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Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-φ-nonexpansive mappings

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Abstract

We consider a hybrid projection method for finding a common element in the fixed point set of an asymptotically quasi-φ-nonexpansive mapping and in the solution set of an equilibrium problem. Strong convergence theorems of common elements are established in a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property.

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1. Introduction and Preliminaries

Let E be a real Banach space, E^* the dual space of E and C a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

In this paper, we consider the following equilibrium problem. Find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote $EP(f)$ the solution set of the equilibrium problem (1.1). That is,

$$EP(f) = \{p \in C : f(p, y) \geq 0 \quad \forall y \in C\}.$$

Given a mapping $Q : C \to E^*$, let

$$f(x, y) = \langle Qx, y - x\rangle \quad \forall x, y \in C.$$

Then $p \in EP(f)$ if and only if p is a solution of the following variational inequality problem. Find p such that

$$\langle Qp, y - p\rangle \geq 0 \quad \forall y \in C. \tag{1.2}$$

Numerous problems in physics, optimization and economics reduce to find a solution of (1.1) (see [1-4]). Let $T : C \to C$ be a mapping.

The mapping T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty}\{||T^{n+1}x - T^n x|| : x \in K\} = 0.$$

The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that

$$\lim_{n\to\infty} x_n = x_0$$

and

$$\lim_{n\to\infty} Tx_n = y_0,$$

then $Tx_0 = y_0$.

A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we denote $F(T)$ the fixed point set of T and denote $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively.

Recall that the mapping T is said to be nonexpansive if

$$|Tx - Ty|| \le ||x - y|| \quad \forall x, y \in C.$$

T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||x - Ty|| \le ||x - y|| \quad \forall x \in F(T), \quad \forall y \in C.$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$||T^n x - T^n y|| \le k_n ||x - y|| \quad \forall x, y \in C, \quad \forall n \ge 1.$$

T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$||x - T^n y|| \le k_n ||x - y|| \quad \forall x \in F(T), \quad \forall y \in C, \forall n \ge 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972. They proved that if C is nonempty bounded closed and convex then every asymptotically nonexpansive self-mapping T on C has a fixed point in uniformly convex Banach spaces. Further, the fixed point set of T is closed and convex.

Recently, many authors considered the problem of finding a common element in the set of fixed points of a nonexpansive mapping and in the set of solutions of the equilibrium problem (1.1) based on iterative methods in the framework of real Hilbert spaces; see, for instance [4,6-14] and the references therein. However, there are few results presented in Banach spaces.

In this paper, we will consider the problem in a Banach space. Before proceeding further, we give some definitions and propositions in Banach spaces.

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle \cdot,\cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $||\frac{x+y}{2}|| < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} ||x_n - y_n|| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $||x_n|| = ||y_n|| = 1$ and

$$\lim_{n\to\infty}\left\|\frac{x_n+y_n}{2}\right\|=1.$$

Let $U_E = \{x \in E : ||x|| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{||x+ty||-||x||}{t} \tag{1.3}$$

exists for each $x, y \in U_E$. It is said to be uniformly smooth if the limit (1.3) is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec-Klee property [15-17], if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $||x_n - x|| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [18] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y)=||x||^2-2\langle x,Jy\rangle+||y||^2 \quad \text{for } x,y\in E. \tag{1.4}$$

Observe that, in a Hilbert space H, (1.4) is reduced to $\varphi(x, y) = ||x-y||^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\varphi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x)=\min_{y\in C}\phi(y,x).$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\varphi(x, y)$ and strict monotonicity of the mapping J (see, for example, [15,17-19]). We know that $\Pi_C = P_C$ in Hilbert spaces. It is obvious from the definition of function φ that

$$(||y||-||x||)^2\le\phi(y,x)\le(||y||+||x||)^2 \quad \forall x,y\in E. \tag{1.5}$$

Remark 1.1. Let E be a reflexive, strictly convex and smooth Banach space. Then for $x, y \in E$, $\varphi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\varphi(x, y) = 0$ then $x = y$. From (1.5), we have $||x|| = ||y||$. This implies that $\langle x, Jy\rangle = ||x||^2 = ||Jy||^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$ (see [15,17]).

