 = \beta_1(x,y) < 1$; $\forall x,y(\neq x) \in A \cup B$ so that proceeding recursively:

$$d\left(T^{n+1}x, T^n x\right) \le \prod_{i=1}^{n} [\beta_i] d(Tx, x) + \varphi(D)\left(\sum_{j=1}^{n} \prod_{\ell=j+1}^{n} [\beta_\ell]\right) \le d(Tx, x) + L\varphi(D),\ \forall x \in A \cup B \quad (2.30)$$

$$D \le \limsup_{n\to\infty} d\left(T^{n+1}x, T^n x\right) \le \frac{\gamma\varphi(D)}{1-\bar{\beta}};\ \forall x \in A \cup B \quad (2.31)$$

where $\bar{\beta} := \lim_{n\to\infty} \left(\prod_{i=1}^{n} [\beta_i]\right)^{1/n} < 1$ and $\exists \lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$; $\forall x \in A \cup B$ if $\phi(D) = D = 0$, and one gets from Lemma 2.1(iii) that $\{T^n x\}_{n\in N_0}$ and $\{T^{n+1}x\}_{n\in N_0}$ are Cauchy sequences which converge to a unique fixed point in $A \cap B$ if A and B are non-empty, closed, and convex and (X,d) is complete [1]. □

Remark 2.9. Note that the constraint (2.1) implies in Lemma 2.1 and Theorem 2.2 that $(1-\alpha-\beta)\ \phi(D) \le (1-\alpha-\beta)D$ what implies $\phi(D) \le D$ if $max\ (\alpha,\beta) > 0$ since $0 \le \alpha + \beta < 1$. However, such a constraint in Lemma 2.3 and Theorem 3.4 implies that $(1-\alpha_0-\beta_0)\ \phi(D) \le (1-\alpha_0-\beta_0)D$. □

3. Properties for the case that *A* and *B* do not intersect

This section considers the contractive conditions (2.1) and (2.21) for the case $A \cap B \ne \emptyset$ For such a case, Lemmas 2.1, 2.3, and 2.8 still hold. However, Theorems 2.2, 2.4, and 2.9 do not further hold since fixed points in $A \cap B$ cannot exist. Thus, the investigation is centred in the existence of best proximity points. It has been proven in [1] that if T: $A \cup B \to A \cup B$ is a cyclic ϕ-contraction with A and B being weakly closed subsets of a reflexive Banach space $(X,\|\ \|)$ then, $\exists (x,y) \in A \times B$ such that $D = d(x,y) = \|x-y\|$ where d:$\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is a norm-induced metric, i.e., x and y are best proximity points. Also, if T:$A \cup B \to A \cup B$ is a cyclic contraction $\exists (x,y) \in A \times B$ such that $D = d(x,y)$ if A is compact and B is approximatively compact with respect to A with both A and B being subsets of a metric space (X,d) (i.e., if $\lim_{n\to\infty} d\left(T^{2n}x,\ y\right) = d\left(B, y\right) := \inf_{z\in B} d\left(z, y\right)$ for some $y \in A$ and $x \in B$ then the sequence $\{T^n x\}_{n\in N_0}$ has a convergent subsequence [14]). Theorem 2.2 extends via Lemma 2.1 as follows for the case when A and B do not intersect, in general:

Theorem 3.1. Assume that T:$A \cup B \to A \cup B$ is a modified weak ϕ-contraction, that is, a cyclic self-map satisfying the contractive condition (2.1) subject to the constraints $min\ (\alpha,\beta) \ge 0$ and $\alpha + \beta < 1$ with ϕ:$\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ being nondecreasing with $\phi(D) = D$. Assume also that A and B are non-empty closed and convex subsets of a uniformly convex Banach space $(X,\|\ \|)$. Then, there exist two unique best proximity points $z \in A$, $y \in B$ of T:$A \cup B \to A \cup B$ such that $Tz = y$, $Ty = z$ to which all the sequences generated by iterations of T:$A \cup B \to A \cup B$ converge for any $x \in A \cup B$ as follows. The sequences $\left\{T^{2n}x\right\}_{n\in N_0}$ and $\left\{T^{2n+1}x\right\}_{n\in N_0}$ converge to z and y for all $x \in A$, respectively, to y and z for all $x \in B$. If $A \cap B \ne \emptyset$ then z = y $\in A \cap B$ is the unique fixed point of T: $A \cup B \to A \cup B$

Proof: If $D = 0$, i.e., A and B intersect then this result reduces to Theorem 2.2 with the best proximity points being coincident and equal to the unique fixed point. Consider the case that A and B do not intersect, that is, $D > 0$ and take $x \in A \cup B$. Assume with no loss in generality that $x \in A$. It follows, since A and B are non-empty and closed, A is convex and Lemma 3.1 (i) that:

$$\left[d\left(T^{2n+1}x, T^{2n}x\right) \to D\, ;\ d\left(T^{2n+1}x, T^{2n+2}x\right) \to D\right] \Rightarrow d\left(T^{2(n+p)}x, T^{2n}x\right) \to 0 \text{ as } n \to \infty \tag{3.1}$$

