# ROOTS OF MAPPINGS FROM MANIFOLDS

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Assume that $f : X \to Y$ is a proper map of a connected $n$-manifold $X$ into a Hausdorff, connected, locally path-connected, and semilocally simply connected space $Y$, and $y_0 \in Y$ has a neighborhood homeomorphic to Euclidean $n$-space. The proper Nielsen number of $f$ at $y_0$ and the absolute degree of $f$ at $y_0$ are defined in this setting. The proper Nielsen number is shown to a lower bound on the number of roots at $y_0$ among all maps properly homotopic to $f$, and the absolute degree is shown to be a lower bound among maps properly homotopic to $f$ and transverse to $y_0$. When $n > 2$, these bounds are shown to be sharp. An example of a map meeting these conditions is given in which, in contrast to what is true when $Y$ is a manifold, Nielsen root classes of the map have different multiplicities and essentialities, and the root Reidemeister number is strictly greater than the Nielsen root number, even when the latter is nonzero.

## 1. Introduction

Let $f : X \to Y$ be a map of topological spaces and $y_0 \in Y$. A point $x \in X$ such that $f(x) = y_0$ is called a *root of $f$ at $y_0$*. In Nielsen root theory, by analogy with Nielsen fixed-point theory, the roots of $f$ are grouped into Nielsen classes, a notion of essentiality is defined, and the Nielsen root number is defined to be the number of essential root classes. The Nielsen root number is a homotopically invariant lower bound for the number of roots of $f$ at $y_0$. When $X$ is noncompact, it is often of more interest to restrict attention to proper maps and proper homotopies, and define a "proper Nielsen root number."

We also consider the topological analog of the case where $y_0$ is a "regular value" of $f$. In this analog, $f$ is said to be "transverse to $y_0$." The map $f$ is transverse to $y_0$ if it has a neighborhood that is evenly covered by $f$. For this purpose, Hopf [7] introduced the notion of "absolute degree" (which we redefine in Section 3 below). For maps of compact oriented manifolds, the absolute degree is the same, up to sign, as the Brouwer degree.

The main objective of this paper is to prove the following two theorems in Nielsen root theory.


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THEOREM 1.1. *Let $f : X \to Y$ be a proper map of a connected $n$-manifold $X$ into a Hausdorff, connected, locally path-connected, and semilocally simply connected space $Y$. Assume $y_0 \in Y$ has a neighborhood homeomorphic to Euclidean $n$-space $\mathbb{R}^n$. Then every map properly homotopic to $f$ and transverse to $y_0$ has at least $\mathcal{A}(f, y_0)$ roots, where $\mathcal{A}(f, y_0)$ denotes the absolute degree of $f$ at $y_0$.*

*Moreover, if $n > 2$, then there is a map properly homotopic to $f$ and transverse to $y_0$ that has exactly $\mathcal{A}(f, y_0)$ roots at $y_0$.*

THEOREM 1.2. *Let $f : X \to Y$ be a proper map of a connected $n$-manifold $X$ into a Hausdorff, connected, locally path-connected, and semilocally simply connected space $Y$. Assume $y_0 \in Y$ has a neighborhood homeomorphic to Euclidean $n$-space $\mathbb{R}^n$. Then every map properly homotopic to $f$ has at least* PNR$(f, y_0)$ *roots at $y_0$, where* PNR$(f, y_0)$ *denotes the proper Nielsen root number of $f$ at $y_0$, and every Nielsen root class of $f$ at $y_0$ with nonzero multiplicity is properly essential.*

*Moreover, if $n > 2$, then here is a map properly homotopic to $f$ that has exactly* PNR$(f, y_0)$ *roots at $y_0$, and a root class of $f$ is properly essential only if it has nonzero multiplicity.*

Each of these theorems is a direct generalization of a theorem that heretofore required $Y$, as well as $X$, to be an $n$-manifold. Those theorems, in their original forms, are due to Hopf [7]. Modern statements and proofs (still requiring $Y$ to be a manifold), as well as a review of the history of the subject are given in Brown and Schirmer [3]. Definitions of the terms "transverse," "absolute degree," "proper Nielsen number," "multiplicity," and "properly essential" are given in Sections 2 and 3 below. Before proceeding to formal definitions, however, we will use the following example to introduce some of these and other concepts from Nielsen root theory, as well as to illustrate Theorems 1.1 and 1.2.

*Example 1.3.* Let $\mathbf{S}^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$ denote the unit sphere in $\mathbb{R}^{n+1}$, and let $S = (0,\dots,0,-1)$ and $N = (0,\dots,0,1)$ denote its south and north poles. Assume $n \geq 2$. For each positive integer $k$, let $k\mathbf{S}^n$ denote the space formed by taking $k$ copies of $\mathbf{S}^n$ and identifying the north pole of each to the south pole of the next. More formally, define an equivalence relation $\approx$ on $\{1,\dots,k\} \times \mathbf{S}^n$ by $(z,N) \approx (z+1,S)$ for $z = 1,\dots,k-1$ and let $k\mathbf{S}^n = \{1,\dots,k\} \times \mathbf{S}^n / \approx$. Thus, in particular, $2\mathbf{S}^n$ is the wedge product of two spheres.

There is a natural map of $\mathbf{S}^n$ onto $2\mathbf{S}^n$ obtained by squeezing the equator of $\mathbf{S}^n$ to a point. We generalize this to a map $g : \mathbf{S}^n \to k\mathbf{S}^n$. First, for each $z = 1,\dots,k$, let

$$X_z = \left\{ (x_1,\dots,x_{n+1}) \in \mathbf{S}^n \;\middle|\; \frac{2(z-1)}{k} - 1 \leq x_{n+1} \leq \frac{2z}{k} - 1 \right\}. \tag{1.1}$$

Define $g_z : X_z \to \mathbf{S}^n$ by

$$g_z(x_1,\dots,x_{n+1}) = \begin{cases} (0,\dots,0,-1) & \text{if } z = 1,\ x_{n+1} = -1, \\ (0,\dots,0,1) & \text{if } z = k,\ x_{n+1} = 1, \\ \left( \sqrt{\dfrac{1-\alpha_z^2(x_{n+1})}{1-x_{n+1}^2}}(x_1,\dots,x_n), \alpha_z(x_{n+1}) \right) & \text{otherwise,} \end{cases} \tag{1.2}$$

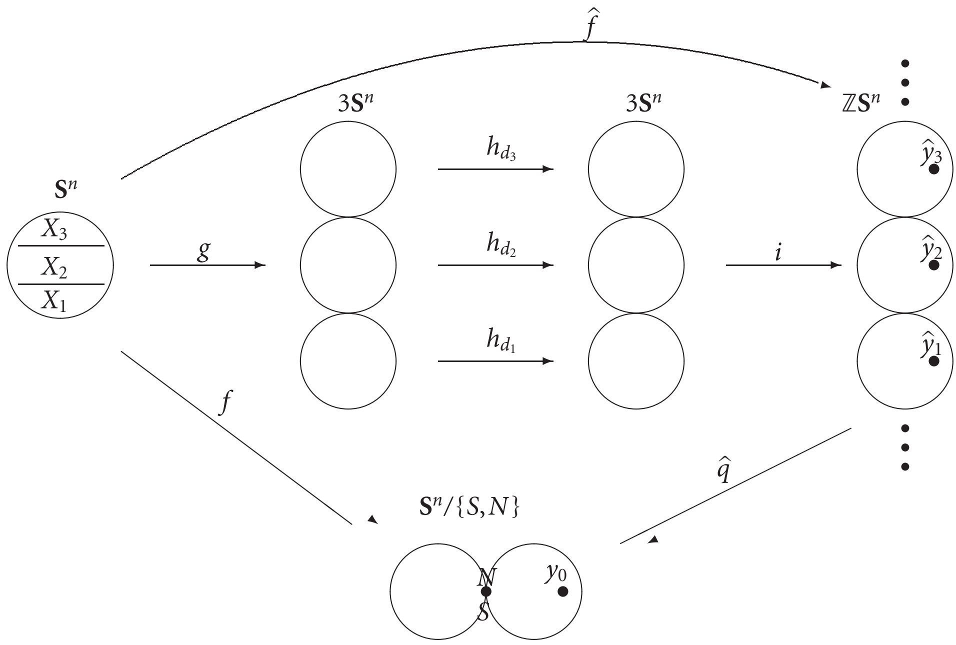



Figure 1.1. Example 1.3 with $k = 3$.

where $\alpha_z(x) = k(x+1) - 2z + 1$. So $g_z$ takes $X_z$ onto $\mathbf{S}^n$ by squeezing the latitudes $x_{n+1} = 2(z-1)/k - 1$ and $x_{n+1} = 2z/k - 1$ to the south and north poles, respectively, and mapping the rest of $X_z$ homeomorphically onto the rest of $\mathbf{S}^n$. Now define $g : \mathbf{S}^n \to k\mathbf{S}^n$ by

$$g(\mathbf{x}) = \left[ (z, g_z(\mathbf{x})) \right] \quad \text{for } \mathbf{x} \in X_z,\ z = 1,\dots,k, \tag{1.3}$$

where the square brackets denote the equivalence class of $(z, g_z(\mathbf{x}))$ in $k\mathbf{S}^n = \{1,\dots,k\} \times \mathbf{S}^n/\approx$.

For every integer $d \in \mathbb{Z}$, let $h_d : \mathbf{S}^n \to \mathbf{S}^n$ be a map with Brouwer degree $d$ that leaves north and south poles fixed. Then, for any sequence $(d_1,\dots,d_k)$ of integers, the map $(z,\mathbf{x}) \mapsto (z, h_{d_z}(\mathbf{x}))$ of $\{1,\dots,k\} \times \mathbf{S}^n$ to itself induces a self-map of $k\mathbf{S}^n$, which we denote $h_{d_1,\dots,d_k} : k\mathbf{S}^n \to k\mathbf{S}^n$.

Now let $\mathbb{Z}\mathbf{S}^n = \mathbb{Z} \times \mathbf{S}^n/\approx$, where $(z,N) \approx (z+1,S)$ for all $z \in \mathbb{Z}$. The inclusion $\{1,\dots,k\} \times \mathbf{S}^n \subset \mathbb{Z} \times \mathbf{S}^n$ induces an injection $i : k\mathbf{S}^n \hookrightarrow \mathbb{Z}\mathbf{S}^n$.

Let $\mathbf{S}^n/\{S,N\}$ denote the space formed from $\mathbf{S}^n$ by identifying the north and south poles. Then the projection $(z,\mathbf{x}) \mapsto \mathbf{x}$ of $\mathbb{Z} \times \mathbf{S}^n$ onto $\mathbf{S}^n$ induces a map $\hat{q} : \mathbb{Z}\mathbf{S}^n \to \mathbf{S}^n/\{S,N\}$, which is easily seen to be a covering; in fact, $\hat{q}$ is the universal covering of $\mathbf{S}^n/\{S,N\}$.

Let $\hat{f} : \mathbf{S}^n \to \mathbb{Z}\mathbf{S}^n$ be the composition $\hat{f} = i \circ h_{d_1,\dots,d_k} \circ g$, and let $f = \hat{q} \circ \hat{f}$. So $\hat{f}$ is a lift of $f$ through $\hat{q}$. Choose a point $y_0 \in \mathbb{Z}/\{S,N\} - \{S,N\}$ and denote the points in $\hat{q}^{-1}(y_0)$ by $\hat{y}_z$, where $\hat{y}_z \in \{z\} \times \mathbf{S}^n$ for each $z \in \mathbb{Z}$. The picture for $k = 3$ is shown in Figure 1.1.

