# ON THE STUDY OF A CLASS OF VARIATIONAL INEQUALITIES VIA LERAY-SCHAUDER DEGREE

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*Received 12 July 2004*

We present existence results for general variational inequalities without monotonicity or coercivity assumptions. It relies on a Leray-Schauder degree approach and provides additional information about the location of solutions.

## 1. Introduction

The study of variational inequalities is very important from a theoretic point of view in mathematics as well as for its various and significant applications in different fields, for instance, in what is called nonsmooth mechanics [1, 3, 10]. Comprehensive treatment of different problems related to variational inequalities and their applications can be found in the monographs [2, 5, 6, 7, 8]. A basic assumption in the results studying the variational inequalities on a Hilbert space is the monotonicity condition, in particular, the ellipticity (or coercivity) hypothesis on the (possibly nonlinear) operator entering the problem. The interest to relax this condition, by imposing other type of assumptions, is a real challenge in the recent developments. The present paper is devoted to this topic, where in place of monotonicity there are supposed suitable assumptions allowing the application of topological degree arguments. Our approach permits to encompass the solvability of cases that were not covered by the previous known results.

We describe the functional setting of the paper. Let $H$ be a real Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$.

Consider the following general assumptions on the data in our variational inequality formulation (see problem (1.3)):

(H1) $\Phi : H \to H$ is a compact mapping, that is, $\Phi$ is continuous and maps the bounded sets onto relatively compact sets;

(H2) $\varphi : H \to \mathbb{R}$ is a convex and continuous function which is bounded from above on the bounded subsets of $H$.

Since a convex and lower semicontinuous function on $H$ is bounded from below by an affine function, it is bounded from below on the bounded subsets of $H$. Hypothesis (H2) ensures thus that the function $\varphi$ is bounded on the bounded subsets of $H$. We stress that


Fixed Point Theory and Applications 2004:4 (2004) 261–271
2000 Mathematics Subject Classification: 49J40, 35K85, 47H11
URL: http://dx.doi.org/10.1155/S1687182004407074

the property of the function $\varphi : H \to \mathbb{R}$ to be bounded from above on the bounded subsets of $H$ as assumed in (H2) is not satisfied, in general, by a convex and continuous function $\varphi$ on $H$. We provide an example in this direction based on private communication with J. Saint Raymond (2004).

*Example 1.1.* Consider the Hilbert space $\ell^2$ and the function $f : \ell^2 \to \mathbb{R}$ defined by

$$f(x) = \sup_{n\geq 0} \left(2n|x_n| - n\right) \quad \forall x \in \ell^2, \tag{1.1}$$

where $x_n$ are the components of $x$. The function $f$ is convex, continuous, and not bounded on the bounded sets. Indeed, $f$ is defined on $\ell^2$ because for any $x \in \ell^2$ the set

$$\{n : 2n|x_n| - n \geq 0\} = \left\{n : |x_n| \geq \frac{1}{2}\right\} \tag{1.2}$$

is finite. The function $f$ is convex, since it is the upper hull of the convex functions $f_n$ on $\ell^2$ given by $f_n(x) = 2n|x_n| - n$. We note that $f$ is zero on the ball centered at 0 and of radius 1/2 because $0 = f_0(x) \leq f(x)$ and $2|x_n| \leq 1$ if $\|x\| < 1/2$. Being bounded on a nonempty open set, the function $f$ is continuous. Finally, it is seen that $f(e_n) = n$, where $e_n$ is the $n$th vector of the canonical basis of $\ell^2$. It turns out that the function $f$ is not bounded from above on the unit sphere in $\ell^2$.

Given $\Phi : H \to H$ and $\varphi : H \to \mathbb{R}$, we formulate now our variational inequality problem: find $\bar{x} \in H$ such that

$$\langle \bar{x} - \Phi(\bar{x}), v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0 \quad \forall v \in H. \tag{1.3}$$

Our approach in studying the variational inequality (1.3) relies on the Leray-Schauder degree theory (see [4, 9]). Assumption (H1) is mainly imposed to fit the setting of the Leray-Schauder degree theory.