(proven in Lemma 3.8 [14]). The same conclusion arises if $x \in B$ since B is convex. Thus, $\left\{T^{2n}x\right\}_{n\in \boldsymbol{N}_0}$ is bounded [Lemma 2.1 (ii)] and converges to some point $z = z(x)$, being potentially dependently on the initial point x, which is in A if $x \in A$, since A is closed, and in B if $x \in B$ since B is closed. Take with no loss in generality the norm-induced metric and consider the associate metric space (X,d) which can be identified with $(X, ||\ ||)$ in this context. It is now proven by contradiction that for every $\varepsilon \in \boldsymbol{R}_+$, there exists $n_0 \in \boldsymbol{N}_0$ such that $d(T^{2m}x,T^{2n+1}x) \le D + \varepsilon$ for all $m > n \ge n_0$. Assume the contrary, that is, given some $\varepsilon \in \boldsymbol{R}_+$, there exists $n_0 \in \boldsymbol{N}_0$ such that $d\left(T^{2m_k}x, T^{2n_k+1}x\right) > D + \varepsilon$ for all $m_k > n_k \ge n_0\ \forall k \in \boldsymbol{N}_0$. Then, by using the triangle inequality for distances:

$$D + \varepsilon < d\left(T^{2m_k}x, T^{2n_k+1}x\right) \le d\left(T^{2m_k}x, T^{2m_k+2}x\right) + d\left(T^{2m_k+2}x, T^{2n_k+1}x\right) \text{ as } n \to \infty \tag{3.2}$$

One gets from (3.1) and (3.2) that

$$\liminf_{k\to\infty}\left(d\left(T^{2m_k}x, T^{2m_k+2}x\right) + d\left(T^{2m_k+2}x, T^{2n_k+1}x\right)\right) = \liminf_{k\to\infty} d\left(T^{2m_k+2}x, T^{2n_k+1}x\right) > D+\varepsilon \tag{3.3}$$

Now, one gets from (3.1), (3.3), $\phi(D) \ge D$, and Lemma 2.1 (i) the following contradiction:

$$\begin{aligned} &D + \varepsilon < \limsup_{k\to\infty} d\left(T^{2m_k+2}x, T^{2n_k+1}x\right) \le \limsup_{n_k\to\infty} d\left(T^{2n_k+2}x, T^{2n_k+1}x\right) + \limsup_{k\to\infty} d\left(T^{2m_k+2}x, T^{2n_k+2}x\right) \\ &= \limsup_{n_k\to\infty} d\left(T^{2n_k+2}x, T^{2n_k+1}x\right) = D \end{aligned} \tag{3.4}$$

As a result, $d(T^{2m}x,T^{2n+1}x) \le D + \varepsilon$ for every given $\varepsilon \in \boldsymbol{R}_+$ and all $m > n \ge n_0$ for some existing $n_0 \in \boldsymbol{N}_0$. This leads by a choice of arbitrarily small ε to

$$D \le \limsup_{n\to\infty} d\left(T^{2m}x, T^{2n+1}x\right) \le D \Rightarrow \exists \lim_{n\to\infty} d\left(T^{2m}x, T^{2n+1}x\right) = D \tag{3.5}$$

But $\left\{T^{2n}x\right\}_{n\in \boldsymbol{N}_0}$ is a Cauchy sequence with a limit $z = T^2z$ in A (respectively, with a limit $y = T^2y$ in B) if $x \in A$ (respectively, if $x \in B$) such that $D = ||Tz\text{-}z|| = d(z,Tz)$ (Proposition 3.2 [14]). Assume on the contrary that $x \in A$ and $\left\{T^{2n}x\right\}_{n\in \boldsymbol{N}_0} \to z \ne T^2z$ as $n \to \infty$ so that $T^2z\text{-}Tz = z\text{-}Tz \ne z\text{-}y$ so that since A is convex and $(X,||\ ||)$ is uniformly convex Banach space, then strictly convex, one has

$$D = d(z, Tz) = d\left(\frac{T^2z + z}{2} - Tz\right) = \left\|\frac{T^2z - Tz}{2} + \frac{z - Tz}{2}\right\| \le \left\|\frac{T^2z - Tz}{2}\right\| + \left\|\frac{z - Tz}{2}\right\| < \frac{D}{2} + \frac{D}{2} = D \tag{3.6}$$