Since both $\mathbf{S}^n$ and $\mathbb{Z}\mathbf{S}^n$ are simply connected, then the images of their fundamental groups under $f$ and $\widehat{q}$, respectively, are (trivially) equal, so $\widehat{q}$ is a *Hopf covering* and $\widehat{f}$ is a *Hopf lift* for $f$. (Terms in italics are from Nielsen root theory, and are reviewed or defined in Section 3 below.) Thus, each of the sets $\widehat{f}^{-1}(\widehat{y}_z)$ is either empty or a Nielsen root class of $f$ at $y_0$. Assume $d_z \neq 0$ for $z = 1,\dots,\ell \leq k$, and $d_z = 0$ for $z = \ell+1,\dots,k$. The *integer root index* $\lambda(f,\widehat{f}^{-1}(\widehat{y}_z))$ for the Nielsen class $f^{-1}(\widehat{y}_z)$ is $d_z$, so each of the classes $\widehat{f}^{-1}(\widehat{y}_z)$ for $1 \leq z \leq \ell$ is essential. For other values of $z$, either $\widehat{f}^{-1}(\widehat{y}_z) = \emptyset$ or $\ell < z \leq k$ and $d_z = 0$. In this last case there is a homotopy, constant on the north and south poles, of $h_{d_z} : \mathbf{S}^n \to \mathbf{S}^n$ to a map $h'$ such that $h'^{-1}(\widehat{y}_0) = \emptyset$. This homotopy can be used to define a homotopy of $\widehat{f}$ to a map $\widehat{f}'$ such that $\widehat{f}'^{-1}(\widehat{y}_z) = \emptyset$. Thus $\widehat{f}^{-1}(\widehat{y}_z)$ is inessential (or empty). It follows that the *Nielsen root number* of $f$ is $\mathrm{NR}(f,y_0) = \ell$. Since $\mathbf{S}^n$ is compact, this is also the *proper Nielsen root number* of $f$, $\mathrm{PNR}(f,y_0)$.

The index for all of $\mathbf{S}^n$ is $\lambda(f,\mathbf{S}^n) = d_1 + \cdots + d_k$. The *multiplicity* of $\widehat{f}^{-1}(\widehat{y}_z)$ is $\mathrm{mult}(f,\widehat{f}^{-1}(\widehat{y}_z),y_0) = |d_z|$, and the *absolute degree of $f$ at $y_0$* is the sum of the multiplicities: $\mathcal{A}(f,y_0) = |d_1| + \cdots + |d_k|$. Every map homotopic to $f$ has at least $\mathrm{NR}(f,y_0) = \ell$ roots at $y_0$. On the other hand, from what we know of maps of spheres, for every $d \neq 0$, there is a map homotopic to $h_d : \mathbf{S}^n \to \mathbf{S}^n$ by a homotopy constant at $S$ and $N$ that has only one root at $\widehat{y}_0$. These maps may be used to define a map homotopic to $f$ that has exactly $\ell = \mathrm{NR}(f,y_0) = \mathrm{PNR}(f,y_0)$ roots. We will see that every map homotopic to $f$ and *transverse* to $y_0$ has at least $\mathcal{A}(f,y_0) = |d_1| + \cdots + |d_k|$ roots. On the other hand, each map $h_d : \mathbf{S}^n \to \mathbf{S}^n$ is homotopic to a map, by a homotopy constant on $S$ and $N$, that is transverse to $\widehat{y}_0$ and has exactly $|d|$ roots. These maps may be used to define a map homotopic to $f$ and transverse to $y_0$ that has exactly $\sum_{z=1}^{k} |d_z| = \mathcal{A}(f,y_0)$ roots.

The *root Reidemeister number* $\mathrm{RR}(f)$ of $f$ is the index in the fundamental group of $\mathbf{S}^n/\{S,N\}$ of the image of the fundamental group of $\mathbf{S}^n$ under $f$. In this example $\mathbf{S}^n$ is simply connected and $\mathbf{S}^n/\{S,N\}$ has infinite cyclic fundamental group, so $\mathrm{RR}(f) = \infty$.

This example is of particular interest because, like maps of closed $n$-manifolds with $n > 2$, $\mathrm{NR}(f,y_0)$ is a *sharp* lower bound on the number of roots of $f'$ at $y_0$ over all maps $f'$ homotopic to $f$, and $\mathcal{A}(f,y_0)$ is a *sharp* lower bound on the number of roots of $f'$ at $y_0$ over all maps $f'$ homotopic to $f$ and transverse to $y_0$. But, unlike maps of manifolds, the root classes may have different multiplicities and some may be inessential while others are essential. Also, in this example, $\mathrm{RR}(f) > \mathrm{NR}(f,y_0)$, whereas for maps of manifolds, $\mathrm{RR}(f) = \mathrm{NR}(f,y_0)$ whenever $\mathrm{NR}(f,y_0) > 0$ (see, e.g., [1, Corollary 3.21]).

The rest of this paper is organized as follows. The next section establishes some notation and conventions, reviews proper maps and homotopies, transversality of a map to a point, and concepts related to the orientation of a manifold. In Section 3, we review basic definitions and results from Nielsen root theory and modify them for the case of proper maps. By the end of Section 3 we will have completed the proof of the first paragraphs in Theorems 1.1 and 1.2: we will have shown that $\mathcal{A}(f,y_0)$ is a lower bound on the number of roots of $f$ for proper maps transverse to $y_0$, and that $\mathrm{PNR}(f,y_0)$ is a lower bound on the number of roots for proper maps $f$—and they are both invariant under proper homotopy. Section 4 is devoted to the problem of isolating roots. In particular, we show that if $f : X \to Y$ is a proper map of a connected $n$-manifold $X$ into a Hausdorff space $Y$

and $y_0 \in Y$ has a neighborhood homeomorphic to Euclidean $n$-space $\mathbb{R}^n$, then there is a map properly homotopic to $f$ and transverse to $y_0$. The last section completes the proofs of Theorems 1.1 and 1.2.

## 2. Preliminaries

**2.1. Miscellaneous conventions and notation.** All spaces are assumed Hausdorff. We say a space is *well connected* if it is connected, locally path-connected, and semilocally simply connected.

Euclidean $n$-space is denoted by $\mathbb{R}^n$, the closed unit ball in $\mathbb{R}^n$ by $\mathbf{B}^n$, the unit interval by $I$, the integers by $\mathbb{Z}$, and the integers modulo 2 by $\mathbb{Z}/2\mathbb{Z}$. For a class $\xi \in \mathbb{Z}/2\mathbb{Z}$, we write $|\xi| = 1$ if $1 \in \xi$, and $|\xi| = 0$ otherwise. Notice that as is the case for ordinary absolute value, $|\xi + \xi'| \le |\xi| + |\xi'|$.

If $S$ is a set, then $\operatorname{card} S$ denotes its cardinality. If $\phi : G \to H$ is an isomorphism, we sometimes write $\phi : G \approx\!\!> H$.

A *path* $A$ in a space $X$ is a map $A : I \to X$. If $x$ is a point in the space $X$, then we also use $x$ to denote the constant path $t \mapsto x$. We use $[A]$ to denote the fixed-endpoint homotopy class of $A$.

A subspace $B \subset X$ of a space $X$ is an *n-ball* if there is a homeomorphism $\phi : \mathbf{B}^n \to B$. A subspace $E \subset X$ is $n$-Euclidean if there is a homeomorphism $\psi : \mathbb{R}^n \to E$.

A *homotopy* $\{h_t : X \to Y \mid t \in I\}$ is a family of maps $h_t : X \to Y$ indexed by $I$ such that the function $(x,t) \mapsto h_t(x)$ is continuous from $X \times I$ to $Y$. We usually denote it more simply by $\{h_t : X \to Y\}$ or even more simply by $\{h_t\}$. The homotopy $\{h_t : X \to Y\}$ is *constant on* $A \subset X$ if $h_t(x) = h_0(x)$ for all $x \in A$ and $t \in I$. It is *constant off of* $A$ if it is constant on $X - A$.

We say that a map $f : (X,A) \to (Y,B)$ *defines* a map $f' : (X',A') \to (Y',B')$ if the two maps are the same except for modifications of domain and codomain—more precisely, if $X' \subset X$, $f(X') \subset Y'$, $f(A') \subset B'$, and $f'(x) = f(x)$ for all $x \in X'$.

If $f : X \to Y$, $\bar{q} : \bar{Y} \to Y$, and $\bar{f} : X \to \bar{Y}$ are maps and $f = \bar{q} \circ \bar{f}$, then $\bar{f}$ is a *lift of $f$ through $p$*.

An inclusion $e : (X - U, B - U) \subset (X,B)$ is an excision in the sense of Eilenberg and Steenrod's axiomatics [5, page 12] if $U$ is open in $X$ and $\operatorname{Cl} U \subset \operatorname{int} B$. Letting $N = X - U$ and $A = X - B$, this is equivalent to saying that $e : (N, N - A) \subset (X, X - A)$ is an excision if $N$ is a closed neighborhood of $\operatorname{Cl} A$. The excision axiom states that $e$ induces homology isomorphisms in all dimensions. Note, however, that if $X$ is normal, as it will be in all our applications, and $N$ is *any* neighborhood of $\operatorname{Cl} A$, then we may find a closed neighborhood $C$ of $\operatorname{Cl} A$ such that $C \subset \operatorname{int} N$. Then the inclusions $e' : (C, C - A) \subset (N, N - A)$ and $e \circ e' : (C, C - A) \subset (X, X - A)$ are both excisions in the above sense and therefore induce homology isomorphisms. It follows that $e : (N, N - A) \subset (X, X - A)$ also induces homology isomorphisms. Therefore, we adopt a somewhat weaker (and more usual) definition of excision: an inclusion $e : (N, N - A) \subset (X, X - A)$ is an *excision* if $N$ is a neighborhood of $\operatorname{Cl} A$. What we call an excision is what Eilenberg and Steenrod call an "excision of type $(E_2)$." Using singular homology, such inclusions induce homology isomorphisms regardless of normality [5, pages 267-268].

**2.2. Proper maps.** A map $f : X \to Y$ is *proper* if $f^{-1}(C)$ is compact whenever $C$ is compact. A homotopy $\{f_t : X \to Y\}$ is *proper* if the map $X \times I \to Y$ given by $(x,t) \mapsto f_t(x)$ is proper. Here are a few elementary results about proper maps and homotopies that we will need.

Theorem 2.1. *In order that a homotopy $\{f_t : X \to Y\}$ be proper it is necessary and sufficient that $\bigcup_{t\in I} f_t^{-1}(C)$ be compact whenever $C \subset Y$ is compact.*

*Proof.* Suppose first that $\{f_t\}$ is proper and that $C \subset Y$ is compact. Then $\{(x,t) \in X \times I \mid f_t(x) \in C\}$ is a compact subset of $X \times I$, and therefore its image under the projection $X \times I \to X$ is compact. But that image is precisely $\bigcup_{t\in I} f_t^{-1}(C)$.

Now suppose that $\bigcup_{t\in I} f_t^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Let $C \subset Y$ be compact. Then $\bigcup_{t\in I} f_t^{-1}(C)$, and therefore $(\bigcup_{t\in I} f_t^{-1}(C)) \times I$, is compact. Now $C$ is compact and therefore closed in $Y$. Since $f_t(x)$ is continuous in $(x,t)$, it follows that $\{(x,t) \in X \times I \mid f_t(x) \in C\}$ is closed. But $\{(x,t) \in X \times I \mid f_t(x) \in C\}$ is easily seen to be a subset of $(\bigcup_{t\in I} f_t^{-1}(C)) \times I$, so as a closed subset of a compact set it is also compact. This shows that $\{f_t\}$ is proper. □

Theorem 2.2. *Suppose $\{f_t : X \to Y\}$ is a homotopy, $f : X \to Y$ is proper, $K \subset X$ is compact, and that $\{f_t\}$ is constant at $f$ off of $K$. Then $\{f_t\}$ is proper.*

*Proof.* Let $C \subset Y$ be compact. Since $\{f_t\}$ is constant at $f$ off of $K$ it is easy to see that $\bigcup_{t\in I} f_t^{-1}(C) = (\bigcup_{t\in I}(f_t|K)^{-1}(C)) \cup f^{-1}(C)$. Since $K$ is compact, then $\{f_t|K\}$ is proper, so by Theorem 2.1 $\bigcup_{t\in I}(f_t|K)^{-1}(C)$ is compact. Since $f$ is proper, $f^{-1}(C)$ is compact. Thus their union $\bigcup_{t\in I} f_t^{-1}(C)$ is compact, so by Theorem 2.1 $\{f_t\}$ is proper. □

Theorem 2.3. *Suppose that $\bar{f} : X \to \bar{Y}$ is a lift of a map $f : X \to Y$ through a covering $\bar{q} : \bar{Y} \to Y$. Then $f$ is proper if and only if $\bar{f}$ is proper.*

Note we do not require $\bar{q}$ to be proper.

*Proof.* Suppose first that $f$ is proper, and let $\bar{C} \subset \bar{Y}$ be compact. Then $\bar{q}(\bar{C})$ is also compact, so since $f$ is proper, then $f^{-1}(\bar{q}(\bar{C}))$ is compact. But it is easily seen that $\bar{f}^{-1}(\bar{C}) \subset f^{-1}(\bar{q}(\bar{C}))$, so, as a closed subset of a compact space, it is compact. Thus $\bar{f}$ is proper.