Several approaches using degree theory have been recently developed so as to study problems like the one given in (1.3), even for general classes of proper, convex, and lower semicontinuous functions $\varphi$ (see [6, 11]), but here we give a more qualitative insight on the topic. Specifically, assumption (H2) allows us to develop a new and powerful continuation result (see Proposition 2.3). Using this continuation result for problem (1.3), we prove several new results guaranteeing the existence of solutions (see Section 3). Some location information on the solution set of problem (1.3) is also available through our results, for example, criteria to have nontrivial solutions. Here the hypotheses (H1) and (H2) play an essential role. Special attention is paid to the situation where the Hilbert space $H$ is finite dimensional. It is worth noting that if $H$ is finite dimensional, then every continuous mapping $\Phi : H \to H$ satisfies assumption (H1), and every continuous and convex function $\varphi : H \to \mathbb{R}$ fulfills assumption (H2). This enables us to have great flexibility in applying our results in the case $H = \mathbb{R}^N$.

Our main argument lies in the use of a nonlinear operator $P_\varphi : H \to H$ which is related to the function $\varphi$ in problem (1.3) and hypothesis (H2).

The rest of the paper is organized as follows. Section 2 contains some preliminary results to set up our topological degree framework. Section 3 is devoted to our existence results for the variational inequality (1.3).

## 2. Preliminary results

This section concerns an auxiliary variational inequality on a real Hilbert space $H$ whose solution will be the main tool in solving problem (1.3). In the sequel we denote by $\mathrm{id}_H$ the identity mapping on $H$. For later use, for any $r > 0$, we denote $B_r := \{x \in H : \|x\| < r\}$, $\bar{B}_r := \{x \in H : \|x\| \le r\}$, and $\partial B_r := \{x \in H : \|x\| = r\}$.

Let $\varphi : H \to \mathbb{R}$ be a convex and continuous function. The notation $\partial\varphi$ stands for the subdifferential of $\varphi$ in the sense of convex analysis, that is, the nonempty set

$$\partial\varphi(x) = \{w \in H : \varphi(v) - \varphi(x) \ge \langle w, v - x\rangle \ \forall v \in H\}. \tag{2.1}$$

The subdifferential $\partial\varphi$ is defined everywhere on $H$ because the function $\varphi$ is convex and continuous on $H$.

For a fixed element $y \in H$, we state the variational inequality problem: find $x \in H$ such that

$$\langle x - y, v - x\rangle + \varphi(v) - \varphi(x) \ge 0 \quad \forall v \in H. \tag{2.2}$$

It is well known that problem (2.2) has a unique solution $x \in H$ (see, e.g., [2, 4, 8]). Therefore the well-defined (nonlinear) operator $P_\varphi : H \to H$ given by

$$P_\varphi(y) = x \quad \forall y \in H, \tag{2.3}$$

where $x \in H$, is the solution to (2.2). We note that $P_0 y = y$ for all $y \in H$.

First we discuss the continuity properties of the nonlinear operator $P_\varphi$ described in (2.2) and (2.3).

Proposition 2.1. *Let $\varphi : H \to \mathbb{R}$ be a convex and continuous function. Then the operator $P_\varphi$ is continuous.*

*Proof.* Let $\{y_n\} \subset H$ be a sequence such that $y_n \to y^*$ as $n \to +\infty$. We claim that $P_\varphi(y_n) \to P_\varphi(y^*)$ in $H$ as $n \to +\infty$. Indeed, denoting $x_n := P_\varphi(y_n)$ and $x^* := P_\varphi(y^*)$, we have from (2.3) and (2.2) that

$$\langle x_n - y_n, v - x_n\rangle + \varphi(v) - \varphi(x_n) \ge 0 \quad \forall v \in H, \tag{2.4}$$

$$\langle x^* - y^*, v - x^*\rangle + \varphi(v) - \varphi(x^*) \ge 0 \quad \forall v \in H. \tag{2.5}$$

If we set $v = x^*$ in (2.4) and $v = x_n$ in (2.5), we obtain

$$\begin{aligned} \langle x_n - y_n, x_n - x^*\rangle - \varphi(x^*) + \varphi(x_n) &\le 0, \\ -\langle x^* - y^*, x_n - x^*\rangle - \varphi(x_n) + \varphi(x^*) &\le 0. \end{aligned} \tag{2.6}$$