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [20] if C contains a sequence $\{x_n\}$ which converges weakly to p such that

$$\lim_{n\to\infty}||x_n-Tx_n||=0.$$

The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping T from C into itself is said to be relatively nonexpansive [21-23] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x)$$

for all $x \in C$ and $p \in F(T)$.

The mapping T is said to be relatively asymptotically nonexpansive [24] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x)$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [21-23].

The mapping T is said to be φ-nonexpansive if

$$\phi(Tx, Ty) \leq \phi(x, y)$$

for all $x, y \in C$.

The mapping T is said to be quasi-φ-nonexpansive [25-27] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x)$$

for all $x \in C$ and $p \in F(T)$.

The mapping T is said to be asymptotically φ-nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \leq k_n \phi(x, y)$$

for all $x, y \in C$.

The mapping T is said to be asymptotically quasi-φ-nonexpansive [27,28] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x)$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.2. The class of (asymptotically) quasi-φ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$. In the framework of Hilbert spaces, (asymptotically) quasi-φ-nonexpansive mappings is reduced to (asymptotically) quasi-nonexpansive mappings (cf. [29-32]).

We assume that f satisfies the following conditions for studying the equilibrium problem (1.1).

(A1): $f(x, x) = 0 \forall x \in C$;
(A2): f is monotone, i.e., $f(x, y) + f(y, x) \leq 0 \forall x, y \in C$;
(A3): $\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y) \forall x, y, z \in C$;
(A4): for each $x \in C$, $y \alpha f(x, y)$ is convex and weakly lower semi-continuous.

Recently, Takahashi and Zembayshi [33] considered the problem of finding a common element in the fixed point set of a relatively nonexpansive mapping and in the solution set of the equilibrium problem (1.1) (cf. [32]).

Theorem TZ. ([33]) *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let f be a bifunction from C × C to* $\mathbb{R}$ *satisfying* (*A*1)-(*A*4) *and let T be a relatively nonexpansive mapping from C into itself such that* $F(T) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x \end{cases} \tag{1.6}$$

for every $n \geq 0$, *where J is the duality mapping on E*, $\{\alpha_n\} \subset [0, 1]$ *satisfies*

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$$

and $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap EP(f)}\, x$, *where* $\Pi_{F(T)\cap EP(f)}$ *is the generalized projection of E onto* $F(T) \cap EP(f)$.

Very recently, Qin et al. [25] further improved Theorem TZ by considering shrinking projection methods which were introduced in [34] for quasi-φ-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space.

Theorem QCK. [25]*Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Ban ach space E. Let f be a bifunction from C × C to* $\mathbb{R}$ *satisfying* (*A*1)-(*A*4) *and let* $T : C \to C$ *be a closed quasi-φ-nonexpansive mappings such that* $\mathcal{F} = F(T) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.7}$$

where J is the duality mapping on E and $\{\alpha_n\}$ *is a sequence in* [0, 1] *satisfying*

$$\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$$

and $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$.

In this paper, we considered the problem of finding a common element in the fixed point set of an asymptotically quasi-φ-nonexpansive mapping which is an another generalization of asymptotically nonexpansive mappings in Hilbert spaces and in the solution set of the equilibrium problem (1.1). The results presented this paper mainly improve the corresponding results announced in [33].

In order to prove our main results, we need the following lemmas.