Now suppose $\bar{f}$ is proper. Let $C \subset Y$ be compact. Then $C$ has a finite covering $\mathcal{K}$ by compact sets each of which is evenly covered by $\bar{q}$. For each $K \in \mathcal{K}$, let $\bar{K}$ be a set mapped homeomorphically onto $K$ by $\bar{q}$. Then each such $\bar{K}$ is compact, so, since $\bar{f}$ is proper, $\bar{f}^{-1}(\bar{K})$ is also compact. Thus $\bigcup_{K\in\mathcal{K}} \bar{f}^{-1}(\bar{K})$ is a finite union of compact sets and is therefore compact. It follows that $f^{-1}(C)$, as a closed subset of the compact set $\bigcup_{K\in\mathcal{K}} \bar{f}^{-1}(\bar{K})$, is compact. Thus $f$ is proper. □

Since a proper homotopy from a space $X$ is a proper map from the space $X \times I$, we have the following corollary.

Corollary 2.4. *Suppose that $\{\bar{f}_t : X \to \bar{Y}\}$ is a lift of a homotopy $\{f_t : X \to Y\}$ through a covering $\bar{q} : \bar{Y} \to Y$. Then $\{f_t\}$ is proper if and only if $\{\bar{f}_t\}$ is proper.*

We leave the proof of the following to the reader.

Theorem 2.5. *A covering map is proper if and only if it is finite sheeted. The composition of proper maps is proper.*

**2.3. Transversality, local homeomorphisms, and isolated roots.** Let $f : X \to Y$ be a map and $y_0 \in Y$. A *root* of $f$ at $y_0$ is a point $x \in X$ such that $f(x) = y_0$. The root $x$ is *isolated* if it has a neighborhood $N$ that contains no other root of $f$ at $y_0$. If all the roots of $f$ are isolated, then $f^{-1}(y_0)$ is discrete, so if $f$ is also proper, then $f^{-1}(y_0)$ is compact and therefore finite.

The map $f$ is a *local homeomorphism* at $x$ if $x_0$ has a neighborhood that is mapped homeomorphically onto a neighborhood of $f(x)$. Clearly, if $f$ is a local homeomorphism at a root $x$, then $x$ is isolated.

A map $f : X \to Y$ is *transverse to* $y_0 \in Y$ if $y_0$ has a neighborhood $N$ for which there is a family $\{N_x \mid x \in f^{-1}(y_0)\}$ of mutually disjoint subsets of $X$ indexed by $f^{-1}(y_0)$ such that $f^{-1}(N) = \bigsqcup_{x \in f^{-1}(y_0)} N_x$, each $N_x$ is a neighborhood of $x \in f^{-1}(y_0)$, and $f$ maps each $N_x$ homeomorphically onto $N$.

The case where $f^{-1}(y_0) = \varnothing$ requires some clarification. If $y_0 \notin \operatorname{Cl} f(X)$, then $y_0$ has a neighborhood $N$ such that $f^{-1}(N)$ is empty and therefore the union of the empty family of sets. Since members of the empty family have (vacuously) any property we want, including being homeomorphic to $N$, it will be convenient to agree that in this case $f$ is (vacuously) transverse to $y_0$. On the other hand, if $y_0 \notin f(X)$, but $y_0 \in \operatorname{Bd} f(X)$, then $f^{-1}(N)$ is nonempty for every neighborhood $N$ of $y_0$, but no subset of $f^{-1}(N)$ is mapped onto $N$ by $f$, so $f$ cannot be transverse to $y_0$.

If $f$ is transverse to $y_0$, then $f$ is a local homeomorphism at each $x \in f^{-1}(y_0)$. The converse is not true. For example, let $f : (-2\pi, 2\pi) \to S^1$ be the exponential map $f(t) = \exp(it)$ from the open interval $(-2\pi, 2\pi)$ to the unit circle in the complex plane. Then $f$ is not transverse to $1 \in S^1$. However, the converse is true under quite general circumstances provided that $f$ is proper.

Theorem 2.6. *Suppose $f : X \to Y$ is a proper map of (Hausdorff) spaces, $y_0 \in Y$ has a compact neighborhood $K \subset Y$, and $f$ is a local homeomorphism at each $x \in f^{-1}(y_0)$. Then $f$ is transverse to $y_0$.*

This theorem with the stronger hypothesis that $X$ and $Y$ are manifolds of the same dimension appears as [2, Lemma 7.5]. However, we will need it now for nonmanifold $Y$.

*Proof.* Since $f$ is proper, then $f^{-1}(K)$ is compact and $f^{-1}(y_0)$ is finite. It is not hard to find an open neighborhood $U \subset K$ of $y_0$, and a family $\{U_x \mid x \in f^{-1}(y_0)\}$ of mutually disjoint open sets $U_x$ such that for each $x \in f^{-1}(y_0)$, $x \in U_x$ and $f$ takes $U_x$ homeomorphically onto $U$. The difficulty is that even though $\bigsqcup_x U_x \subset f^{-1}(U)$, in general, $\bigsqcup_x U_x \neq f^{-1}(U)$. To remedy this, let $\mathscr{C}$ be the family of all closed neighborhoods $C \subset U$ of $y_0$. Since $K$ is compact Hausdorff, it is not hard to show that $\mathscr{C} \neq \varnothing$ and $\bigcap_{C \in \mathscr{C}} C = y_0$. Thus, since $f^{-1}(y_0) \subset \bigsqcup_x U_x$, we have $\bigcap_{C \in \mathscr{C}}(f^{-1}(C) - \bigsqcup_x U_x) = f^{-1}(\bigcap_{C \in \mathscr{C}} C) - \bigsqcup_x U_x = \varnothing$. Since $f^{-1}(K)$ is compact, this shows that the family $\{(f^{-1}(C) - \bigsqcup_x U_x) \mid C \in \mathscr{C}\}$ cannot have the finite intersection property, so there is a finite subfamily $\mathscr{C}' \subset \mathscr{C}$ such that $\bigcap_{C \in \mathscr{C}'}(f^{-1}(C) - \bigsqcup_x U_x) = \varnothing$, and therefore $f^{-1}(\bigcap_{C \in \mathscr{C}'} C) \subset \bigsqcup_x U_x$. It follows that $\bigcap_{C \in \mathscr{C}'} C$ is a neighborhood of $y_0$ such that $f^{-1}(\bigcap_{C \in \mathscr{C}'} C) = \bigsqcup_x (U_x \cap f^{-1}(\bigcap_{C \in \mathscr{C}'} C))$ and for each $x \in f^{-1}(y_0)$, $f$ maps the neighborhood $U_x \cap f^{-1}(\bigcap_{C \in \mathscr{C}'} C)$ of $x$ homeomorphically onto the neighborhood $\bigcap_{C \in \mathscr{C}'} C$ of $y_0$. Hence, $f$ is transverse to $y_0$. □

### 2.4. Orientation

*Definition 2.7.* A topological space $Y$ is *locally n-Euclidean at* $y_0 \in Y$ if $y_0$ has a neighborhood $E$ homeomorphic to Euclidean $n$-space $\mathbb{R}^n$. If $Y$ is $n$-Euclidean at $y_0$, then by excision $H_p(Y,Y-y_0;\mathbb{Z}) \approx H_p(E,E-y_0;\mathbb{Z})$ is trivial for $p \neq n$ and infinite cyclic for $p = n$. A generator of $H_n(Y,Y-y_0;\mathbb{Z})$ is called a *local orientation of* $Y$ *at* $y_0$.

Throughout the rest of this subsection, let $X$ be an $n$-manifold, that is, a paracompact (and Hausdorff) space that is $n$-Euclidean at each of its points. Then an orientation of $X$ is, roughly speaking, a continuous choice of local orientation at each point $x \in X$. In order to make this definition precise, we follow Dold [4, pages 251–259] and use the *orientation bundle* $p^{\mathbb{OB}} : \mathbb{OB}(X) \to X$, the *orientation manifold* $\widetilde{X}$, and the *orientation covering* $\widetilde{p} : \widetilde{X} \to X$ of $X$. The following description also draws on [2, pages 5–8]. (However, in both of these references, $\widetilde{X}$ is used to denote what we are now calling $\mathbb{OB}(X)$, and $\widetilde{X}(1)$ is used to denote the orientation manifold, which we will now denote more simply by $\widetilde{X}$.)

As a set, $\mathbb{OB}(X) = \bigcup_{x\in X} H_n(X,X-x;\mathbb{Z})$, and as a function, $p^{\mathbb{OB}}(\xi) = x$ for all $\xi \in H_n(X,X-x;\mathbb{Z})$ and $x \in X$. To describe the topology on $X^{\mathbb{OB}}$, let $U \subset X$ be the interior of an $n$-ball in $X$. Then, for any $x \in U$, $X - U$ is a deformation retract of $X - x$, so the inclusion $i_{Ux} : (X,X-U) \subset (X,X-x)$ induces an isomorphism $i_{Uxn} : H_n(X,X-U;\mathbb{Z}) \xrightarrow{\approx} H_n(X,X-x;\mathbb{Z})$. Therefore, we may define a bijection $\phi_U : U \times H_n(X,X-U;\mathbb{Z}) \to (p^{\mathbb{OB}})^{-1}(U)$ by $\phi(x,\xi) = i_{Uxn}(\xi)$. Give $U$ the subspace topology, $H_n(X,X-U;\mathbb{Z})$ the discrete topology, and $U \times H_n(X,X-U;\mathbb{Z})$ the product topology. Then the topology on $\mathbb{OB}(X)$ is characterized by the property that $\phi_U$ is a homeomorphism for every such $U \subset X$. With this topology, $p^{\mathbb{OB}} : \mathbb{OB}(X) \to X$ is a covering.

For each $x \in X$, the group $H_n(X,X-x;\mathbb{Z})$ has two possible generators; let $\widetilde{X}$ denote the subspace of $\mathbb{OB}(X)$ consisting of all these generators, two for each $x \in X$, and let $\widetilde{p} : \widetilde{X} \to X$ be the restriction of $p^{\mathbb{OB}}$ to $\widetilde{X}$. Then $\widetilde{p} : \widetilde{X} \to X$ is a two-sheeted covering called the *orientation covering* of $X$. The space $\widetilde{X}$ is an $n$-manifold called the *orientation manifold* of $X$. An *orientation* of $X$ is a section $s_X : X \to \widetilde{X}$ of $\widetilde{p}$. The manifold $X$ is *orientable* if it has an orientation, otherwise it is *nonorientable*. A manifold $X$, together with an orientation $s_X : X \to \widetilde{X}$, is an *oriented* manifold.

The orientation manifold of $\widetilde{X}$ is $\widetilde{\widetilde{X}}$. It has a *canonical orientation* $s_{\widetilde{X}} : \widetilde{X} \to \widetilde{\widetilde{X}}$ defined as follows: let $\widetilde{x} \in \widetilde{X}$, $x = \widetilde{p}(\widetilde{x})$, let $U$ be an evenly covered connected open neighborhood of $x$, and $\widetilde{U}$ the component of $\widetilde{p}^{-1}(U)$ containing $\widetilde{x}$. Construct the diagram

$$(\widetilde{X},\widetilde{X}-\widetilde{x}) \overset{\widetilde{e}}{\supset} (\widetilde{U},\widetilde{U}-\widetilde{x}) \xrightarrow{\widetilde{p}_U} (U,U-x) \overset{e}{\subset} (X,X-x), \tag{2.1}$$

where $\widetilde{p}_U$ is defined by $\widetilde{p}$. The inclusions are excisions and $\widetilde{p}_U$ is a homeomorphism, so we may define $s_{\widetilde{X}}(\widetilde{x}) = \widetilde{e}_n \circ \widetilde{p}_{Un}^{-1} \circ e_n^{-1}(\widetilde{x})$, where $\widetilde{e}_n$, $\widetilde{p}_{Un}$, and $e_n$ are the induced $n$-dimensional homology isomorphisms. Thus, the orientation manifold is always orientable.

If $s_X : X \to \widetilde{X}$ is an orientation, then so is $-s_X$, and both $s_X$ and $-s_X$ are homeomorphisms onto their images. Thus, if $X$ is connected, then $X$ is nonorientable if and only if $\widetilde{X}$ is connected.