We derive

$$\|x_n - x^*\|^2 \le \|x_n - x^*\| \, \|y_n - y^*\|. \tag{2.7}$$

It follows that $x_n \to x^*$ in $H$ as $n \to +\infty$, and the conclusion is achieved. □

Proposition 2.2. *Let $G : [0,T] \times H \to H$ be a continuous function on $[0,T] \times H$, with a number $T > 0$, and let $\varphi : H \to \mathbb{R}$ satisfy hypothesis (H2). Then the mapping*

$$(\lambda, y) \in [0,T] \times H \longmapsto P_{\lambda\varphi}(G(\lambda, y)) \in H \tag{2.8}$$

*is continuous on $[0,T] \times H$, where $P_{\lambda\varphi}$ is the nonlinear operator introduced by (2.2) and (2.3).*

*Proof.* We check the continuity at an arbitrary point $(\lambda^*, y^*) \in [0,T] \times H$. Consider the convergent sequences $\{y_n\} \subset H$ and $\{\lambda_n\} \subset [0,T]$ with $y_n \to y^*$ in $H$ and $\lambda_n \to \lambda^*$ in $\mathbb{R}$ as $n \to +\infty$. We have to show that $P_{\lambda_n\varphi}(G(\lambda_n, y_n)) \to P_{\lambda^*\varphi}(G(\lambda^*, y^*))$ in $H$ as $n \to +\infty$. Denote $x_n := P_{\lambda_n\varphi}(G(\lambda_n, y_n))$ and $x^* := P_{\lambda^*\varphi}(G(\lambda^*, y^*))$. By the definition of the mapping $P_{\lambda\varphi}$ in (2.2) and (2.3) it is known that

$$\langle x_n - G(\lambda_n, y_n), v - x_n\rangle + \lambda_n\varphi(v) - \lambda_n\varphi(x_n) \geq 0 \quad \forall v \in H, \tag{2.9}$$

$$\langle x^* - G(\lambda^*, y^*), v - x^*\rangle + \lambda^*\varphi(v) - \lambda^*\varphi(x^*) \geq 0 \quad \forall v \in H. \tag{2.10}$$

We first prove that the sequence $\{x_n\}$ is bounded. To this end, suppose, on the contrary, that along a relabeled subsequence one has $\|x_n\| \to +\infty$ as $n \to +\infty$. Setting $v = 0$ in (2.9), we obtain

$$-\langle x_n - G(\lambda_n, y_n), x_n\rangle + \lambda_n[\varphi(0) - \varphi(x_n)] \geq 0. \tag{2.11}$$

This leads to

$$1 \leq \frac{||G(\lambda_n, y_n)||}{||x_n||} + \frac{\lambda_n}{||x_n||^2}[\varphi(0) - \varphi(x_n)]. \tag{2.12}$$

For $n$ large enough, we may admit that $1/\|x_n\| \in (0,1]$. Using the convexity of $\varphi$ we obtain

$$\varphi\left(\frac{x_n}{||x_n||}\right) \leq \frac{1}{||x_n||}\varphi(x_n) + \left(1 - \frac{1}{||x_n||}\right)\varphi(0) \tag{2.13}$$

or, equivalently,

$$\frac{\varphi(0) - \varphi(x_n)}{||x_n||} \leq \varphi(0) - \varphi\left(\frac{x_n}{||x_n||}\right). \tag{2.14}$$

Combining with (2.12) implies

$$1 \leq \frac{||G(\lambda_n, y_n)||}{||x_n||} + \lambda_n\frac{\varphi(0) - \varphi(x_n/||x_n||)}{||x_n||}. \tag{2.15}$$

Since the function $\varphi$ is convex and continuous on the whole space $H$, it turns out $\varphi$ is bounded from below on the bounded subsets of $H$. Consequently, in conjunction with assumption (H2), one has that $\varphi$ is bounded on the bounded subsets of $H$. This ensures that

$$\lim_{n \to +\infty} \frac{\varphi(x_n/||x_n||)}{||x_n||} = 0. \tag{2.16}$$

Passing to the limit as $n \to +\infty$ in (2.15) and using the continuity of $G$, we arrive at contradiction. Therefore the sequence $\{x_n\}$ is bounded in $H$.