Lemma 1.3. [18]*Let C be a nonempty closed convex subset of a smooth Banach space E and* $x \in E$. *Then* $x_0 = \Pi_C x$ *if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

Lemma 1.4. [18]*Let E be a reflexive, strictly convex and smooth Banach space, C a nonempty closed convex subset of E and* $x \in E$. *Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 1.5. *Let E be a strictly convex and smooth Banach space, C a nonempty closed convex subset of E and* $T : C \to C$ *a quasi-φ-nonexpansive mapping. Then* $F(T)$ *is a closed convex subset of C.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$. Then we have to prove that $p \in F(T)$ for the closedness of $F(T)$. From the definition of T, we have

$$\phi(p_n, Tp) \leq \phi(p_n, p),$$

which implies that $\varphi(p_n, Tp) \to 0$ as $n \to \infty$. Note that

$$\phi(p_n, Tp) = ||p_n||^2 - 2\langle p_n, J(Tp)\rangle + ||Tp||^2.$$

Letting $n \to \infty$ in the above equality, we see that $\varphi(p, Tp) = 0$. This shows that $p = Tp$.

Next, we show that $F(T)$ is convex. To end this, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0, 1)$, putting $p_3 = tp_1 + (1 - t)p_2$, we prove that $Tp_3 = p_3$. Indeed, from the definition of φ, we see that

$$\begin{aligned}
\phi(p_3, Tp_3) &= ||p_3||^2 - 2\langle p_3, J(Tp_3)\rangle + ||Tp_3||^2 \\
&= ||p_3||^2 - 2\langle tp_1 + (1-t)p_2, J(Tp_3)\rangle + ||Tp_3||^2 \\
&= ||p_3||^2 - 2t\langle p_1, J(Tp_3)\rangle - 2(1-t)\langle p_2, J(Tp_3)\rangle + ||Tp_3||^2 \\
&\leq ||p_3||^2 + t\phi(p_1, p_3) + (1-t)\phi(p_2, p_3) - t||p_1||^2 - (1-t)||p_2||^2 \\
&= ||p_3||^2 - 2\langle tp_1 + (1-t)p_2, Jp_3\rangle) - ||p_3||^2 \\
&= ||p_3||^2 - 2\langle p_3, Jp_3\rangle - ||p_3||^2 \\
&= 0.
\end{aligned}$$

This implies that $p_3 \in F(T)$. This completes the proof.

Now we will improve the above Lemma 1.6 as follows.

Lemma 1.6. *Let E be a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property, C a nonempty closed convex subset of E and* $T : C \to C$ *a closed and asymptotically quasi-φ-nonexpansive mapping. Then* $F(T)$ *is a closed convex subset of C.*

Proof. It is easy to check that the closedness of $F(T)$ can be deduced from the closedness of T. We mainly show that $F(T)$ is convex. To end this, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0, 1)$, putting $p_3 = tp_1 + (1 - t)p_2$, we prove that $Tp_3 = p_3$.

Indeed, from the definition of φ, we see that

$$\begin{aligned}
\phi(p_3, T^n p_3) &= ||p_3||^2 - 2\langle p_3, J(T^n p_3)\rangle + ||T^n p_3||^2 \\
&= ||p_3||^2 - 2\langle tp_1 + (1-t)p_2, J(T^n p_3)\rangle + ||T^n p_3||^2 \\
&= ||p_3||^2 - 2t\langle p_1, J(T^n p_3)\rangle - 2(1-t)\langle p_2, J(T^n p_3)\rangle + ||T^n p_3||^2 \\
&= ||p_3||^2 + t\phi(p_1, T^n p_3) + (1-t)\phi(p_2, T^n p_3) - t||p_1||^2 - (1-t)||p_2||^2 \\
&\leq ||p_3||^2 + k_n t\phi(p_1, p_3) + k_n(1-t)\phi(p_2, p_3) - t||p_1||^2 - (1-t)||p_2||^2 \\
&= (k_n - 1)\left(t||p_1||^2 + (1-t)||p_2||^2 - ||p_3||^2\right).
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(p_3, T^n p_3) = 0.$$