Suppose $U \subset X$ an open subset of the $n$-manifold $X$. Then $U$ is also an $n$-manifold. For each $x \in U$, the excision $e_x : (U, U - x) \subset (X, X - x)$ induces an isomorphism $e_{xn} : H_n(U, U - x; \mathbb{Z}) \approx\!> H_n(X, X - x; Z)$. If $s_X : X \to \widetilde{X}$ is an orientation of $X$, then we may define an orientation $s_U : U \to \widetilde{U}$ by $s_U(x) = e_{xn}^{-1}(s_X(x))$. The orientation $s_U$ is called, with only a slight abuse of terminology, the *restriction* of $s_X$ to $U$.

Let $h : X \to X$ be a homeomorphism. Then $h$ induces a homeomorphism $\widetilde{h} : \widetilde{X} \to \widetilde{X}$, given by $\widetilde{h}(\widetilde{x}) = h_{xn}(\widetilde{x})$, where for each $x \in X$, $h_x : (X, X - x) \to (X, X - h(x))$ is defined by $h$ and $h_{xn}$ is the induced homology isomorphism. Now suppose $X$ has an orientation $s_X : X \to \widetilde{X}$. If $\widetilde{h} \circ s_X(x) = s_X \circ h(x)$, for all $x \in X$, then $h$ is *orientation-preserving*. If $\widetilde{h} \circ s_X(x) = -s_X \circ h(x)$ for all $x \in X$, then $h$ is *orientation-reversing*. If $X$ is connected, then these are the only possibilities. As an important example, it is easy to show (using the canonical orientation $s_{\widetilde{X}}$ defined above) that the map $\widetilde{x} \mapsto -\widetilde{x}$ is always an orientation-reversing homeomorphism of $\widetilde{X}$.

Let $A$ be a loop in an $n$-manifold $X$, and let $\widetilde{A}$ be a lift of $A$ to a path in $\widetilde{X}$. Then either $\widetilde{A}(1) = \widetilde{A}(0) \in H_n(X, X - A(0))$, so $\widetilde{A}$ is a loop, or $\widetilde{A}(1) = -\widetilde{A}(0)$, so $\widetilde{A}$ is not a loop. In the first case we say that $A$ is *orientation-preserving*, and in the second case, $A$ is *orientation-reversing*. It is easy to show that $X$ is orientable if and only if all of its loops are orientation-preserving.

*Definition 2.8.* Suppose $f : X \to Y$ is a map. Then $f$ is called *orientable* if there is no orientation-reversing loop $A$ in $X$ such that $f \circ A$ is contractible. It is called *nonorientable* if $f \circ A$ is contractible for some orientation-reversing loop $A$ in $X$.

Note that this definition agrees with the usual definition of map orientability [3, Definition 2.1] in the case where $Y$ is also an $n$-manifold, but requires only $X$ to be a manifold—$Y$ can be arbitrary.

Let $K \subset X$ be a compact subset of an oriented $n$-manifold $X$ with orientation $s_X : X \to \widetilde{X}$. Then there is an unique element $o_K \in H_n(X, X - K)$ such that for every $x \in K$ the homomorphism $H_n(X, X - K; \mathbb{Z}) \to H_n(X, X - x; \mathbb{Z})$ induced by the inclusion takes $o_K$ to $s_X(x)$. The element $o_K$ is called the *fundamental class around* $K$.

Let $f : X \to Y$ be a map from an oriented $n$-manifold $X$ to an oriented $n$-manifold $Y$ with orientation $s_Y : Y \to \widetilde{Y}$, and suppose that $f^{-1}(y_0)$ is compact for some $y_0 \in Y$. Then $f$ defines a map $f' : (X, X - f^{-1}(y_0)) \to (Y, Y - y_0)$ that induces a homomorphism $f'_n : H_n(X, X - f^{-1}(y_0); \mathbb{Z}) \to H_n(Y, Y - y_0; \mathbb{Z})$. The *degree of $f$ over $y_0$* is the integer $\deg_{y_0}(f)$ defined by the equation $f'_n(o_{f^{-1}(y_0)}) = \deg_{y_0}(f) s_Y(y_0)$. If $Y$ is connected and $f$ proper, then $\deg_{y_0} f$ is independent of the choice of $y_0$ and is called the *degree of $f$* and denoted by $\deg f$. This is a direct generalization of the notion of Brouwer degree for maps of connected compact oriented $n$-manifolds.

## 3. Elementary Nielsen root theory for proper maps

This section has three purposes. First, it serves as a summary of the elementary Nielsen root theory that we will need in the sequel. A more leisurely treatment of that theory, together with proofs of the assertions made here without proof, may be found in [1].

The second purpose is to modify that theory for the case of proper maps; in particular, to define "proper essentiality," the "proper Nielsen root number," and an "integer proper root index" for proper maps $f : X \to Y$ of an $n$-manifold into a space $Y$ that is $n$-Euclidean at a point $y_0 \in Y$. The third is to extend the definitions of "multiplicity" of a root class and "absolute degree" of a proper map $f : X \to Y$ of $n$-manifolds to situations in which $Y$ is $n$-Euclidean at $y_0$ but not necessarily a manifold.

**3.1. Nielsen root classes and the (proper) Nielsen root number.** Let $f : X \to Y$ be a map and $y_0 \in Y$. Two roots $x$ and $x'$ are *Nielsen root equivalent* if there is a path $A$ in $X$ from $x$ to $x'$ such that $[f \circ A] = [y_0]$. This is indeed an equivalence relation, and an equivalence class is called a *Nielsen root class of* $f$ *at* $y_0$, although this will frequently be shortened to *Nielsen class* or *Nielsen class of* $f$, and so forth. The set of Nielsen root classes of $f$ at $y_0$ is denoted by $f^{-1}(y_0)/N$.

Now let $\{f_t : X \to Y\}$ be a homotopy and $y_0 \in Y$. A root $x_0$ of $f_0$ at $y_0$ is $\{f_t\}$*-related* to a root $x_1$ of $f_1$ at $y_0$ if there is a path $A$ in $X$ from $x_0$ to $x_1$ such that the path $\{f_t(A(t))\}$ is fixed-endpoint-homotopic to $y_0$. If one root in a Nielsen class $\alpha_0$ of $f_0$ is $\{f_t\}$-related to a root in a Nielsen class $\alpha_1$ of $f_1$, then every root in $\alpha_0$ is $\{f_t\}$-related to every root in $\alpha_1$. In this case we say that $\alpha_0$ *is* $\{f_t\}$*-related to* $\alpha_1$. The $\{f_t\}$ relation among root classes is one-to-one in the sense that each root class of $f_0$ is $\{f_t\}$-related to at most one root class of $f_1$ and each root class of $f_1$ has at most one root class of $f_0$ related to it.

A root class $\alpha_0$ of $f : X \to Y$ at $y_0 \in Y$ is called *essential* if given any homotopy $\{h_t : X \to Y\}$ with $h_0 = f$, there is a root class $\alpha_1$ of $h_1$ at $y_0$ to which $\alpha_0$ is related. The number of essential root classes of a map $f : X \to Y$ at $y_0$ is the *Nielsen root number of* $f$ *at* $y_0$ and is denoted by $\mathrm{NR}(f, y_0)$. We modify these definitions for proper maps as follows.

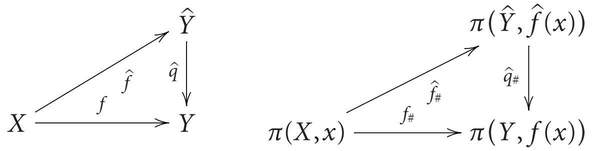
*Definition 3.1.* A root class $\alpha_0$ of a proper map $f : X \to Y$ at $y_0 \in Y$ is called *properly essential* if given any proper homotopy $\{h_t : X \to Y\}$ with $h_0 = f$, there is a root class $\alpha_1$ of $h_1$ at $y_0$ to which $\alpha_0$ is related. The number of properly essential root classes of a proper map $f : X \to Y$ at $y_0$ is the *proper Nielsen root number of* $f$ *at* $y_0$ and is denoted by $\mathrm{PNR}(f, y_0)$.

Clearly, every essential root class is properly essential, so $\mathrm{NR}(f, y_0) \le \mathrm{PNR}(f, y_0)$. It can happen, however, that $\mathrm{NR}(f, y_0) < \mathrm{PNR}(f, y_0)$. Later, in Example 3.11, we show that if $f$ is the identity on $\mathbb{R}^n$, then $\mathrm{PNR}(f, y_0) = 1$ but $\mathrm{NR}(f) = 0$.

The following theorem is an easy consequence of the preceding discussion.

Theorem 3.2. *Let* $f : X \to Y$ *be a map and let* $y_0 \in Y$. *Then* $\mathrm{NR}(f, y_0)$ *is a homotopy invariant of* $f$ *and* $\mathrm{NR}(f, y_0) \le \operatorname{card} f^{-1}(y_0)$. *If* $f$ *is proper, then* $\mathrm{PNR}(f, y_0)$ *is a proper homotopy invariant of* $f$ *and* $\mathrm{PNR}(f, y_0) \le \operatorname{card} f^{-1}(y_0)$.

**3.2. Hopf coverings and lifts.** Let $f : X \to Y$ be a map of well-connected spaces, and let $x \in X$. Then, from covering space theory, there is a covering $\hat{q} : \hat{Y} \to Y$ such that for any $\hat{y} \in \hat{q}^{-1}(f(x))$ we have $\operatorname{im} \hat{q}_\# = \operatorname{im} f_\#$, where $f_\# : \pi(X, x) \to \pi(Y, f(x))$ and $\hat{q}_\# : \pi(\hat{Y}, \hat{y}) \to \pi(Y, f(x))$ are the induced fundamental group homomorphisms. Moreover, there is a lift $\hat{f} : X \to \hat{Y}$ of $f$ through $\hat{q}$, and $\hat{f}_\# : \pi(X, x) \to \pi(\hat{Y}, \hat{f}(x))$ is an epimorphism. Here are the diagrams:



(3.1)

We call $\widehat{q}$ and $\widehat{f}$ a *Hopf covering* and *Hopf lift* for $f$, since Hopf was the first to exploit $\widehat{q}$ and $\widehat{f}$ in root theory. The covering $\widehat{q}$ is unique up to covering space isomorphism and does not depend upon the choice of $x \in X$. The covering $\widehat{q}$ is also a Hopf covering for any map homotopic to $f$. The lift $\widehat{f}$ is unique up to deck transformation, that is, if $\widehat{f}'$ is another lift of $f$ through $\widehat{q}$, then $\widehat{f}' = h \circ \widehat{f}$, where $h$ is a deck transformation for the covering $\widehat{q}$.

The importance of $\widehat{q}$ and $\widehat{f}$ for root theory is the following. Let $y_0 \in Y$. A nonempty subset $\alpha \subset X$ is a Nielsen root class of $f$ at $y_0$ if and only if $\alpha = \widehat{f}^{-1}(\widehat{y})$ for some $\widehat{y} \in \widehat{q}^{-1}(y_0)$. Moreover, if $\{h_t\}$ is a homotopy with $f = h_0$, then we may lift $\{h_t\}$ to a homotopy $\{\widehat{h}_t\}$ beginning at $\widehat{f}$, and a root class $\alpha_0$ of $f$ at $y_0$ is $\{h_t\}$-related to a root class $\alpha_1$ of $h_1$ if and only if $\alpha_0 = \widehat{f}^{-1}(\widehat{y})$ and $\alpha_1 = \widehat{h}_1^{-1}(\widehat{y})$ for the same $\widehat{y} \in \widehat{q}^{-1}(y_0)$. It follows that a root class $\widehat{f}^{-1}(\widehat{y})$ is essential if and only if $\widehat{h}_1^{-1}(\widehat{y}) \neq \emptyset$ for every homotopy $\{\widehat{h}_t\}$ beginning at $\widehat{f}$. Also, using Corollary 2.4, if $f$ is a proper map, then a root class $\widehat{f}^{-1}(\widehat{y})$ is properly essential if and only if $\widehat{h}_1^{-1}(\widehat{y}) \neq \emptyset$ for every proper homotopy $\{\widehat{h}_t\}$ beginning at $\widehat{f}$.

### 3.3. Admissible pairs

*Definition 3.3.* Let $X$ and $Y$ be spaces and $y_0 \in Y$. A pair $(f,A)$ is *admissible for* $X$, $Y$, $y_0$ if $f : X \to Y$ is a map, $A \subset X$, and $A$ has a closed neighborhood $C$ such that $C - A$ has no roots of $f$ at $y_0$. If, in addition, $f$ is proper, then $(f,A)$ is *properly admissible*.