which is a contradiction so that $z = T^2z$ is a best approximation point in A of $T{:}A \cup B \to A \cup B$. In the same way, $\left\{T^{2n}x\right\}_{n\in \boldsymbol{N}_0}$ is a Cauchy sequence with a limit $T^2y = y \in B$ which is a best approximation point in B of $T{:}A \cup B \to A \cup B$ if $x \in B$ since B is convex and $(X,||\ ||)$ is strictly convex. We prove now that $y = Tz$. Assume, on the contrary that $y \ne Tz$ with $y = T^2y, Tz = T^3z \in B$, $z = T^2z \in A$, $d(z,y) > D$, $d(Tz,Ty) \ge D$, $d(Tz,z) = d(Ty,y) = D$, and $\phi(D) = D$. One gets from (2.1) since $\phi{:}\ \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing the following contradiction:

$$\begin{aligned} &D < d(z, y) = d\left(T^2z, T^2y\right) \le \alpha\left[\frac{d\left(T^2z, Tz\right) d\left(T^2y, Ty\right)}{d(Tz, Ty)} - \varphi\left(\frac{d\left(T^2z, Tz\right) d\left(T^2y, Ty\right)}{d(Tz, Ty)}\right)\right] \\ &+\beta\left(d(Tz, Ty) - \varphi\left(d(Tz, Ty)\right)\right) + D(\alpha + \beta) D + (1 - \alpha - \beta) D = D \end{aligned} \tag{3.7}$$

Thus, $z = Ty = T^2z = T^3y$ and $y = Tz = T^2y = T^3z$ are the best proximity points of T: $A \cup B \to A \cup B$ in A and B. Finally, we prove that the best proximity points $z \in A$ and $y \in B$ are unique. Assume that $z_1 (\neq z_2) \in A$ are two distinct best proximity points of T:$A \cup B \to A \cup B$ in A. Thus, $Tz_1 (\neq Tz_2) \in B$ are two distinct best proximity points in B. Otherwise, $Tz_1 = Tz_2 \Rightarrow T^2z_1 = T^2z_2 \Rightarrow z_1 = z_2$, since z_1 and z_1 are best proximity points, contradicts $z_1 \neq z_2$. One gets from Lemma 2.1(i) and $d(Tz_1,T^2z_2) = d(Tz_2,T^2z_1) = d(z_1,Tz_2) = d(z_2,Tz_1) = D$. Through a similar argument to that concluding with (3.6) with the convexity of A and the strict convexity of $(X,\|\ \|)$, guaranteed by its uniform convexity, one gets the contradiction:

$$\leq \left\| \frac{T^2z_1 - Tz_1}{2} \right\| + \left\| \frac{z_2 - Tz_2}{2} \right\| < \frac{D}{2} + \frac{D}{2} = D \tag{3.8}$$

since $T^2z_1\text{-}Tz_1 \neq Tz_1\text{-}z_1$. Thus, z_1 is the unique best proximity point in A while Tz_1 is the unique best proximity point in B. □

In a similar way, Theorem 2.4 extends via Lemma 2.3 as follows from the modification (2.12) of the contractive condition (2.1):

Theorem 3.2. Assume the following hypotheses:

(1) T:$A \cup B \to A \cup B$ is a modified weak ϕ-contraction, that is, a cyclic self-map satisfying the contractive condition (2.12) subject to the constraints *min* $(\alpha_0,\beta_0) \geq 0$, *min* $(\alpha_0,\beta_0) > 0$, and $\alpha_0 + \beta_0 \leq 1$.

(2) ϕ: $\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing subject to $\varphi\left(d\left(Tx,x\right)\right) \geq d\left(Tx,x\right) - \frac{1-\alpha}{1-\alpha-\beta}M_0$; $\forall x \in A \cup B$ and $\phi(D) = (1 + \alpha + \beta - \alpha_0 - \beta_0)D$ for some non-negative real constants $M_0 \leq \frac{1-\alpha-\beta}{1-\alpha}D$, $0 \leq \alpha \leq \alpha_0$ and $0 \leq \beta \leq \beta_0$ with $\alpha + \beta < 1$.

(3) A and B are non-empty closed and convex subsets of a uniformly convex Banach space $(X,\|\ \|)$.

Then, there exist two unique best proximity points $z \in A$, $y \in B$ of T:$A \cup B \to A \cup B$ such that $Tz = y$, $Ty = z$ to which all the sequences generated by iterations of T:$A \cup B \to A \cup B$ converge for any $x \in A \cup B$ as follows. The sequences $\left\{T^{2n}x\right\}_{n\in \boldsymbol{N}_0}$ and $\left\{T^{2n+1}x\right\}_{n\in \boldsymbol{N}_0}$ converge to z and y for all $x \in A$, respectively, to y and z for all $x \in B$ If $A \cap B \neq \emptyset$ then $z = y \in A \cap B$ is the unique fixed point of T:$A \cup B \to A \cup B$.