Setting now $v = x^*$ in (2.9) and $v = x_n$ in (2.10) allows to write

$$\begin{gathered}\langle x_n - G(\lambda_n, y_n), x_n - x^*\rangle - \lambda_n \varphi(x^*) + \lambda_n \varphi(x_n) \leq 0, \\ -\langle x^* - G(\lambda^*, y^*), x_n - x^*\rangle - \lambda^* \varphi(x_n) + \lambda^* \varphi(x^*) \leq 0.\end{gathered} \tag{2.17}$$

It follows that

$$||x_n - x^*||^2 \leq ||G(\lambda_n, y_n) - G(\lambda^*, y^*)|| \, ||x_n - x^*|| + (\lambda_n - \lambda^*)\varphi(x^*) + (\lambda^* - \lambda_n)\varphi(x_n). \tag{2.18}$$

The continuity of $G$ gives $\|G(\lambda_n, y_n) - G(\lambda^*, y^*)\| \to 0$, while the boundedness of the sequence $\{x_n\}$ combined with assumption (H2) guarantees that the sequence $\{\varphi(x_n)\}$ is bounded. It is then clear that (2.18) yields $x_n \to x^*$ as $n \to +\infty$, which completes the proof. □

The following technical result is useful for the computations involving the Leray-Schauder degree in the next section. We recall that, given a compact mapping $\Psi : \bar{B}_r \to H$ such that $0 \notin (\mathrm{id}_H - \Psi)(\partial B_r)$, there exists the Leray-Schauder degree $\deg(\mathrm{id}_H - \Psi, B_r, 0)$ of $\mathrm{id}_H - \Psi$ in $B_r$ with respect to 0 (see, e.g., [4, 9]).

Proposition 2.3. *Assume that conditions (H1) and (H2) on the mappings $\Phi : H \to H$ and $\varphi : H \to \mathbb{R}$, respectively, are fulfilled. If there exists a compact mapping $\chi : H \to H$ and a number $r > 0$ such that*

$$\langle x - \Phi(x), \chi(x) - x\rangle + \varphi(\chi(x)) - \varphi(x) < 0 \quad \forall x \in H, \ \|x\| = r, \tag{2.19}$$

*then the following equality holds:*

$$\deg(\mathrm{id}_H - P_\varphi \Phi, B_r, 0) = \deg(\mathrm{id}_H - \chi, B_r, 0). \tag{2.20}$$

*Proof.* Notice that the mapping $P_\varphi \Phi$ is compact being the composition of the continuous mapping $P_\varphi$ (cf. Proposition 2.1) and the compact one $\Phi$ (cf. (H1)). So the mapping $\mathrm{id}_H - P_\varphi \Phi$ is of the form required in the definition of the Leray-Schauder degree (see [4, 9]). Let $h : [0,1] \times \bar{B}_r \to H$ be the mapping defined by

$$h(\lambda, y) = y - P_{\lambda\varphi}(\lambda\Phi(y) + (1-\lambda)\chi(y)) \quad \forall (\lambda, y) \in [0,1] \times \bar{B}_r. \tag{2.21}$$

Applying Proposition 2.2 with $G : [0,1] \times H \to H$ given by $G(\lambda, y) = \lambda\Phi(y) + (1-\lambda)\chi(y)$, for all $(\lambda, y) \in [0,1] \times H$, we infer that $h$ is a continuous mapping. Moreover, since $\Phi$ and $\chi$ are compact, for each $\lambda \in [0,1]$, the mapping $y \mapsto P_{\lambda\varphi}(\lambda\Phi(y) + (1-\lambda)\chi(y))$ is compact too.

We claim that

$$h(\lambda,x) \neq 0 \quad \forall \lambda \in [0,1],\ \forall x \in \partial B_r. \tag{2.22}$$

Arguing by contradiction, suppose that there exist $x \in H$, with $\|x\| = r$, and $\lambda \in [0,1]$ such that $h(\lambda,x) = 0$. This reads as

$$x = P_{\lambda\varphi}\big(\lambda\Phi(x) + (1-\lambda)\chi(x)\big). \tag{2.23}$$

We first remark that

$$\lambda > 0. \tag{2.24}$$

If not, we have $\lambda = 0$ and equality (2.23) reduces to $x = P_0(\chi(x)) = \chi(x)$, which contradicts assumption (2.19). Thus (2.24) holds true.