From (1.5), we see that

$$\lim_{n\to\infty} ||T^n p_3|| = ||p_3||. \tag{1.8}$$

It follows that

$$\lim_{n\to\infty} ||J(T^n p_3)|| = ||Jp_3||. \tag{1.9}$$

This shows that the sequence $\{J(T^n p_3)\}$is bounded. Note that E^* is reflexive; we may, without loss of generality, assume that $J(T^n p_3) \rightharpoonup e^* \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $e \in E$ such that $Je = e^*$. It follows that

$$\begin{aligned}\phi(p_3, T^n p_3) &= ||p_3||^2 - 2\langle p_3, J(T^n p_3)\rangle + ||T^n p_3||^2 \\ &= ||p_3||^2 - 2\langle p_3, J(T^n p_3)\rangle + ||JT^n p_3||^2\end{aligned}$$

Taking lim $\inf_{n\to\infty}$ on the both sides of above equality, we obtain that

$$\begin{aligned}0 &\geq ||p_3||^2 - 2\langle p_3, e^*\rangle + ||e^*||^2 \\ &= ||p_3||^2 - 2\langle p_3, Je\rangle + ||Je||^2 \\ &= ||p_3||^2 - 2\langle p_3, Je\rangle + ||e||^2 \\ &= \phi(p_3, e).\end{aligned}$$

This implies that $p_3 = e$, that is, $Jp_3 = e^*$. It follows that $J(T^n p_3) \rightharpoonup Jp_3 \in E^*$.

In view of the Kadec-Klee property of E^* and (1.9), we have

$$\lim_{n\to\infty} ||J(T^n p_3) - Jp_3|| = 0.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous, we see that $T^n p_3 \rightharpoonup p_3$. By virtue of the Kadec-Klee property of E and (1.8), we have $T^n p_3 \to p_3$ as $n \to \infty$. Hence

$$TT^n p_3 = T^{n+1} p_3 \to p_3$$

as $n \to \infty$. In view of the closedness of T, we can obtain that $p_3 \in F(T)$. This shows that $F(T)$ is convex. This completes of proof

Lemma 1.7. [35,36]*Let E be a smooth and uniformly convex Banach space and let r >0. Then there exists a strictly increasing, continuous and convex function* $g : [0, 2r] \to R$ *such that* $g(0) = 0$ *and*

$$||tx + (1-t)y||^2 \leq t||x||^2 + (1-t)||y||^2 - t(1-t)g(||x-y||)$$

for all $x, y \in B_r = \{x \in E : ||x|| \leq r\}$ *and* $t \in [0, 1]$.

Lemma 1.8. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $r > 0$ *and* $x \in E$. *Then we have the followings.*

(a): ([1]) *There exists* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

(b): *(Refs.* [25,33]*) Define a mapping* $T_r : E \to C$ *by*

$$S_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \quad \forall y \in C \right\}.$$

Then the following conclusions hold:

(1): S_r *is single-valued;*
(2): S_r *is a firmly nonexpansive-type mapping, i.e., for all* $x, y \in E$,

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \leq \langle S_r x - S_r y, Jx - Jy\rangle$$

(3): $F(S_r) = EP)(f)$;
(4): S_r *is quasi-φ-nonexpansive;*
(5): $\varphi(q, S_r x) + \varphi(S_r x, x) \leq \varphi(q, x)$, $\forall q \in F(S_r)$;
(6): $EP(f)$ *is closed and convex.*

2. Main Results

Theorem 2.1. *Let E be a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property and C a nonempty closed convex subset of E. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4) *and* $T : C \to C$ *a closed and asymptotically quasi-φ-nonexpansive mapping. Assume that T is asymptotically regular on C and* $\mathcal{F} = F(T) \cap EF(f)$*is nonempty and bounded. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \mathcal{F}\}$*for each* $n \geq 1$, $\{\alpha_n\}$ *is a real sequence in* $[0, 1]$ *such that* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\{r_n\}$ *is a real sequence in* $[a, \infty)$, *where a is some positive real number and J is the duality mapping on E. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$, *where* $\Pi_{\mathcal{F}}$*is the generalized projection from E onto* $\mathcal{F}$.