The following theorem gives some important examples of (properly) admissible pairs. Its proof is easy and therefore omitted.

Theorem 3.4. *Let $f : X \to Y$ be a map and $y_0 \in Y$; then*

(1) $(f,X)$, $(f,\emptyset)$ *are admissible;*
(2) *if both $(f,A_1)$ and $(f,A_2)$ are admissible, then so are $(f,A_1 \cap A_2)$ and $(f,A_1 \cup A_2)$;*
(3) $(f,f^{-1}(y_0))$ *is admissible;*
(4) *for any Nielsen root class $\alpha$ of $f$ at $y_0$, $(f,\alpha)$ is admissible;*
(5) *if $U \subset X$ is open and* $\operatorname{Bd} U$ *has no roots of $f$ at $y_0$, then $(f,U)$ is admissible.*

*If $f$ is proper, then each of the above admissible pairs is properly admissible.*

Theorem 3.5. *Suppose $X$ is normal and $(f,A)$ is admissible for $X$, $Y$, $y_0$. Then* $\operatorname{Cl} A$ *has a neighborhood $N$ such that $N - A$ has no roots of $f$ at $y_0$. The inclusion $(N,N-A) \subset (X,X-A)$ is an excision in the sense of* Section 2.1.

*Proof.* Since $(f,A)$ is admissible, then $A$ has a closed neighborhood $C$ such that $C - A$ is root-free. Then $C$ and $(X - \operatorname{int} C) \cap f^{-1}(y_0)$ are disjoint closed sets. Hence, by normality, they have disjoint neighborhoods $N$ and $N'$, respectively. The neighborhood $N$ is the desired neighborhood of $\mathrm{Cl}A$. The fact that $(N, N-A) \subset (X, X-A)$ is an excision is immediate from Section 2.1. □

### 3.4. Proper root indices

*Definition 3.6.* Let $X$ and $Y$ be topological spaces and $y_0 \in Y$. A *(proper) root index* for $X$, $Y$, $y_0$ is a function $\omega$ from the set of (properly) admissible pairs for $X$, $Y$, $y_0$ into an abelian group satisfying the following.

(1) (Additivity) If $A \subset X$ and $A_1, \ldots, A_n$ are subsets of $A$ such that
  (a) $(f, A)$ is (properly) admissible and $(f, A_i)$ is (properly) admissible for each $i$,
  (b) $f^{-1}(y_0) \cap (A - \bigcup_i A_i) = \varnothing$,
  (c) $A_i \cap A_j = \varnothing$ for $i \neq j$,
  then $\omega(f, A) = \sum_i \omega(f, A_i)$.
(2) (Homotopy) If $\{f_t : X \to Y\}$ is a (proper) homotopy, $A$ is open in $X$, and $(f_t, A)$ is (properly) admissible for all $t \in I$, then $\omega(f_0, A) = \omega(f_1, A)$.

Theorem 3.7. *Let $\{f_t : X \to Y\}$ be a proper homotopy, let $y_0 \in Y$, let $\omega$ be a proper root index for $X$, $Y$, $y_0$, and suppose that $\alpha_0$ is a Nielsen root class of $f_0$. If $\alpha_0$ is $\{f_t\}$-related to a root class $\alpha_1$ of $f_1$ at $y_0$, then $\omega(f_0, \alpha_0) = \omega(f_1, \alpha_1)$. If $\alpha_0$ is not $\{f_t\}$-related to any root class of $f_1$ at $y_0$, then $\omega(f_0, \alpha_0) = 0$.*

*Proof.* See [1, Theorem 4.6] for a proof. Theorem 4.6 of [1] assumes that $X$ is compact. However, the proof is structured in such a way that it is still valid for noncompact $X$ provided $\{f_t\}$ is proper. □

Corollary 3.8. *Let $f : X \to Y$ be a proper map, $y_0 \in Y$, $\alpha$ a Nielsen root class of $f$ at $y_0$, and $\omega$ a proper root index for $X$, $Y$, $y_0$. Then $\omega(f, \alpha) \neq 0$ implies that $\alpha$ is properly essential.*

The following theorem allows us to construct a proper root index $\omega$ by defining $\omega(f, A)$ for properly admissible pairs $(f, A)$ for which $\mathrm{Cl}A$ is compact, and then extending it automatically to all properly admissible pairs.

Theorem 3.9. *Let $X$ and $Y$ be topological spaces and $y_0 \in Y$, and let $\omega$ be a function into an abelian group from the set of all properly admissible pairs $(f, A)$ for $X$, $Y$, $y_0$ such that $\mathrm{Cl}A$ is compact. Suppose that $\omega$ satisfies conditions (1) and (2) of Definition 3.6 whenever the sets $A$ and $A_i$ have compact closure. Then $\omega$ has a unique extension to a proper root index for $X$, $Y$, $y_0$.*

*Proof.* Let $(f, A)$ be properly admissible for $X$, $Y$, $y_0$. Then $(f, f^{-1}(y_0) \cap A)$ is properly admissible. Since $f$ is proper, then $f^{-1}(y_0)$ is compact, so $f^{-1}(y_0) \cap A$ has compact closure. Thus $\omega(f, f^{-1}(y_0) \cap A)$ is well defined, so we may define $\omega'$ by

$$\omega'(f, A) = \omega(f, f^{-1}(y_0) \cap A), \tag{3.2}$$

for every pair $(f, A)$ that is properly admissible for $X$, $Y$, $y_0$. If $\mathrm{Cl}A$ is compact, then $\omega(f, A)$ is already defined, and by additivity (with $n = 1$ and $A_1 = f^{-1}(y_0) \cap A$) we have $\omega(f, A) = \omega(f, f^{-1}(y_0) \cap A)$, so $\omega'$ is in fact an extension of $\omega$. Moreover, if $\omega'$ is to be a proper root index, then additivity demands that $\omega'(f,A) = \omega'(f, f^{-1}(y_0) \cap A) = \omega(f, f^{-1}(y_0) \cap A)$. So the extension is unique. It remains to show that $\omega'$ is a proper root index.

Additivity of $\omega'$ follows easily from the additivity of $\omega$, so we omit its proof. For homotopy, suppose that $\{f_t : X \to Y\}$ is a proper homotopy, $A$ is open in $X$, and that $(f_t, A)$ is admissible for every $t \in I$. Let $V$ be an open neighborhood of $y_0$ with compact closure and let $U = \bigcup_{t\in I} f_t^{-1}(V)$. Then, for each $t \in I$, $U$ is an open neighborhood of $f_t^{-1}(y_0)$, and therefore $f_t^{-1}(y_0) \cap \operatorname{Bd} U = \emptyset$, so $(f_t, U)$, and therefore $(f_t, U \cap A)$, is properly admissible for each $t \in I$. Also $U = \bigcup_{t\in I} f_t^{-1}(V) \subset \bigcup_{t\in I} f_t^{-1}(\operatorname{Cl} V)$, so, by Theorem 2.1, $U$, and therefore $U \cap A$, has compact closure. Thus $\omega(f_t, U \cap A)$ is well defined for all $t \in I$ and

$$\begin{aligned}\omega'(f_0,A) &= \omega(f_0, f_0^{-1}(y_0) \cap A) = \omega(f_0, A \cap U) \\ &= \omega(f_1, A \cap U) = \omega(f_0, f_0^{-1}(y_0) \cap A) = \omega'(f_1, A).\end{aligned} \tag{3.3}$$

The first and last equality follow from the definition of $\omega'$. The second equality follows from additivity of $\omega$ and the fact that $(A \cap U) - (f_0^{-1}(y_0) \cap A)$ is root-free. The third equality follows from the homotopy property for $\omega$. □

We apply this theorem for the case where $X$ is a (not necessarily compact) orientable $n$-manifold, and $Y$ is a topological space that is $n$-Euclidean at a point $y_0 \in Y$.

Theorem and Definition 3.10. *Suppose $X$ is an orientable $n$-manifold and $Y$ is a topological space that is $n$-Euclidean at $y_0 \in Y$. Let $s_X : X \to \widetilde{X}$ be an orientation of $X$ and let $\nu \in H_n(Y, Y - y_0; Z)$ be a local orientation of $Y$ at $y_0$. Define an integer-valued proper root index (relative to these orientations) $\lambda$ for $X$, $Y$, $y_0$ as follows.*

*Let $(f,A)$ be properly admissible for $X$, $Y$, $y_0$ with* Cl$A$ *compact. Let $N \subset X$ be any neighborhood of* Cl$A$ *such that $N - A$ is root-free, and let $K \subset X$ be any compact set containing $A$. Let $o_K \in H_n(X, X - K)$ be the fundamental class of $X$ around $K$ (relative to the orientation $s_X$). Construct the diagram*

$$(X, X-K) \overset{i_K}{\subset} (X, X-A) \overset{e}{\supset} (N, N-A) \xrightarrow{f'} (Y, Y-y_0), \tag{3.4}$$

*where $f'$ is the map defined by $f$. Then $e$ is an excision and therefore induces homology isomorphisms in all dimensions, so there exists a homomorphism $f'_n \circ e_n^{-1} \circ i_{Kn} : H_n(X, X - K; \mathbb{Z}) \to H_n(Y, Y - y_0; \mathbb{Z})$. Define the integer $\lambda(f,A)$ by*

$$f'_n \circ e_n^{-1} \circ i_{Kn}(o_K) = \lambda(f,A)\nu. \tag{3.5}$$

*Then $\lambda(f,A)$ is independent of the choice of $K$ and $N$—subject only to the conditions that $N$ be a neighborhood of* Cl$A$ *and that $K$ be a compact set containing $A$. Moreover the integer-valued function $\lambda$, defined on the set of all properly admissible pairs $(f,A)$ for which $A$ has compact closure, extends uniquely to an integer-valued root index for $X$, $Y$, $y_0$ which will be called* the integer root index for $X$, $Y$, $y_0$.

*Proof.* We first show independence from $K$. So let $K'$ be another compact set containing $A$. Then $K \cap K'$ is also a compact superset of $A$ and we have the following commutative diagram of inclusions:

$$\begin{array}{ccc} (X,X-K) & & \\ \Big\downarrow j_K & \searrow^{j_K} & \\ (X,X-(K\cap K')) & \xrightarrow{i_{K\cap K'}} & (X,X-A) \\ \Big\uparrow j_{K'} & \nearrow_{i_{K'}} & \\ (X,X-K') & & \end{array} \qquad (3.6)$$

By the characterization of fundamental class, we easily have $j_{Kn}(o_k) = o_{K\cap K'} = j_{K'n}(o_{K'})$. Therefore, by commutativity,

$$i_{Kn}(o_K) = i_{K\cap K'n}(o_{K\cap K}) = i_{K'n}(o'_K), \qquad (3.7)$$

so $f'_n \circ e_n^{-1} \circ i_{Kn}(o_K) = f'_n \circ e_n^{-1} \circ i_{K'n}(o_{K'})$. Therefore $\lambda(f,A)$ is independent of the choice of $K$.

The proof that $\lambda$ is independent of the choice of $N$ and that it satisfies the additivity and homotopy for admissible pairs $(f,A)$ in which $A$ has compact closure is very similar to the proofs of the corresponding facts in [1, Theorem and Definition 4.10], and will therefore be omitted. By Theorem 3.9, $\lambda$ has a unique extension to a root index for $X$, $Y$, $y_0$. □

*Example 3.11.* Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map and $y_0 \in \mathbb{R}^n$. Then $y_0$ is the only root of $f$ at $y_0$, and therefore $\{y_0\}$ is the only Nielsen root class of $f$ at $y_0$. Choose an orientation $s_{\mathbb{R}^n} : \mathbb{R}^n \to \widetilde{\mathbb{R}}^n$ of $\mathbb{R}^n$ and choose the local orientation at $y_0$ to be $\nu = s_{\mathbb{R}^n}(y_0)$. To compute $\lambda(f, y_0)$ relative to these orientations, let $N = \mathbb{R}^n$ and $K = \{y_0\}$ in the above definition. Then $f'$, $e$, and $i_K$ are the identity map on $(\mathbb{R}^n, \mathbb{R}^n - y_0)$, so $f'_n \circ e_n^{-1} \circ i_{Kn}$ is the identity on $H_n(\mathbb{R}^n, \mathbb{R}^n - y_0; \mathbb{Z}) \approx \mathbb{Z}$. Also, $o_K = \nu$. Hence, $\lambda(f, \{y_0\}) = 1 \neq 0$. It follows from Corollary 3.8 that $\{y_0\}$ is properly essential, and therefore $\mathrm{PNR}(f, y_0) = 1$.