Outline of proof: It is similar to that of Theorem 3.1 since (3.1) to (3.3) still hold, (3.4) and (3.5) still hold as well from Lemma 2.3(ii) as well as the results from the contradictions (3.6)-(3.8). □

The following result may be proven using identical arguments to those used in the proof of Theorem 3.1 by using Lemma 2.8 starting with its proven convergence property (2.23) for distances:

Theorem 3.3. Assume that T:$A \cup B \to A \cup B$ is a cyclic self-map satisfying the contractive condition (2.21) with *min* $(\alpha,\beta) \geq 0$, $\alpha + \beta < 1$, and ϕ: $\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ is non-decreasing having a finite limit $\lim_{x\to\infty} \phi\left(x\right) = \bar{\phi}$ and subject to $\phi(0) = 0$ Assume also that ϕ: $\boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$ satisfies $\limsup_{x\to+\infty}\left(x - \varphi\left(x\right)\right) > \frac{\varphi\left(D\right)}{1-\alpha-\beta}$. Finally, assume that A and B are non-empty closed and convex subsets of a uniformly convex Banach space $(X,\|\ \|)$. Then, there exist two unique best proximity points $z \in A$, $y \in B$ of T:$A \cup B \to A \cup B$

such that $Tz = y$, $Ty = z$ to which all the sequences generated by iterations of $T:A \cup B \to A \cup B$ converge for any $x \in A \cup B$ as follows. The sequences $\left\{T^{2n}x\right\}_{n\in \mathbf{N}_0}$ and $\left\{T^{2n+1}x\right\}_{n\in \mathbf{N}_0}$ converge to z and y for all $x \in A$, respectively, to y and z for all $x \in B$. If $A \cap B \neq \emptyset$ then $z = y \in A \cap B$ is the unique fixed point of $T:A \cup B \to A \cup B$. □

Example 3.4. The first contractive condition (2.1) is equivalent to

$$d\left(Tx, T^2x\right) \leq \frac{1}{1-\alpha}\left(\beta d\left(x, Tx\right) + \varphi\left(D\right) - \alpha\varphi\left(d\left(Tx, T^2x\right)\right) - \beta\varphi\left(d\left(x, Tx\right)\right)\right). \quad (3.9)$$

To fix ideas, we first consider the trivial particular case $\phi(x) \equiv 0$ ($\Rightarrow \phi(D) = 0$); $\forall x \in \mathbf{R}_{0+}$. This figures out that $T:A \cup B \to A \cup B$ is a strict contraction if $A \cap B$ is non-empty and closed, $min\ (\alpha,\beta) \geq 0$, and $\alpha + \beta < 1$. Then, it is known from the contraction principle that there is a unique fixed point in $A \cap B$. Note that in this case ϕ: $\mathbf{R}_{0+} \to 0$. If $\alpha + \beta = 1$ then $T:A \cup B \to A \cup B$ is non-expansive fulfilling $d(T^{p+1}x, T^p x) = d(x, Tx)$; $\forall x \in A \cup B$, $\forall p \in \mathbf{z}_{0+}$. The convergence to fixed points cannot be proven. It is of interest to see if $T:A \cup B \to A \cup B$ being a weak contraction with ϕ: $\mathbf{R}_{0+} \to \mathbf{R}_{0+}$ being non-decreasing guarantees the convergence to a fixed point if $\alpha + \beta = 1$ and $\phi(0) = D = 0$ according to the modified contractive condition (2.12). In this case, if $\phi(x) > 0$; $\forall x \in \mathbf{R}_+$ then convergence to a fixed point is still potentially achievable since

$$d\left(Tx, T^2x\right) \leq d\left(x, Tx\right) - \frac{1}{1-\alpha}\left(\alpha\varphi\left(d\left(Tx, T^2x\right)\right) + \beta\varphi\left(d\left(x, Tx\right)\right)\right) < d\left(x, Tx\right) \text{ if } x \neq Tx. \quad (3.10)$$

Now, consider the discrete scalar dynamic difference equation of respective state and control real sequences $\{x_k\}_{k\in Z_{0+}}$ and $\{u_k\}_{k\in Z_{0+}}$ and dynamics and control parametrical real sequences $\{a_k\}_{k\in Z_{0+}}$ and $\{b_k \neq 0\}_{k\in Z_{0+}}$, respectively:

$$x_{k+1} = a_k x_k + b_k u_k + \eta_k;\ \forall k \in \mathbf{Z}_{0+},\ x_0 \in \mathbf{R} \quad (3.11)$$