Proof. First, we show that C_n is closed and convex by induction on $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some integer m. For $z \in C_m$, we see that $\varphi(z, u_m) \leq \varphi(z, x_m) + (k_m - 1)M_m$ is equivalent to

$$2\langle z, Jx_m - Ju_m\rangle \leq ||x_m||^2 - ||u_m||^2 + (k_m - 1)M_m.$$

It is easy to see that C_{m+1} is closed and convex. This proves that C_n is closed and convex for each $n \geq 1$. This in turn shows that $\prod_{C_{n+1}} x_0$ is well defined. Putting $u_n = S_{r_n} y_n$, we from Lemma 1.8 see that S_{r_n} is quasi-φ-nonexpansive.

Now, we are in a position to prove that $\mathcal{F} \subset C_n$ for each $n \geq 1$. Indeed, $\mathcal{F} \subset C_1 = C$ is obvious. Suppose that $\mathcal{F} \subset C_m$ for some positive integer m. Then, $\forall w \in \mathcal{F} \subset C_m$, we have

$$\begin{aligned}
\phi(w, u_m)+ &= \phi(w, S_{r_m}y_m) \\
&\le \phi(w, y_m) \\
&= \phi(w, J^{-1}[\alpha_m Jx_m + (1-\alpha_m)JT^m x_m]) \\
&= ||w||^2 - 2\langle w, \alpha_m Jx_m + (1-\alpha_m)JT^m x_m\rangle \\
&\quad + ||\alpha_m Jx_m + (1-\alpha_m)JT^m x_m||^2 \\
&\le ||w||^2 - 2\alpha_m\langle w, Jx_m\rangle - 2(1-\alpha_m)\langle w, JT^m x_m\rangle + \alpha_m||x_m||^2 \\
&\quad + (1-\alpha_m)||T^m x_m||^2 \\
&= \alpha_m\phi(w, x_m) + (1-\alpha_m)\phi(w, T^m x_m) \\
&\le \alpha_m\phi(w, x_m) + (1-\alpha_m)k_m\phi(w, x_m) \\
&= \phi(w, x_m) + (1-\alpha_m)(k_m-1)\phi(w, x_m) \\
&\le \phi(w, x_m) + (k_m-1)M_m,
\end{aligned} \tag{2.1}$$

which shows that $w \in C_{m+1}$. This implies that $\mathcal{F} \subset C_n$ for each $n \ge 1$.

On the other hand, it follows from Lemma 1.4 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \le \phi(w, x_0) - \phi(w, x_n) \le \phi(w, x_0),$$

for each $w \in \mathcal{F} \subset C_n$ and for each $n \ge 1$. This shows that the sequence $\varphi(x_n, x_0)$ is bounded. From (1.5), we see that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup p$. Not that C_n is closed and convex for each $n \ge 1$. It is easy to see that $p \in C_n$ for each $n \ge 1$. Note that

$$\phi(x_n, x_0) \le \phi(p, x_0).$$

It follows that

$$\phi(p, x_0) \le \liminf_{n\to\infty} \phi(x_n, x_0) \le \limsup_{n\to\infty} \phi(x_n, x_0) \le \phi(p, x_0).$$

This implies that

$$\lim_{n\to\infty} \phi(x_n, x_0) = \phi(p, x_0).$$

Hence, we have $||x_n|| \to ||p||$ as $n \to \infty$. In view of the Kadec-Klee property of E, we obtain that $x_n \to p$ as $n \to \infty$.