On the other hand, (2.23) expresses that

$$\langle x - \lambda\Phi(x) - (1-\lambda)\chi(x), v - x\rangle + \lambda\varphi(v) - \lambda\varphi(x) \geq 0 \quad \forall v \in H. \tag{2.25}$$

For $v = \chi(x)$, it is seen that

$$\lambda\big[\langle x - \Phi(x), \chi(x) - x\rangle + \varphi(\chi(x)) - \varphi(x)\big] \geq (1-\lambda)\|\chi(x) - x\|^2 \geq 0. \tag{2.26}$$

In view of (2.24) we derive

$$\langle x - \Phi(x), \chi(x) - x\rangle + \varphi(\chi(x)) - \varphi(x) \geq 0. \tag{2.27}$$

This contradicts assumption (2.19). Property (2.22) is established.

On the basis of (2.22), the homotopy invariance property of the Leray-Schauder degree implies

$$\begin{aligned} \deg\big(\mathrm{id}_H - P_\varphi\Phi, B_r, 0\big) &= \deg\big(h(1,\cdot), B_r, 0\big) = \deg\big(h(0,\cdot), B_r, 0\big) \\ &= \deg\big(\mathrm{id}_H - P_0\chi, B_r, 0\big) = \deg\big(\mathrm{id}_H - \chi, B_r, 0\big). \end{aligned} \tag{2.28}$$

The proof is thus complete. □

## 3. Existence theorems

Our first main existence result in studying problem (1.3) is the following.

THEOREM 3.1. *Assume that (H1), (H2) hold and that*

(H3) *there exists $r > 0$ such that*

$$\langle x - \Phi(x), x\rangle + \varphi(x) - \varphi(0) > 0 \quad \forall x \in H,\ \|x\| = r. \tag{3.1}$$

*Then problem (1.3) has at least a solution in $B_r$, that is, there exists $\bar{x} \in B_r$ such that*

$$\langle \bar{x} - \Phi(\bar{x}), v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0 \quad \forall v \in H. \tag{3.2}$$

*Proof.* Assumption (H3) entails that relation (2.19) is fulfilled for $\chi = 0$. Consequently, Proposition 2.3 can be applied with $\chi = 0$. Thus we have

$$\deg\left(\mathrm{id}_H - P_\varphi\Phi, B_r, 0\right) = \deg\left(\mathrm{id}_H, B_r, 0\right) = 1 \neq 0. \tag{3.3}$$

A basic property of Leray-Schauder degree ensures that there exists $\bar{x} \in B_r$ verifying $\bar{x} = P_\varphi(\Phi(\bar{x}))$. Taking into account (2.2), it follows that $\bar{x}$ solves problem (1.3). □

Theorem 3.1 yields a sufficient condition for the existence of nontrivial solutions in solving problem (1.3).

Corollary 3.2. *Assume that the hypotheses of Theorem 3.1 hold and, in addition, there exists a point $v_0 \in H \setminus \{0\}$ such that*

$$\langle \Phi(0), v_0 \rangle > \varphi(v_0) - \varphi(0). \tag{3.4}$$

*Then problem (1.3) has at least a nontrivial solution in $B_r$.*

*Proof.* Applying Theorem 3.1, we find $\bar{x} \in B_r$ verifying (1.3). In view of (3.4), one obtains that $\bar{x} \neq 0$. □

The next result provides verifiable conditions under which Theorem 3.1 can be applied.

Corollary 3.3. *Suppose that conditions (H1) and (H2) are verified as well as $0 \in \partial\varphi(0)$ and that*

(H3′) *there exists $r > 0$ such that*

$$\langle x - \Phi(x), x \rangle > 0 \quad \forall x \in H,\ \|x\| = r. \tag{3.5}$$

*Then problem (1.3) has at least a solution in $B_r$.*

*Proof.* The result follows from Theorem 3.1 observing that assumptions (H3′) and $0 \in \partial\varphi(0)$ imply (H3). □

A second main existence result in solving problem (1.3) is now given.