where $\{\bar{x}_k\}_{k\in Z_{0+}}$, of general term defined by $\bar{x}_k := (x_0,\ x_1,\ \ldots, x_k)$, is a sequence of real kth tuples built with state values up till the kth sampled value such that the real sequence $\{\eta_k\}_{k\in Z_{0+}}$ with $\eta_k = \eta_k\ (\bar{x}_k)$ is related to non-perfectly modeled effects which can include, for instance, contributions of unmodeled dynamics (if the real order of the difference equation is larger than one), parametrical errors (for instance, the sequences of parameters are not exactly known), and external disturbances. It is assumed that upper- and lower-bounding real sequences $\{\bar{\eta}_k\}_{k\in Z_{0+}}$ and $\left\{\bar{\eta}_k^0\right\}_{k\in Z_{0+}}$ are known which satisfy $\bar{\eta}_k = \bar{\eta}_k\ (\bar{x}_k) \geq \eta_k \geq \bar{\eta}_k^0 = \bar{\eta}_k^0\ (\bar{x}_k)$; $\forall k \in \mathbf{z}_{0+}$. Define a 2-cyclic self-mapping $T:A \cup B \to A \cup B$ with $T(A) \subseteq B$ and $T(B) \subseteq A$ for some sets $A \subset \mathbf{R}_{0+} := \{z \in \mathbf{R}: z \geq 0\}$ and $B \subset \mathbf{R}_{0-} := \{z \in \mathbf{R}: z \leq 0\}$ being non-empty bounded connected sets containing $\{0\}$, so that $D = 0$, such that $Tx_k = x_{k+1}$; $\forall k \in \mathbf{z}_{0+}$ for the control sequence $\{u_k\}_{k\in Z_{0+}}$ lying in some appropriate class to be specified later on. Note from (3.11) that

$$\begin{aligned} x_{k+2} &= a_k x_{k+1} + b_{k+1}u_{k+1} + \eta_{k+1} \\ &= a_{k+1}a_k x_k + a_{k+1}b_k u_k + b_{k+1}u_{k+1} + a_{k+1}\eta_k + \eta_{k+1};\ \forall k \in \mathbf{Z}_{0+},\ x_0 \in A \cup B \end{aligned} \quad (3.12)$$

.

An equivalent expression to (3.9) if $\phi(D) = D = 0$ is by using the Euclidean distance:

$$\alpha\varphi\left(|x_{k+1}|+|x_{k+2}|\right)+\beta\varphi\left(|x_k|+|x_{k+1}|\right)\leq\beta\left(|x_k|+|x_{k+1}|\right)-(1-\alpha)\left(|x_{k+1}|+|x_{k+2}|\right);\ \forall k\in\mathbf{Z}_{0+}\quad(3.13)$$

.

Consider different cases as follows by assuming with no loss in generality that the parametrical sequences $\{a_k\}_{k\in\mathbf{Z}_{0+}}$ and $\{b_k\}_{k\in\mathbf{Z}_{0+}}$ are positive:

(a) $D = 0$ Then

$$x_{k+2}=a_{k+1}a_kx_k+a_{k+1}b_ku_k+b_{k+1}u_{k+1}+a_{k+1}\eta_k+\eta_{k+1};\ \forall k\in\mathbf{Z}_{0+},\ x_0\in A\cup B\quad(3.14)$$

Note that if $x_k \geq 0$ then $x_{k+1} \leq 0$ and $x_{k+2} \leq 0$ if

$$u_k\leq-\frac{a_kx_k+\bar{\eta}_k}{b_k}\leq 0;\, u_{k+1}\geq\frac{a_{k+1}\left(\bar{\eta}_k-a_kx_k-b_ku_k\right)+\bar{\eta}_{k+1}}{b_{k+1}};\ \forall k\in\mathbf{Z}_{0+}\quad(3.15)$$

If $x_k \leq 0$ then $x_{k+1} \geq 0$ and $x_{k+2}\leq 0$ if

$$u_k\geq\frac{\bar{\eta}_k-a_kx_k}{b_k};\, u_{k+1}\leq-\frac{a_{k+1}\left(\bar{\eta}_k+a_kx_k+b_ku_k\right)+\bar{\eta}_{k+1}}{b_{k+1}};\ \forall k\in\mathbf{Z}_{0+}\quad(3.16)$$

.

Thus, if $x_0 \geq 0$ then the control law is

$$u_{2k}\leq-\frac{a_{2k}x_{2k}+\bar{\eta}_{2k}}{b_{2k}}\leq 0;\, u_{2k+1}\geq\frac{a_{2k+1}\left(\bar{\eta}_{2k}-a_{2k}x_{2k}-b_{2k}u_{2k}\right)+\bar{\eta}_{2k+1}}{b_{2k+1}};\ \forall k\in\mathbf{Z}_{0+}\quad(3.17)$$

and if $x_0 < 0$ then

$$u_{2k}\geq\frac{\bar{\eta}_{2k}-a_{2k}x_{2k}}{b_{2k}};\, u_{2k+1}\leq-\frac{a_{2k+1}\left(\bar{\eta}_{2k}+a_{2k}x_{2k}+b_{2k}u_{2k}\right)+\bar{\eta}_{2k+1}}{b_{2k+1}};\ \forall k\in\mathbf{Z}_{0+}\quad(3.18)$$