Next, we show that $p \in F(T)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \prod_{C_{n+1}} x_0 \in C_n$. It follows that

$$\begin{aligned}
\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_0) \\
&\le \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\
&= \phi(x_{n+1}, x_0) - \phi(x_n, x_0).
\end{aligned}$$

Letting $n \to \infty$, we obtain that $\varphi(x_{n+1}, x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) + (k_n - 1)M_n.$$

It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{2.2}$$

From (1.5), we see that

$$||u_n|| \to ||p|| \quad \text{as } n \to \infty. \tag{2.3}$$

It follows that

$$||Ju_n|| \to ||Jp|| \quad \text{as } n \to \infty. \tag{2.4}$$

This implies that $\{Ju_n\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Ju_n \rightharpoonup x^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an $x \in E$ such that $Jx = x^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, u_n) &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Ju_n\rangle + ||u_n||^2 \\ &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Ju_n\rangle + ||Ju_n||^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ the both sides of above equality yields that

$$\begin{aligned}0 &\geq ||p||^2 - 2\langle p, x^*\rangle + ||x^*||^2 \\ &= ||p||^2 - 2\langle p, Jx\rangle + ||Jx||^2 \\ &= ||p||^2 - 2\langle p, Jx\rangle + ||x||^2 \\ &= \phi(p, x).\end{aligned}$$

That is, $p = x$, which in turn implies that $x^* = Jp$. It follows that $Ju_n \rightharpoonup Jp \in E^*$. From (2.4) and E^* has the Kadec-Klee property, we obtain that

$$Ju_n - Jp \to 0 \quad \text{as } n \to \infty.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $u_n \rightharpoonup p$. From (2.3) and E has the Kadec-Klee property, we obtain that

$$u_n \to p \quad \text{as } n \to \infty. \tag{2.5}$$

Note that

$$||x_n - u_n|| \leq ||x_n - p|| + ||p - u_n||.$$

It follows that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{2.6}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} ||Jx_n - Ju_n|| = 0. \tag{2.7}$$

Let $r = \sup_{n\geq 0}\{||x_n||, ||T^n x_n||\}$. Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 1.7, we see that

$$\begin{aligned}\phi(w, u_n) &= \phi(w, S_{r_n} y_n) \\ &\leq \phi(w, y_n) \\ &= \phi(w, J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JT^n x_n]) \\ &= ||w||^2 - 2\langle w, \alpha_n Jx_n + (1-\alpha_n)JT^n x_n\rangle + ||\alpha_n Jx_n + (1-\alpha_n)JT^n x_n||^2 \\ &\leq ||w||^2 - 2\alpha_n\langle w, Jx_n\rangle - 2(1-\alpha_n)\langle w, JT^n x_n\rangle + \alpha_n ||x_n||^2 \\ &\quad + (1-\alpha_n)||T^n x_n||^2 - \alpha_n(1-\alpha_n)g(||Jx_n - JT^n x_n||) \\ &= \alpha_n\phi(w, x_n) + (1-\alpha_n)\phi(w, T^n x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - JT^n x_n||) \\ &\leq \alpha_n\phi(w, x_n) + (1-\alpha_n)k_n\phi(w, x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - JT^n x_n||) \\ &\leq \phi(w, x_n) + (k_n - 1)M_n - \alpha_n(1-\alpha_n)g(||Jx_n - JT^n x_n||).\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n)g(||Jx_n - JT^n x_n||) \le \phi(w, x_n) - \phi(w, u_n) + (k_n - 1)M_n.$$

On the other hand, we have

$$\begin{aligned}\phi(w, x_n) - \phi(w, u_n) &= ||x_n||^2 - ||u_n||^2 - 2\langle w, Jx_n - Ju_n\rangle \\ &\le ||x_n - u_n||(||x_n|| + ||u_n||) + 2||w||\, ||Jx_n - Ju_n||.\end{aligned}$$