On the other hand, let $y_1 \in \mathbb{R}^n$ be distinct from $y_0$, and let $\{h_t\}$ be the straight line homotopy from $f$ to the constant map into $y_1$, $h_t(x) = (1-t)x + ty_1$. Then $h_1^{-1}(y_0) = \emptyset$, so $\mathrm{NR}(h_1, y_0) = 0$. Since NR is a homotopy invariant, then $\mathrm{NR}(f, y_0) = \mathrm{NR}(h_1, y_0) = 0$. This example shows that $\mathrm{PNR}(f, y_0)$ can be strictly less than $\mathrm{NR}(f, y_0)$.

*Remark 3.12.* If $X$ is compact, then we may take $K = X$ in Theorem and Definition 3.10. In this case the homomorphism $f_{Nn} \circ e_{Nn}^{-1} \circ i_{Kn} : H_n(X, X-K; \mathbb{Z}) \to H_n(Y, Y - y_0; \mathbb{Z})$ is the homomorphism $L_n(f,A) : H_n(X;\mathbb{Z}) \to H_n(Y, Y - y_0; \mathbb{Z})$ of [1, Theorem and Definition 4.12], and therefore $\lambda$ is the same as the integer-valued index defined in [1, Theorem and Definition 4.14].

*Remark 3.13.* If $Y$ is also an oriented manifold, then $\lambda(f, f^{-1}(y_0)) = \deg_{y_0} f$, the *degree of $f$ along $y_0$*. And when $Y$ is connected (as we usually assume), then this number is the same for all $y_0 \in Y$ and is *the degree of $f$*, $\deg f$. (This generalizes Brouwer degree from maps of compact oriented manifolds to proper maps of arbitrary oriented manifolds.)

By additivity, $\lambda(f, f^{-1}(y_0)) = \lambda(f, X)$. Thus, $\lambda(f, X) = \deg f$ whenever $Y$ is an oriented connected manifold.

We have an alternative description of $\lambda(f, A)$ in terms of degree.

Theorem 3.14. *Suppose $X$ is an orientable $n$-manifold and $Y$ is $n$-Euclidean at $y_0 \in Y$. Choose an orientation $s_X : X \to \widetilde{X}$ of $X$ and a local orientation $\nu \in H_n(Y, Y - y_0)$ of $Y$ at $y_0$. Let $\lambda$ be the integer root index for $X$, $Y$, $y_0$ relative to these orientations. Let $E \subset Y$ be a Euclidean neighborhood of $y_0$, and let $s_E : E \to \widetilde{E}$ be the orientation of $E$ such that $j_n(s_E(y_0)) = \nu$, where $j_n$ is induced by the inclusion $j : n(E, E - y_0) \subset (Y, Y - y_0)$. Now suppose that $(f, A)$ is properly admissible for $X$, $Y$, $y_0$. Then there is an open neighborhood $U$ of $f^{-1}(y_0) \cap A$ such that $f(U) \subset E$ and $U - (f^{-1}(y_0) \cap A)$ has no roots of $f$ at $y_0$. Let $s_U : U \to \widetilde{U}$ be the restriction of $s_X$ to $U$. Then relative to the orientations $s_U$ and $s_E$,*

$$\lambda(f, A) = \deg_{y_0} f_{UE}, \tag{3.8}$$

*where $f_{UE} : U \to E$ is defined by $f$.*

*Proof.* By additivity, we have $\lambda(f, A) = \lambda(f, f^{-1}(y_0) \cap A)$, so it suffices to show that $\lambda(f, f^{-1}(y_0) \cap A) = \deg_{y_0} f_{UE}$. Notice that $f^{-1}(y_0 \cap A) = f_{UE}^{-1}(y_0)$, so we will show that $\lambda(f, f_{UE}^{-1}(y_0)) = \deg_{y_0} f_{UE}$.

Since $f$ is proper, then $f^{-1}(y_0)$ is compact, and since $(f, A)$ is admissible, we have $f^{-1}(y_0) \cap A = f^{-1}(y_0) \cap \operatorname{Cl} A$ which is closed in $X$ and therefore closed in the compact set $f^{-1}(y_0)$. It follows that $f_{UE}^{-1}(y_0) = f^{-1}(y_0) \cap A$ is compact. Hence, in order to compute $\lambda(f, f_{UE}^{-1}(y_0))$ we may use $f_{UE}^{-1}(y_0)$ for the set $K$ of Theorem and Definition 3.10. We may also use $U$ in place of $N$. Now consider the diagram

$$\begin{array}{ccccc}
(X, X - f_{UE}^{-1}(y_0)) & \overset{e}{\supset} & (U, U - f_{UE}^{-1}(y_0)) & \xrightarrow{f'} & (Y, Y - y_0) \\
 & & \Big\| & & \cup j \\
 & & (U, U - f_{UE}^{-1}(y_0)) & \xrightarrow{f'_{UE}} & (E, E - y_0) \\
 & & \cup & & \cup \\
 & & U & \xrightarrow{f_{UE}} & E
\end{array} \tag{3.9}$$

where $f'$, $f'_{UE}$, and $f_{UE}$ are defined by $f$ and all other maps are the indicated inclusions. By the definition of $\lambda$, we have

$$f'_n \circ e_n^{-1}(o_{X, f_{UE}^{-1}(y_0)}) = \lambda(f, f_{UE}^{-1}(y_0))\nu, \tag{3.10}$$

where $o_{X, f_{UE}^{-1}(y_0)}$ is the fundamental class of $X$ around $f_{UE}^{-1}(y_0)$. By the definition of $\deg_{y_0} f_{UE}$ we have

$$f'_{UEn}(o_{U, f_{UE}^{-1}(y_0)}) = (\deg_{y_0} f_{UE}) s_E(y_0), \tag{3.11}$$

where $o_{U,f_{UE}^{-1}(y_0)}$ is the fundamental class of $U$ around $f_{UE}^{-1}(y_0)$. Applying $j_n$ to both sides of the last equality and making use of commutativity,

$$\begin{aligned} f'_n(o_{U,f_{UE}^{-1}(y_0)}) &= j_n \circ f'_{UEn}(o_{U,f_{UE}^{-1}(y_0)}) \\ &= (\deg_{y_0} f_{UE})\, j_n(s_E(y_0)) = (\deg_{y_0} f_{UE})\nu. \end{aligned} \tag{3.12}$$

Hence, it remains to show that $o_{U,f_{UE}^{-1}(y_0)} = e_n^{-1}(o_{X,f_{UE}^{-1}(y_0)})$. To do so, let $x$ be an arbitrary point in $f_{UE}^{-1}(y_0)$ and consider the diagram

$$\begin{array}{ccc} (X, X - f_{UE}^{-1}(y_0)) & \overset{e}{\supset} & (U, U - f_{UE}^{-1}(y_0)) \\ \cap i_x & & \cap k_x \\ (X, X - x) & \overset{e_x}{\supset} & (U, U - x) \end{array} \tag{3.13}$$

Then $i_{xn}(e_n(o_{U,f_{UE}^{-1}(y_0)})) = e_{xn}(k_{xn}(o_{U,f_{UE}^{-1}(y_0)})) = e_{xn}(s_U(x)) = s_X(x)$. The first equality follows from commutativity, the second from the characterization of the fundamental class $o_{U,f_{UE}^{-1}(y_0)}$, and the third from the fact that $s_U$ is the restriction of $s_X$. Hence, from the characterization of the fundamental class $o_{X,f_{UE}^{-1}(y_0)}$, we have $e_n(o_{U,f_{UE}^{-1}(y_0)}) = o_{X,f_{UE}^{-1}(y_0)}$, and therefore $o_{U,f_{UE}^{-1}(y_0)} = e_n^{-1}(o_{X,f_{UE}^{-1}(y_0)})$. □

The integer-valued root index $\lambda$ is defined using homology with integer coefficients. We now state a completely parallel theorem/definition of a $\mathbb{Z}/2\mathbb{Z}$-valued index. The definition applies to nonorientable as well as orientable manifolds $X$. It is also somewhat simpler, since the local groups $H_n(X, X - x; \mathbb{Z}/2\mathbb{Z})$ have unique generators, so we need not worry about choice of orientation.

Theorem and Definition 3.15. *Suppose $X$ is an $n$-manifold and $Y$ is a topological space that is $n$-Euclidean at $y_0 \in Y$. Define a $\mathbb{Z}/2\mathbb{Z}$-valued proper root index $\lambda_2$ for $X$, $Y$, $y_0$ as follows. Let $(f, A)$ be properly admissible for $X$, $Y$, $y_0$ with* Cl*$A$ compact. Let $N \subset X$ be any neighborhood of* Cl*$A$ such that $N - A$ is root-free, and let $K \subset X$ be any compact set containing $A$. Let $o_{K2} \in H_n(X, X - K; \mathbb{Z}/2\mathbb{Z})$ be the $\mathbb{Z}/2\mathbb{Z}$ fundamental class of $X$ around $K$. Consider the diagram*

$$(X, X - K) \overset{i_K}{\subset} (X, X - A) \overset{e}{\supset} (N, N - A) \xrightarrow{f'} (Y, Y - y_0), \tag{3.14}$$

*where $f'$ is the map defined by $f$. Then $e$ is an excision and therefore induces homology isomorphisms in all dimensions, so there exists a homomorphism $f'_n \circ e_n^{-1} \circ i_{Kn} : H_n(X, X - K; \mathbb{Z}/2\mathbb{Z}) \to H_n(Y, Y - y_0; \mathbb{Z}/2\mathbb{Z})$. Define $\lambda_2(f, A) \in \mathbb{Z}/2\mathbb{Z}$ by*

$$f'_n \circ e_n^{-1} \circ i_{Kn}(o_K) = \lambda_2(f, A)\nu, \tag{3.15}$$

*where $\nu$ generates $H_n(Y, Y - y_0; \mathbb{Z}/2\mathbb{Z})$. Then $\lambda_2(f, A)$ is independent of the choice of $K$ and $N$—subject only to the conditions that $N$ be a neighborhood of* Cl*$A$ and that $K$ be a compact set containing $A$. Moreover, the $\mathbb{Z}/2\mathbb{Z}$-valued function $\lambda_2$, defined on the set of all properly* *admissible pairs* $(f, A)$ *for which* $A$ *has compact closure, extends uniquely to an integer mod root index* $\lambda_2$ *for* $X, Y, y_0$ *which is called the integer mod two root index for* $X$, $Y$, $y_0$.

The proof of Theorem and Definition 3.15 is completely parallel to that of Theorem and Definition 3.10, so we leave its proof as well as formulating the $\mathbb{Z}/2\mathbb{Z}$ parallels to Remarks 3.12 and 3.13 and Theorem 3.14 to the reader.

**3.5. Nielsen classes in the orientation manifold.** In this subsection, we examine the relation between Nielsen root classes of a map $f : X \to Y$ of a nonorientable manifold $X$ and the classes of $f \circ \tilde{p}$, where $\tilde{p} : \tilde{X} \to X$ is the orientation covering of $X$. So, throughout this subsection, let $f : X \to Y$ be a map of a connected nonorientable $n$-manifold $X$ into a well-connected space $Y$, let $\tilde{p} : \tilde{X} \to X$ be the orientation covering of $X$, let $\hat{q} : \hat{Y} \to Y$ and $\hat{f} : X \to \hat{Y}$ be a Hopf covering and lift for $f$, and let $\bar{q} : \bar{Y} \to \hat{Y}$ and $\bar{f} : \tilde{X} \to \bar{Y}$ be a Hopf covering and lift for $\hat{f} \circ \tilde{p} : \tilde{X} \to \hat{Y}$. Choose a base point $\tilde{x}_0 \in \tilde{X}$ and base the fundamental groups of $\tilde{X}$, $X$, $Y$, $\hat{Y}$, and $\bar{Y}$ at $\tilde{x}_0$, $\tilde{p}(\tilde{x}_0)$, $f \circ \tilde{p}(\tilde{x}_0)$, $\hat{f} \circ \tilde{p}(\tilde{x}_0)$, and $\bar{f}(\tilde{x}_0)$, respectively. We then have the following diagram of maps and diagram of induced fundamental group homomorphisms:

$$
\begin{array}{ccc}
 & & \bar{Y} \\
 & \overset{\bar{f}}{\nearrow} & \downarrow \bar{q} \\
\tilde{X} & & \hat{Y} \\
\tilde{p} \downarrow & \overset{\hat{f}}{\nearrow} & \downarrow \hat{q} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & \pi(\bar{Y}) \\
 & \overset{\bar{f}_\#}{\nearrow} & \downarrow \bar{q}_\# \\
\tilde{\pi}(X) & & \pi(\hat{Y}) \\
\tilde{p}_\# \downarrow & \overset{\hat{f}_\#}{\nearrow} & \downarrow \hat{q}_\# \\
\pi(X) & \xrightarrow{\ f_\#\ } & \pi(Y)
\end{array}
\tag{3.16}
$$

Theorem 3.16. *Referring to the above diagram,* $\hat{q} \circ \bar{q}$ *and* $\bar{f}$ *are a Hopf covering and lift for* $f \circ \tilde{p}$. *If* $f$ *is orientable, then* $\bar{q}$ *is a double covering. If* $f$ *is nonorientable, then* $\bar{q}$ *is a single covering (homeomorphism), so* $\hat{q}$ *and* $\hat{f} \circ \tilde{p}$ *are a Hopf covering and lift for* $f \circ \tilde{p}$.