The stabilization and convergence of the state sequence to zero is achieved by using a control sequence that makes compatible (3.16) and (3.17) with (3.13). First, assume $x_0 \leq 0$ and rewrite the controls (3.17) in equivalent equality form as:

$$u_{2k}=-\frac{a_{2k}x_{2k}+\bar{\eta}_{2k}+\varepsilon_{2k}}{b_{2k}};\, u_{2k+1}=\frac{a_{2k+1}\left(\bar{\eta}_{2k}-a_{2k}x_{2k}-b_{2k}u_{2k}\right)+\bar{\eta}_{2k+1}+\varepsilon_{2k+1}}{b_{2k+1}};\ \forall k\in\mathbf{Z}_{0+}\quad(3.19)$$

for any non-negative real sequence $\{\varepsilon_k\}_{k\in\mathbf{Z}_{0+}}$ to be defined so that (3.13) holds. Then (3.11) and (3.14) lead to:

$$\begin{aligned}&-\varepsilon_{2k}^0=\bar{\eta}_{2k}^0-\bar{\eta}_{2k}-\varepsilon_{2k}\leq x_{2k+1}=a_{2k}x_{2k}+b_{2k}u_{2k}+\eta_{2k}=\eta_{2k}-\bar{\eta}_{2k}-\varepsilon_{2k}\leq-\varepsilon_{2k}\leq 0\\&\Rightarrow\ \varepsilon_{2k}^0\geq|x_{2k+1}|\geq\varepsilon_{2k};\ \forall k\in\mathbf{Z}_{0+}\end{aligned}\quad(3.20)$$

$$\begin{aligned}&\varepsilon_{2k+1}^0=2\left(a_{2k+1}\bar{\eta}_{2k}+\bar{\eta}_{2k+1}\right)+\varepsilon_{2k+1}\\&\geq x_{2k+2}=a_{2k+1}a_{2k}x_{2k}+a_{2k+1}b_{2k}u_{2k}+b_{2k+1}u_{2k+1}+a_{2k+1}\eta_{2k}+\eta_{2k+1}\\&=a_{2k+1}\left(\bar{\eta}_{2k}+\eta_{2k}\right)+\bar{\eta}_{2k+1}+\eta_{2k+1}+\varepsilon_{2k+1}\geq\varepsilon_{2k+1}\geq 0\\&\Rightarrow\ \varepsilon_{2k+1}^0\geq|x_{2k+2}|\geq\varepsilon_{2k+1};\ \forall k\in\mathbf{Z}_{0+}\end{aligned}\quad(3.21)$$

for the given controls (3.19). Then, (3.13) becomes for $x_0 \in A$:

$$\alpha\varphi\left(|x_{2k+1}|+|x_{2k+2}|\right)+\beta\varphi\left(|x_{2k}|+|x_{2k+1}|\right)+(1-\alpha)\left(|x_{2k+1}|+|x_{2k+2}|\right)\leq\beta\left(|x_{2k}|+|x_{2k+1}|\right);\ \forall k\in\mathbf{Z}_{0+}\quad(3.22)$$

which is guaranteed from (3.20) and (3.21), without a need for directly testing the solution of the difference equation, if the sequence $\{u_k\}_{k\in\mathbf{Z}_{0+}}$ can be chosen to have zero limit while satisfying:

$$\alpha\varphi\left(\varepsilon_{2k}^{0}+\varepsilon_{2k+1}^{0}\right)+\beta\varphi\left(\varepsilon_{2k-1}^{0}+\varepsilon_{2k}^{0}\right)+(1-\alpha)\left(\varepsilon_{2k}^{0}+\varepsilon_{2k+1}^{0}\right)\leq\beta\left(\varepsilon_{2k-1}+\varepsilon_{2k}\right);\ \forall k\in\boldsymbol{Z}_{0+}\quad(3.23)$$

for some upper-bounding sequence $\left\{\varepsilon_k^0\right\}_{k\in\boldsymbol{Z}_{0+}}$ satisfying (3.20) and (3.21) and some given non-decreasing function $\phi:\boldsymbol{R}_{0+}\to\boldsymbol{R}_{0+}$. This implies that $x_k\to 0$ as $k\to\infty$, which is the unique fixed point of $T{:}A\cup B\to A\cup B$, by using the proposed control law (3.19). Note the following:

(1) Even, although $\{\varepsilon_k\}_{k\in\boldsymbol{Z}_{0+}}$ converges to zero, it is not required for the contribution of the non-perfectly modeled part of the model to converge to zero. It can suffice, for instance, $\eta_{2k}\to\bar{\eta}_{2k}$; $\eta_{2k+1}\to-\left(\bar{\eta}_{2k+1}+2a_{2k+1}\bar{\eta}_{2k}\right)$ as $k\to\infty$ It is not necessary that $\{\eta_k\}_{k\in\boldsymbol{Z}_{0+}}$ be convergent fulfilling $|\bar{\eta}_k|\to|\eta_k|\to\hat{\eta}<\infty$ as $k\to\infty$ for some non-negative real $\hat{\eta}=\hat{\eta}\left(\bar{x}_k\right)$. However, there are particular cases in this framework as, for instance, $|\bar{\eta}_k|\to|\eta_k|\to 0$ as $k\to\infty$ or $\eta_k\to\hat{\eta}>0$; $a_{2k+1}\to 1$ as $k\to\infty$.