It follows from (2.6) and (2.7) that

$$\phi(w, x_n) - \phi(w, u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

In view of $\lim_{n\to\infty} (k_n\text{-}1)\, M_n = 0$ and (2.8) and the assumption $\liminf_{n\to\infty} \alpha_n\,(1 - \alpha_n) > 0$, we see that

$$g(||Jx_n - JT^n x_n||) \to 0 \quad \text{as } n \to \infty.$$

It follows from the property of g that

$$||Jx_n - JT^n x_n|| \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

Since $x_n \to p$ as $n \to\infty$ and $J : E \to E^*$ is demi-continuous, we obtain that $Jx_n \to Jp \in E^*$. Note that

$$|||Jx_n|| - ||Jp||| = |||x_n|| - ||p||| \le ||x_n - p||.$$

This implies that $||Jx_n|| \to ||Jp||$ as $n \to \infty$. Since E^* has the Kadec-Klee property, we see that

$$||Jx_n - Jp|| \to 0 \quad \text{as } n \to \infty. \tag{2.10}$$

Note that

$$||JT^n x_n - Jp|| \le ||JT^n x_n - Jx_n|| + ||Jx_n - Jp||.$$

From (2.9) and (2.10), we obtain at

$$\lim_{n\to\infty} ||JT^n x_n - Jp|| = 0. \tag{2.11}$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $T^n x_n \to p$. On the other hand, we have

$$|\,||T^n x_n|| - ||p||\,| = |\,||JT^n x_n|| - ||Jp||\,| \le ||JT^n x_n - Jp||.$$

In view of (2.11), we obtain that $||T^n x_n|| \to ||p||$ as $n \to \infty$. Since E has the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty} ||T^n x_n - p|| = 0. \tag{2.12}$$

Note that

$$||T^{n+1} x_n - p|| \le ||T^{n+1} x_n - T^n x_n|| + ||T^n x_n - p||.$$

It follows from the asymptotic regularity of T and (2.12) that

$$\lim_{n\to\infty} ||T^{n+1} x_n - p|| = 0.$$

That is, $TT^n x_n - p \to 0$ as $n \to \infty$: It follows from the closedness of T that $Tp = p$:

Next, we show that $p \in EF(f)$: From (2.1), we have

$$\phi(w, y_n) \le \phi(w, x_n) + (k_n - 1)M_n. \tag{2.13}$$

In view of $u_n = S_{r_n}y_n$ and Lemma 1.8, we obtain

$$\begin{aligned}\phi(u_n, y_n) &= \phi(S_{r_n}y_n, y_n)\\ &\le \phi(w, y_n) - \phi(w, S_{r_n}y_n)\\ &\le \phi(w, x_n) - \phi(w, S_{r_n}y_n) + (k_n - 1)M_n\\ &= \phi(w, x_n) - \phi(w, u_n) + (k_n - 1)M_n.\end{aligned} \tag{2.14}$$

It follows from (2.8) that

$$\phi(u_n, y_n) \to 0 \quad \text{as } n \to \infty.$$

From (1.5), we see that $||u_n|| - ||y_n|| \to 0$ as $n \to \infty$. In view of $u_n \to p$ as $n \to \infty$, we have

$$||y_n|| - ||p|| \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

It follows that

$$||Jy_n|| - ||Jp|| \to 0 \quad \text{as } n \to \infty. \tag{2.16}$$

Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup q^* \in E^*$: In view of $J(E) = E^*$, we see that there exists $q \in E$ such that $Jq = q^*$. It follows that

$$\begin{aligned}\phi(u_n, y_n) &= ||u_n||^2 - 2\langle u_n, Jy_n\rangle + ||y_n||^2\\ &= ||u_n||^2 - 2\langle u_n, Jy_n\rangle + ||Jy_n||^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ the both sides of above equality yields that

$$\begin{aligned}0 &\ge ||p||^2 - 2\langle p, q^*\rangle + ||q^*||^2\\ &= ||p||^2 - 2\langle p, Jq\rangle + ||Jq||^2\\ &= ||p||^2 - 2\langle p, Jq\rangle + ||q||^2\\ &= \phi(p, q).\end{aligned}$$