*Proof.* To prove the first statement we have

$$
\begin{aligned}
\operatorname{im}(\hat{q} \circ \bar{q})_\# &= \hat{q}_\#(\operatorname{im}\bar{q}_\#) = \hat{q}_\#\Big(\operatorname{im}(\hat{f} \circ \tilde{p})_\#\Big) \\
&= \operatorname{im}(\hat{q}_\# \circ \hat{f}_\# \circ \tilde{p}_\#) = \operatorname{im}(f_\# \circ \tilde{p}_\#) = \operatorname{im}(f \circ \tilde{p})_\#.
\end{aligned}
\tag{3.17}
$$

The second equality follows from the fact that $\bar{q}$ is a Hopf covering for $\hat{f} \circ \tilde{p}$, and the fourth follows from commutativity. Thus $\hat{q} \circ \bar{q}$ and $\bar{f}$ are a Hopf covering and lift for $f \circ \tilde{p}$.

To prove the rest of the theorem, note that the sequence

$$
1 \longrightarrow \ker \hat{f}_\# \longrightarrow \pi(X) \xrightarrow{\hat{f}_\#} \pi(\hat{Y}) \longrightarrow 1
\tag{3.18}
$$

is exact and therefore induces an exact sequence

$$1 \longrightarrow \frac{\ker \widehat{f}_\#}{\operatorname{im} \widetilde{p}_\# \cap \ker \widehat{f}_\#} \longrightarrow \frac{\pi(X)}{\operatorname{im} \widetilde{p}_\#} \longrightarrow \frac{\pi(\widehat{Y})}{\operatorname{im} \widehat{f}_\# \circ \widetilde{p}_\#} \longrightarrow 1. \tag{3.19}$$

Since $\widehat{q}_\#$ is a monomorphism, then $\ker \widehat{f}_\# = \ker \widehat{q}_\# \circ \widehat{f}_\# = \ker f_\#$, and since $\bar{q}$ is a Hopf covering for $\widehat{f} \circ \widetilde{p}$, then $\operatorname{im} \widehat{f}_\# \circ \widetilde{p}_\# = \operatorname{im} \bar{q}_\#$. Making these substitutions, the exact sequence becomes

$$1 \longrightarrow \frac{\ker f_\#}{\operatorname{im} \widetilde{p}_\# \cap \ker f_\#} \longrightarrow \frac{\pi(X)}{\operatorname{im} \widetilde{p}_\#} \longrightarrow \frac{\pi(\widehat{Y})}{\operatorname{im} \bar{q}_\#} \longrightarrow 1. \tag{3.20}$$

Now suppose $f$ is orientable. Then $\ker f_\# \subset \operatorname{im} \widetilde{p}_\#$, so the group $\ker f_\# / \operatorname{im} \widetilde{p}_\# \cap \ker f_\#$ is trivial, and therefore, by exactness, $\pi(X)/\operatorname{im} \widetilde{p}_\# \to \pi(\widehat{Y})/\operatorname{im} \bar{q}_\#$ is an isomorphism. Since $\pi(X)/\operatorname{im} \widetilde{p}_\#$ is of order 2, then so is $\pi(\widehat{Y})/\operatorname{im} \bar{q}_\#$, and therefore $\bar{q}$ is a double covering.

Finally, suppose $f$ is nonorientable. Then $\ker f_\# \not\subset \operatorname{im} \widetilde{p}_\#$, so the group $\ker f_\# / (\operatorname{im} \widetilde{p}_\# \cap \ker f_\#)$ is not trivial, and therefore, by exactness, the epimorphism $\pi(X)/\operatorname{im} \widetilde{p}_\# \to \pi(\widehat{Y})/\operatorname{im} \bar{q}_\#$ is not an isomorphism. Since $\pi(X)/\operatorname{im} \widetilde{p}_\#$ is of order 2, this implies that $\pi(\widehat{Y})/\operatorname{im} \bar{q}_\#$ has order 1, and therefore $\bar{q}$ is a single covering. □

Now let $y_0 \in Y$.

Theorem 3.17. *Suppose the map $f : X \to Y$ is orientable. Then, for any Nielsen root class $\alpha$ of $f$ at $y_0 \in Y$, $\widetilde{p}^{-1}(\alpha) = \widetilde{\alpha} \sqcup (-\widetilde{\alpha})$, where both $\widetilde{\alpha}$ and $-\widetilde{\alpha}$ are Nielsen root classes of $f \circ \widetilde{p}$ at $y_0$. If $\widetilde{\alpha}$ is (properly) essential, then so is $\alpha$.*

*Proof.* Since $\alpha$ is a Nielsen root class of $f$ at $y_0$, there is a $\widehat{y} \in \widehat{q}^{-1}(y_0)$ such that $\widehat{f}^{-1}(\widehat{y}) = \alpha$. Since $f$ is orientable, then from Theorem 3.16 $\bar{q}^{-1}(\widehat{y}) = \{\bar{y}, \bar{y}'\}$ for two distinct points $\bar{y}$ and $\bar{y}'$. Let $\widetilde{\alpha} = \bar{f}^{-1}(\bar{y})$ and $\widetilde{\alpha}' = \bar{f}^{-1}(\bar{y}')$, so $\widetilde{p}^{-1}(\alpha) = \widetilde{\alpha} \sqcup \widetilde{\alpha}' \neq \emptyset$, and each of $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ is either a Nielsen root class of $f \circ \widetilde{p}$ at $y_0$ or empty. To complete the proof of the first statement, it remains to show that $-\widetilde{\alpha} = \widetilde{\alpha}'$. So let $\widetilde{x} \in \widetilde{\alpha}$, then $\bar{q} \circ \bar{f}(-\widetilde{x}) = \widehat{f} \circ \widetilde{p}(-\widetilde{x}) = \widehat{f} \circ \widetilde{p}(\widetilde{x}) = \bar{q} \circ \bar{f}(\widetilde{x}) = \widehat{y}$, so $-\widetilde{x} \in \widetilde{\alpha} \sqcup \widetilde{\alpha}'$. Let $\widetilde{A}$ be any path from $\widetilde{x}$ to $-\widetilde{x}$. Then $\widetilde{p} \circ \widetilde{A}$ is an orientation-reversing loop in $X$, so, since $f$ is orientable, we cannot have $[f \circ \widetilde{p} \circ \widetilde{A}] = [y_0]$. It follows that $-\widetilde{x} \notin \widetilde{\alpha}$, and therefore $-\widetilde{x} \in \widetilde{\alpha}'$. Thus, $-\widetilde{\alpha} \subset \widetilde{\alpha}'$. Similarly, $-\widetilde{\alpha}' \subset \widetilde{\alpha}$, and therefore $\widetilde{\alpha}' = -(-\widetilde{\alpha}') \subset -\widetilde{\alpha}$, so $-\widetilde{\alpha} = \alpha'$.

To prove the last statement, we prove its contrapositive. So suppose that $\alpha$ is (properly) inessential; we will show that $\widetilde{\alpha}$ is also (properly) inessential. Since $\alpha = \widehat{f}^{-1}(\widehat{y})$ is (properly) inessential, there is a (proper) homotopy $\{\widehat{h}_t : X \to \widehat{Y}\}$ beginning at $\widehat{f}$ such that $\widehat{h}_1^{-1}(\widehat{y}) = \emptyset$. Lift $\{\widehat{h}_t \circ \widetilde{p}\}$ to a (proper) homotopy $\{\bar{h}_t : \widetilde{X} \to \bar{Y}\}$ beginning at $\bar{f} \circ \widetilde{p}$. Then $\widetilde{\alpha} = \bar{f}^{-1}(\bar{y})$. But $\bar{h}_1^{-1}(\bar{y}) \subset \bar{f}^{-1}(\bar{q}^{-1}(\widehat{y})) = \widetilde{p}^{-1}(\widehat{h}_1^{-1}(\widehat{y})) = \emptyset$. Thus $\widetilde{\alpha}$ is (properly) inessential. □

The following theorem is an easy consequence of Theorem 3.16, so we omit its proof.

Theorem 3.18. *Suppose the map $f : X \to Y$ is nonorientable. Then, for any Nielsen root class $\alpha$ of $f$ at $y_0 \in Y$, $\widetilde{\alpha} = \widetilde{p}^{-1}(\alpha)$ is a root class of $f \circ \widetilde{p}$, and for this class, $\widetilde{\alpha} = -\widetilde{\alpha}$.*

**3.6. Multiplicity and absolute degree.** We are finally in a position to define multiplicity and absolute degree.

*Definition 3.19.* Let $f : X \to Y$ be a proper map of a connected $n$-manifold $X$ into a well-connected space $Y$ that is locally $n$-Euclidean at the point $y_0 \in Y$. Then, for any Nielsen root class $\alpha$ of $f$ at $y_0$, define the *multiplicity* of $\alpha$, denoted by $\text{mult}(f,\alpha,y_0)$, as follows.

(1) If $X$ is orientable, then

$$\text{mult}(f,\alpha,y_0) = |\lambda(f,\alpha)|. \tag{3.21}$$

(2) If $X$ is nonorientable, but $f$ is orientable, then according to Theorem 3.17 there is a root class $\widetilde{\alpha}$ of $f \circ \widetilde{p} : \widetilde{X} \to Y$ such that $\widetilde{p}^{-1}(\alpha) = \widetilde{\alpha} \sqcup (-\widetilde{\alpha})$. Then,

$$\text{mult}(f,\alpha,y_0) = |\lambda(f \circ \widetilde{p},\widetilde{\alpha})| = |\lambda(f \circ \widetilde{p},-\widetilde{\alpha})|. \tag{3.22}$$

(3) If neither $X$ nor $f$ is orientable, then

$$\text{mult}(f,\alpha,y_0) = |\lambda_2(f,\alpha)|. \tag{3.23}$$

*Remark 3.20.* Since we use the absolute value of $\lambda$ in case (1), the definition in case (1) is independent of the choice of orientations used to define $\lambda$. In the second case, since the map $\widetilde{x} \mapsto -\widetilde{x}$ is an orientation-reversing homeomorphism, it is easy to see that $\lambda(f \circ \widetilde{p},\alpha) = -\lambda(f \circ \widetilde{p},-\alpha)$, so the definition of multiplicity is independent of the choice of $\widetilde{\alpha}$ versus $-\widetilde{\alpha}$. Thus multiplicity is well defined.

*Remark 3.21.* In [3, page 57], Brown and Schirmer define multiplicity using the notion of degree. Using Theorem 3.14, their definition of multiplicity is easily seen to coincide with ours in cases (1) and (3). Case (2) is a bit more complicated, however. In this case they first show that $\alpha$ has an orientable open neighborhood $U$ containing no roots of $f$, other than those in $\alpha$, that is mapped by $f$ into a connected orientable open neighborhood $V$ of $y_0$. Then $f$ defines a map $f_{UV} : U \to V$. In general, however, $U$ is not connected, so different orientations of $U$ may differ by more than just a sign. They describe an "orientation procedure" for orienting $U$, and define $\text{mult}(f,\alpha,y_0) = |\deg_{y_0} f_{UV}|$. It can be shown that their procedure for finding an oriented neighborhood $U$ of $\alpha$ is equivalent to the following: since, by Theorem 3.17, $\widetilde{\alpha} \neq -\widetilde{\alpha}$, we can find a neighborhood $\widetilde{U}$ of $\widetilde{\alpha}$ disjoint from $-\widetilde{U}$ that is mapped by $f \circ \widetilde{p}$ into a Euclidean neighborhood $E$ of $y_0$. Then, since $\widetilde{p}$ is a double covering, $\widetilde{p}$ maps $\widetilde{U}$ homeomorphically onto a neighborhood $U$ of $\alpha$. We orient $U$ by first restricting an orientation of $\widetilde{X}$ to $\widetilde{U}$, and then using the homeomorphism $\widetilde{p}|\widetilde{U}$ to orient $U$. We now have (using Theorem 3.14) $|\lambda(f \circ \widetilde{p},\widetilde{\alpha})| = |\deg_{y_0}(f \circ \widetilde{p})_{\widetilde{U}E}| = |\deg_{y_0} f_{UE}|$, so the two definitions of multiplicity are consistent.