(2) The constraints (3.23) imply $\phi(x)=0$ for $x\in[0,x_0]$ and some $x_0\in\boldsymbol{R}_{0+}$ but not that $\phi:\boldsymbol{R}_{0+}\to\boldsymbol{R}_{0+}$ is strictly increasing or that $\phi(x)=0$ if and only if $x=0$.

If $x_0\in B$, then $x_0<0$, take $u_0\geq\dfrac{a_0|x_0|+\hat{\eta}_0}{b_0}$ leading to $x_1\geq 0$ The above stabilization/convergence condition (3.23) still holds with the replacement $k\to k$-1 for any $k\in\boldsymbol{z}_+$.

(b) Now, assume that $D>0$, $A{:=}\{z\in\boldsymbol{R}_+:z\geq D/2\}$, and $B{:=}\{z\in\boldsymbol{R}_+:z\leq -D/2\}$ are bounded subsets of $\boldsymbol{R}$ and reconsider the above Case b modified so that $T{:}A\cup B\to A\cup B$ the sequence $\{\varepsilon_k\}_{k\in\boldsymbol{Z}_{0+}}$ is subject to $\varepsilon_{2k}\geq D/2$, $\varepsilon_{2k}\to D/2$ as $k\to\infty$ and $\phi(D)=D=dist(A,B)$ Also, the stabilization constraints (3.22) and (3.23) become modified as follows:

$$\begin{aligned}&\alpha\varphi\left(|x_{2k+1}|+|x_{2k+2}|\right)+\beta\varphi\left(|x_{2k}|+|x_{2k+1}|\right)+(1-\alpha)\left(|x_{2k+1}|+|x_{2k+2}|\right)\\&\leq D+\beta\left(|x_{2k}|+|x_{2k+1}|\right);\forall k\in Z_{0+}\end{aligned}\quad(3.24)$$

$$\alpha\varphi\left(\varepsilon_{2k}^{0}+\varepsilon_{2k+1}^{0}\right)+\beta\varphi\left(\varepsilon_{2k-1}^{0}+\varepsilon_{2k}^{0}\right)+(1-\alpha)\left(\varepsilon_{2k}^{0}+\varepsilon_{2k+1}^{0}\right)\leq D+\beta\left(\varepsilon_{2k-1}+\varepsilon_{2k}\right),\ \forall k\in\boldsymbol{Z}_{0+}\quad(3.25)$$

the second one being a sufficient condition for the first one to hold. Note that x_{2k} and x_{2k+1} both converge to best proximity points as $k\to\infty$ If $x_0\geq D/2$ then $x_{2k}\to D/2$ and $x_{2k+1}\to D/2$ as $k\to\infty$ and if $x_0\leq -D/2$ then $x_{2k}\to -D/2$ and $x_{2k+1}\to D/2$. Note that Case a is a particular version of Case b for $D=0$.

(c) The conditions (3.23) and (3.25) can be generalized to the nonlinear potentially non-perfectly modeled difference equation:

$$x_{k+1}=a_k g\left(x_k\right)+b_k u_k+\eta_k;\ \forall k\in\boldsymbol{Z}_{0+},\ x_0\in\boldsymbol{R}^n\quad(3.26)$$

for some function $g:\boldsymbol{R}\to\boldsymbol{R}$ leading to the nonlinear real sequence $\left\{g_k=g\left(x_k\right)\right\}_{k\in\boldsymbol{Z}_{0+}}$. Proceed by replacing the controls (3.19) by their counterparts obtained correspondingly with right-hand side replacements $x_k\to g_k=g(x_k)$ by choosing the sequence $\{\varepsilon_k\}_{k\in\boldsymbol{Z}_{0+}}$ with $\phi:[D,\infty)\to[D,\infty)$ satisfying $\phi(x)=D$ for $x\in[D,D+x_0]$ and some $x_0\in\boldsymbol{R}_{0+}$ so that (3.25) holds.