That is, $p = q$, which in turn implies that $q^* = Jp$. It follows that $Jy_n \rightharpoonup Jp \in E^*$. From (2.16) and E^* has the Kadec-Klee property, we obtain that

$$Jy_n - Jp \to 0 \quad \text{as } n \to \infty.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_n \rightharpoonup p$. From (2.15) and E has the Kadec-Klee property, we obtain that

$$y_n \to p \quad \text{as } n \to \infty. \tag{2.17}$$

Note that

$$||u_n - y_n|| \le ||u_n - p|| + ||p - y_n||.$$

It follows from (2.5) and (2.17) that

$$\lim_{n\to\infty} ||u_n - y_n|| = 0. \tag{2.18}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} ||Ju_n - Jy_n|| = 0.$$

From the assumption $r_n \geq a$, we see that

$$\lim_{n\to\infty} \frac{||Ju_n - Jy_n||}{r_n} = 0. \tag{2.19}$$

In view of $u_n = S_{r_n} y_n$, we see that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from the condition ($A2$) that

$$||y - u_n|| \frac{||Ju_n - Jy_n||}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in the above inequality, we from conditions ($A4$) and (2.19) obtain that

$$f(y, p) \leq 0, \quad \forall y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1 - t)p$. It follows that $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from conditions ($A1$) and ($A4$) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, p) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \geq 0.$$

Letting $t \downarrow 0$, from condition ($A3$), we obtain that $f(p, y) \geq 0$ $\forall y \in C$: This implies that $p \in EP(f)$. This shows that $p \in \mathcal{F} = EP(f) \cap F(T)$.

Finally, we prove that $p = \Pi_{\mathcal{F}} x_0$. From $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n.$$

Since $\mathcal{F} \subset C_n$ for each $n \geq 1$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in \mathcal{F}. \tag{2.20}$$

Letting $n \to \infty$ in (2.20), we see that

$$\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in \mathcal{F}.$$

In view of Lemma 1.3, we can obtain that $x_n = \Pi_{C_n} x_0$. This completes the proof.

Remark 2.2. Theorem 2.1 improves Theorem QCK in the following aspects:

(a) From a uniformly smooth and uniformly convex space to a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property;
(b) From a quasi-φ-nonexpansive mapping to an asymptotically quasi-φ-non-expansive mapping.

From the definition of quasi-φ-nonexpansive mappings, we see that every quasi-φ-nonexpansive mapping is asymptotically quasi-φ-nonexpansive with the constant sequence {1}. From the proof of Theorem 2.1, we have the following results immediately.

Corollary 2.3. *Let E be a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property and C a nonempty closed convex subset of E. Let f be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4) *and* $T : C \to C$ *a closed and quasi-φ-nonexpansive mapping. Assume that* $\mathcal{F} = F(T) \cap EF(f)$ *is nonempty.*

Let $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* [0, 1] *such that* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $\{r_n\}$ *is a real sequence in* $[a, \infty)$, *where a is some positive real number and J is the duality mapping on E. Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{\mathcal{F}} x_0$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from E onto* $\mathcal{F}$.

Remark 2.4. Corollary 2.3 improves Theorem TZ in the following aspects.

(a) For the framework of spaces, we extend the space from a uniformly smooth and uniformly convex space to a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property (note that every uniformly convex Banach space has the Kadec-Klee property).

(b) For the mappings, we extend the mapping from a relatively nonexpansive mapping to a quasi-φ-nonexpansive mapping (we remove the restriction $\widetilde{F}(T) = F(T)$, where $\widetilde{F}(T)$ denotes the asymptotic fixed point set).

(c) For the algorithms, we remove the set "W_n" in Theorem TZ.

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Competing interests
The author declares that they have no competing interests.

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