Theorem 3.4. *Assume that (H1), (H2) hold and that*

(H3″) *there exists $r > 0$ such that*

$$\begin{gathered} \varphi(\Phi(x)) - \varphi(x) < \left\|\Phi(x) - x\right\|^2 \quad \forall x \in H,\ \|x\| = r, \\ \deg\left(\mathrm{id}_H - \Phi, B_r, 0\right) \neq 0. \end{gathered} \tag{3.6}$$

*Then problem (1.3) has at least a solution in $B_r$.*

*Proof.* We apply Proposition 2.3 with $\chi = \Phi$. This is possible because relation (2.19) is fulfilled for $\chi = \Phi$. It turns out from Proposition 2.3 that

$$\deg\left(\mathrm{id}_H - P_\varphi\Phi, B_r, 0\right) = \deg\left(\mathrm{id}_H - \Phi, B_r, 0\right). \tag{3.7}$$

According to assumption (H3″), we infer that

$$\deg\left(\mathrm{id}_H - P_\varphi\Phi, B_r, 0\right) \neq 0. \tag{3.8}$$

It follows that there exists $\bar{x} \in B_r$ such that $\bar{x} = P_\varphi(\Phi(\bar{x}))$. This allows us to conclude. □

Theorem 3.4 gives rise to the following result.

Corollary 3.5. *Assume the hypotheses of Theorem 3.4 hold and that there exists a point $v_0 \in H \setminus \{0\}$ satisfying (3.4). Then problem (1.3) admits at least a nontrivial solution in $B_r$.*

*Proof.* The existence of a solution follows from Theorem 3.4. The obtained solution $\bar{x} \in B_r$ of problem (1.3) is nontrivial because (3.4) prevents having $\bar{x} = 0$. □

We have the following significant case of Theorem 3.4.

Corollary 3.6. *Suppose that (H1) holds and that*

(H2′) $\varphi : H \to \mathbb{R}$ *is convex and Lipschitz continuous with Lipschitz constant $K > 0$, that is,*

$$\left|\varphi(x) - \varphi(y)\right| \leq K\|x - y\| \quad \forall x, y \in H; \tag{3.9}$$

(H3‴) *there exists $r > 0$ such that*

$$\begin{gathered} ||x - \Phi(x)|| > K \quad \forall x \in H \text{ with } \|x\| = r, \\ \deg\left(\mathrm{id}_H - \Phi, B_r, 0\right) \neq 0. \end{gathered} \tag{3.10}$$

*Then problem (1.3) has at least a solution in $B_r$.*

*Proof.* It is worth noting that because a Lipschitz continuous function is bounded on bounded sets, assumption (H2′) assures that (H2) is satisfied. We see again from (H2′) that

$$\begin{aligned} \varphi(\Phi(x)) - \varphi(x) - ||\Phi(x) - x||^2 &\leq K||x - \Phi(x)|| - ||x - \Phi(x)||^2 \\ &= ||x - \Phi(x)||\left(K - ||x - \Phi(x)||\right). \end{aligned} \tag{3.11}$$

Thus, due to (H3‴), we have

$$\varphi(\Phi(x)) - \varphi(x) - ||\Phi(x) - x||^2 < 0 \quad \forall x \in H,\ \|x\| = r. \tag{3.12}$$

Since (H3″) holds, the conclusion follows from Theorem 3.4. □

We point out a relevant special case of Corollary 3.6.