(d) Consider the nth-order nonlinear dynamic system:

$$x_{k+1}=A_k x_k+B_k u_k;\ \forall k\in\boldsymbol{Z}_{0+},\ x_0\in\boldsymbol{R}^n\quad(3.27)$$

for some matrix function sequences sampling point-wise defined by $A_k = A_k(x_k)$ and $B_k = B_k(x_k)$ of images in $\boldsymbol{R}^{n\times n}$ and $\boldsymbol{R}^{n\times m}$, respectively; $\forall k \in \boldsymbol{Z}_{0+}$. Proceeding recursively with (3.27) over n consecutive samples, one gets

$$x_{(k+1)n} = \varphi_k x_{kn} + \Gamma_k \bar{u}_{kn};\ \forall k \in \boldsymbol{Z}_{0+},\ x_0 \in \boldsymbol{R}^n \tag{3.28}$$

with $\Phi_{\mathrm{k}} = \Phi_{\mathrm{k}}(x_{kn})$ and $\Gamma_k = \Gamma_k(x_{kn})$ as:

$$\varphi_k := \prod_{i=kn}^{(k+1)n-1} [A_i];\ \Gamma_k := \left[B_{(k+1)n-1} \vdots A_{(k+1)n-1} B_{(k+1)n-2} \vdots \cdots \vdots \prod_{j=kn+1}^{(k+1)n-1} \left[A_j\right] B_{kn} \right] \tag{3.29}$$

with the extended nm control real vector sequence over n consecutive samples being defined by $\overline{u_{kn}} = \overline{u_{kn}}\,(x_k) := \left(u_{(k+1)n-1}, u_{(k+1)n-2}, \ldots, u_{kn}\right)^T$. Consider solutions of (3.28) lying alternately in a non-empty closed bounded connected subset A of the first closed orthant of $\boldsymbol{R}^n$and in $B = -A$ for each couple of subsequent samples for some extended control sequence $\{u_k\}_{k\in Z_{0+}}$ in $\boldsymbol{R}^m$, for some integer $1 \le m \le n$, assumed to exist. A unique such a control sequence exist, if for instance, the controllability condition *rank* $\Gamma_k = n$; $\forall k \in \boldsymbol{z}_{0+}$ holds for each matrix sequence $\Lambda_k = \Lambda_k(x_k)$ by achieving:

$$x_{(k+1)n} = \varphi_k x_{kn} + \Gamma_k \bar{u}_{kn} = -\Lambda_k x_{kn};\ \forall k \in \boldsymbol{Z}_{0+},\ x_0 \in A \tag{3.30}$$

with $\Lambda_k = \Lambda_k(x_{kn})$; $\forall k \in \boldsymbol{z}_{0+}$ defining some prefixed positive real matrix sequence taking values in $\boldsymbol{R}^{n\times \mathrm{n}}$with at least a non-zero entry per row. The closed-loop control objective (3.30) is achievable by the feedback control sequence:

$$\bar{u}_{kn} = -\Gamma_k^T \left(\Gamma_k \Gamma_k^T\right) (\varphi_k + \Lambda_k)\, x_{kn},\ \forall k \in \boldsymbol{Z}_{0+}; x_0 \in A \tag{3.31}$$

Thus, a modified constraint of the type (3.22), or (3.23), ensures that the solution of (3.28), subject to the extended control (3.31), lies alternately in A and B for each two consecutive samples for $x_0 \in A$ and converges to zero, while a modification of (3.24), or (3.25), ensures that the solution lies alternately in B and A and converges to zero, provided that $\Lambda_k x_0 \in A \cup (-A)$; $\forall x_0 \in A \cup (-A)$, $\forall k \in \boldsymbol{z}_{0+}$, i.e., $A \cup (-A)$ is Λ_k-invariant, $\forall k \in \boldsymbol{z}_{0+}$. Furthermore, A and $\underline{\mathrm{B}}$ are both Λ_{2k}-invariant. Such a modifications are got directly by replacing $x(\cdot) \to \Lambda(\cdot)x(\cdot)$, $\varepsilon(\cdot) \to \Lambda(\cdot)\varepsilon(\cdot)$ Note that the constraints (3.22), (3.23), (3.24), and (3.25) now become n-vector constraints. The Euclidean distances are now replaced by any Minkowski distance of order p (p-norm-induced distance for some real $p \ge 1$) in $\boldsymbol{R}^n$ as for instance, 1-norm-induced distance $d_1\left(x, y\right) = \sum_{i=1}^n \left|x_i - y_i\right|$, 2-norm-induced (i.e., Euclidean) distance $d_2\left(x, y\right) = \left(\sum_{i=1}^n \left|x_i - y_i\right|^2\right)^{1/2}$, p-norm-induced distance $d_p\left(x, y\right) = \left(\sum_{i=1}^n \left|x_i - y_i\right|^p\right)^{1/p}$, or infinity-norm-induced distance $d_\infty\left(x, y\right) = \lim_{p\to\infty} \left(\sum_{i=1}^n \left|x_i - y_i\right|^p\right)^{1/p} = \max_{1\le i\le n} \left(\left|x_i - y_i\right|\right)$.

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Authors' contributions
Both the authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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