Theorem 3.7. *Assume that (H2′) holds and that*

(H4) $A : H \to H$ *is a linear topological isomorphism such that* $\mathrm{id}_H - A$ *is a compact mapping.*

*Then, for every* $f \in H$, *there exists* $\bar{x} \in H$ *such that*

$$\langle A\bar{x} - f, v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0, \quad \forall v \in H. \tag{3.13}$$

*Proof.* Fix $f \in H$. According to assumption (H4), the mapping $\Phi : H \to H$ defined by

$$\Phi(x) = x - Ax + f, \quad \forall x \in H, \tag{3.14}$$

is compact, so condition (H1) is verified. Since $A$ is invertible, there exists a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in H$. Fix a number

$$r > \max\left\{\frac{K + \|f\|}{c}, \|A^{-1}f\|\right\}, \tag{3.15}$$

where $K > 0$ is the Lipschitz constant in (H2′). For $\|x\| = r$, it is seen from (3.15) that

$$\|x - \Phi(x)\| = \|Ax - f\| \geq \|Ax\| - \|f\| \geq c\|x\| - \|f\| = cr - \|f\| > K. \tag{3.16}$$

It follows that the first part of condition (H3‴) in Corollary 3.6 is fulfilled. We introduce the mapping $h : [0,1] \times \bar{B}_r \to H$ by

$$h(\lambda, x) = Ax - \lambda f \quad \forall \lambda \in [0,1], \ \forall x \in \bar{B}_r. \tag{3.17}$$

We remark that

$$h(\lambda, x) \neq 0 \quad \forall \lambda \in [0,1], \ \forall x \in H, \text{ with } \|x\| = r. \tag{3.18}$$

Indeed, suppose on the contrary that there exist $\lambda \in [0,1]$ and $x \in H$ with $\|x\| = r$ such that $Ax = \lambda f$. By (3.15), it is known that $\|x\| = \lambda\|A^{-1}f\| < r$, which is a contradiction. Thus the homotopy invariance property of the Leray-Schauder degree can be applied to obtain

$$\begin{aligned} \deg\left(\mathrm{id}_H - \Phi, B_r, 0\right) &= \deg\left(A - f, B_r, 0\right) = \deg\left(h(1,\cdot), B_r, 0\right) \\ &= \deg\left(h(0,\cdot), B_r, 0\right) = \deg\left(A, B_r, 0\right) \neq 0 \end{aligned} \tag{3.19}$$

since $A \in \mathrm{Isom}(H)$. So the second part of (H3‴) is valid too. Therefore the hypotheses of Corollary 3.6 are satisfied. Applying Corollary 3.6 leads to the desired conclusion. □

Corollary 3.8. *Let* $X$ *and* $Y$ *be Hilbert spaces, with* $Y$ *finite dimensional. Suppose that* $T : Y \to Y$ *is a linear invertible mapping and* $\varphi : X \times Y \to \mathbb{R}$ *is a function verifying (H2′) with* $H = X \times Y$. *Then, for any* $(f, g) \in X \times Y$, *there exists* $(\bar{x}, \bar{y}) \in X \times Y$ *such that*

$$\langle \bar{x} - f, v - \bar{x}\rangle + \langle T\bar{y} - g, w - \bar{y}\rangle + \varphi(v, w) - \varphi(\bar{x}, \bar{y}) \geq 0 \quad \forall (v, w) \in X \times Y. \tag{3.20}$$

*Proof.* Let $H := X \times Y$. Then the operator $A : H \to H$ defined by

$$A(x,y) = (x, Ty), \quad \forall (x,y) \in H, \tag{3.21}$$

is a linear topological isomorphism. Since

$$(\mathrm{id}_H - A)(x,y) = (x,y) - (x,Ty) = (0, y - Ty), \quad \forall (x,y) \in H, \tag{3.22}$$

and $Y$ is finite dimensional, it follows that the mapping $\mathrm{id}_H - A$ is compact. The application of Theorem 3.7 completes the proof. □

We illustrate the above result with an application in the finite-dimensional setting.

Corollary 3.9. *Suppose that $T \in \mathbb{R}^{N\times N}$ is a real nonsingular matrix and $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a function verifying (H2′) with $H = \mathbb{R}^N$. Then, for any $g \in \mathbb{R}^N$, there exists $\bar{x} \in \mathbb{R}^N$ such that*

$$\langle T\bar{x} - g, v - \bar{x}\rangle + \varphi(v) - \varphi(\bar{x}) \geq 0 \quad \forall v \in \mathbb{R}^N. \tag{3.23}$$

*Proof.* It suffices to apply Corollary 3.8 for $X = \{0\}$, $Y = \mathbb{R}^N$, $f = 0$. □

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