Theorem 3.22. *Let $\{h_t : X \to Y\}$ be a proper homotopy, where $X$ is a connected $n$-manifold and $Y$ is a well-connected space that is $n$-Euclidean at $y_0 \in Y$, and suppose that $\alpha_0$ is a Nielsen root class of $h_0$ at $y_0$. If $\alpha_0$ is $\{h_t\}$-related to a Nielsen root class $\alpha_1$ of $h_1$ at $y_0$, then* $\text{mult}(h_0,\alpha_0,y_0) = \text{mult}(h_1,\alpha_1,y_0)$. *If $\alpha_0$ is not $\{h_t\}$-related to a Nielsen root class of $h_1$, then* $\text{mult}(h_0,\alpha_0,y_0) = 0$.

*Proof.* In cases (1) and (3) of Definition 3.19, this follows directly from the definition, Theorem 3.7, and the fact that $\lambda$ and $\lambda_2$ are proper root indices. So assume $X$ is nonorientable but $h_0$ (and therefore $h_1$) is orientable and write $\widetilde{p}^{-1}(\alpha_0) = \widetilde{\alpha}_0 \sqcup (-\widetilde{\alpha}_0)$. If $\alpha_0$ is not essential, then by Theorem 3.17 neither is $\widetilde{\alpha}_0$, so we have $\operatorname{mult}(f,\alpha_0,y_0) = |\lambda(h_0 \circ \widetilde{p},\widetilde{\alpha}_0)| = 0$. On the other hand, it is easy to show (using Hopf coverings and Theorems 3.16 and 3.17) that if $\alpha_0$ is $\{h_t\}$-related to $\alpha_1$, then $\widetilde{\alpha}_0$ is $\{h_t \circ \widetilde{p}\}$-related to a class $\widetilde{\alpha}_1$, where $\widetilde{p}(\widetilde{\alpha}_1) = \alpha_1$. In this case we have $\operatorname{mult}(f,\alpha_0,y_0) = |\lambda(h_0 \circ \widetilde{p},\widetilde{\alpha}_0)| = |\lambda(h_1 \circ \widetilde{p},\widetilde{\alpha}_1)| = \operatorname{mult}(f,\alpha_1,y_0)$. □

Corollary 3.23. *Let $\{h_t : X \to Y\}$ be a proper homotopy, where $X$ is a connected $n$-manifold and $Y$ is a well-connected space that is $n$-Euclidean at $y_0 \in Y$. Then the $\{h_t\}$ relation defines a bijection from the set of root classes of $h_0$ with nonzero multiplicity onto the set of those of $h_1$.*

Corollary 3.24. *Let $\alpha$ be a Nielsen root class at $y_0$ of a proper map $f : X \to Y$ of an $n$-manifold $X$ into a well-connected space $Y$ that is $n$-Euclidean at $y_0 \in Y$. Then* $\operatorname{mult}(f,\alpha, y_0) \neq 0$ *implies that $\alpha$ is properly essential.*

We will see later that at least for $n > 2$, we also have the converse: if $\alpha$ is properly essential, then $\operatorname{mult}(f,\alpha,y_0) \neq 0$.

*Definition 3.25.* Let $f : X \to Y$ be a proper map of an $n$-manifold $X$ into a space $Y$ that is locally $n$-Euclidean at the point $y_0 \in Y$. Then the *absolute degree* of $f$ at $y_0$ is the sum of the multiplicities of all the root classes of $f$ at $y_0$. It is denoted by $\mathcal{A}(f,y_0)$:

$$\mathcal{A}(f,y_0) = \sum_{\alpha \in f^{-1}(y_0)/N} \operatorname{mult}(f,\alpha,y_0). \tag{3.24}$$

As an immediate consequence of Theorem 3.22 and Corollary 3.23 we have the following corollary.

Corollary 3.26. *Let $f : X \to Y$ be a proper map of a connected $n$-manifold $X$ into a well-connected space $Y$ that is $n$-Euclidean at $y_0$. Then $\mathcal{A}(f,y_0) = \mathcal{A}(g,y_0)$ for every map $g$ properly homotopic to $f$.*

As an easy consequence of the fact that $\widetilde{p} : \widetilde{X} \to X$ is a double covering, Theorem 3.17, and Definitions 3.19 and 3.25, we have the following theorem.

Theorem 3.27. *Let $f : X \to Y$ be an orientable proper map of a connected nonorientable $n$-manifold $X$ into a well-connected space $Y$ that is locally $n$-Euclidean at the point $y_0 \in Y$, and let $\widetilde{p} : \widetilde{X} \to X$ be the orientation covering. Then* $\operatorname{card}(f \circ \widetilde{p})^{-1}(y_0) = 2\operatorname{card} f^{-1}(y_0)$ *and $\mathcal{A}(f \circ \widetilde{p}, y_0) = 2\mathcal{A}(f,y_0)$.*

We are now ready to show that $\mathcal{A}(f)$ is a lower bound on the number of roots of transverse maps.

Theorem 3.28. *Let $f : X \to Y$ be a proper map of a connected $n$-manifold $X$ into a well-connected space $Y$ that is $n$-Euclidean at $y_0$. Then every map properly homotopic to $f$ and transverse to $y_0$ has at least $\mathcal{A}(f,y_0)$ roots.*

*Proof.* Suppose that $g$ is properly homotopic to $f$ and transverse to $y_0$. We distinguish three cases.
*Case 1* ($X$ orientable). Let $\alpha$ be a root class of $g$. Then

$$\operatorname{mult}(g,\alpha,y_0) = |\lambda(g,\alpha)| = \left|\sum_{x\in\alpha}\lambda(g,x)\right| \le \sum_{x\in\alpha}|\lambda(g,x)| = \operatorname{card}\alpha. \tag{3.25}$$

The first equality is by definition of $\lambda$, the second follows from additivity of $\lambda$, and the last from the fact that $g$ is a local homeomorphism at each $x \in \alpha$, and therefore $\lambda(g,x) = \pm 1$. When we sum this inequality over all Nielsen root classes $\alpha$ of $g$, we have $\mathcal{A}(g,y_0) \le \operatorname{card} g^{-1}(y_0)$. But $\mathcal{A}(f,y_0) = \mathcal{A}(g,y_0)$ since $f$ and $g$ are properly homotopic. Thus $\mathcal{A}(f,y_0) \le \operatorname{card} g^{-1}(y_0)$.
*Case 2* ($X$ nonorientable but $f$ orientable). Let $\tilde{p} : \tilde{X} \to X$ be the orientation covering. Since $\tilde{p}$ has only two sheets, then it is proper, so $f \circ \tilde{p}$ and $g \circ \tilde{p}$ are properly homotopic. Since $\tilde{p}$ is a covering and $g$ is transverse to $y_0$, it follows easily that $g \circ \tilde{p}$ is a local homeomorphism at each of its roots at $y_0$, and therefore, since it is proper, $g \circ \tilde{p}$ is transverse to $y_0$. Thus, using Theorem 3.27 together with Case 1, we have

$$\mathcal{A}(f,y_0) = \frac{1}{2}\mathcal{A}(f \circ \tilde{p}, y_0) \le \frac{1}{2}\operatorname{card}(g \circ \tilde{p})^{-1}(y_0) = \operatorname{card} g^{-1}(y_0). \tag{3.26}$$

*Case 3* (neither $X$ nor $f$ orientable). The proof is the same as in Case 1, but uses $\lambda_2$ in place of $\lambda$. □

## 4. Isolating roots

This section is devoted to the following theorem and its corollaries.

Theorem 4.1. *Let $f : X \to Y$ be a map from an $n$-manifold $X$ into a space $Y$ that is locally $n$-Euclidean at $y_0$, and let $N \subset Y$ be any neighborhood of $y_0$. Then $f$ is homotopic to a map that is a local homeomorphism at each of its roots at $y_0$ by a homotopy that is constant outside of $f^{-1}(N)$.*

*Proof.* Let $E$ be a Euclidean neighborhood of $y_0$ such that $E \subset N$. The proof proceeds in two stages. In the first stage we approximate $f^{-1}(E)$ by a polyhedron and the map $f$ by a simplicial approximation and use this approximation to get a new map $g$ homotopic to $f$ such that $g^{-1}(y_0)$ is covered by a disjoint union of open sets $U \subset g^{-1}(E)$ each of which is contained in the interior of an $n$-ball $B$. In the second stage we use triangulations of the balls $B$ to get a map homotopic to $g$, and therefore $f$, that is a local homeomorphism at each of its roots at $y_0$. All of the homotopies will be constant outside of $f^{-1}(E)$, and therefore outside of $f^{-1}(N)$.

In the following, if $s$ is a simplex in a simplicial complex $K$, then $\operatorname{st}_K s$ denotes the open star of $s$ in $K$—the union of all open simplices including $s$ that have $s$ for a face. If $v_0,\dots,v_k$ are vertices in $K$, then $\langle v_0,\dots,v_k\rangle$ denotes the open simplex whose vertices are $v_0,\dots,v_k$.
*Stage 1.* Let $\psi : \mathbb{R}^n \to E$ be a homeomorphism and $K_E$ a simplicial complex such that $\mathbb{R}^n = |K_E|$. Then $\{\psi, K_E\}$ is a triangulation of $E$. We may assume that $\psi^{-1}(y_0)$ is in an open $n$-simplex $s$ of $K_E$ because if it is not, taking $z_0$ to be a point that is in an open $n$-simplex,

we may define $\psi'$ by $\psi'(z) = \psi(\psi^{-1}(y_0) + z - z_0)$ so that $\{\psi', K_E\}$ is a triangulation of $E$ and $\psi'(z_0) = y_0$.

The collection $\{\mathrm{st}_{K_E} v \mid v \text{ a vertex of } K_E\}$ is an open cover of $\mathbb{R}^n$, so $\{f^{-1}(\psi(\mathrm{st}_{K_E} v)) \mid v$ a vertex of $K_E\}$ is an open cover of $f^{-1}(E)$. Now let $\mathcal{W}$ be an open cover of $f^{-1}(E)$ with the following properties.

(1) $\mathcal{W}$ is a refinement of $\{f^{-1}(\psi(\mathrm{st}_{K_E} v)) \mid v \text{ a vertex of } K_E\}$.
(2) For each $W \in \mathcal{W}$, there is an $n$-ball $B$ such that $\mathrm{Cl}\, W \subset \mathrm{int}\, B$.
(3) The nerve of $\mathcal{W}$ has dimension $n$ or less.

Construct a family $\{\gamma_W \mid W \in \mathcal{W}\}$ of maps $\gamma_W : f^{-1}(E) \to I$ with the following properties.

(1) $W = \{x \in f^{-1}(E) \mid \gamma_W(x) > 0\}$.
(2) $\sum_{W \in \mathcal{W}} \gamma_W(x) = 1$ for all $x \in f^{-1}(E)$.

We may construct such a family by first defining $\gamma'_W(x)$ to be the distance from $x$ to $X - W$, and then letting $\gamma_W(x) = \gamma'_W(x) / \sum_{V \in \mathcal{W}} \gamma'_V(x)$.

Now define a map $\nu : f^{-1}(E) \to |\mathrm{Nerve}\,\mathcal{W}|$ by

$$\nu(x) = \sum_{\{W \in \mathcal{W} \mid \gamma_W(x) > 0\}} \gamma_W(x) W. \tag{4.1}$$

For each $W \in \mathcal{W}$, select a vertex $v$ of $K_E$ such that $W \subset f^{-1}(\psi(\mathrm{st}_{K_E} v))$, and let $\mu(W) = v$. Then $\mu$ extends to a simplicial map $\mu : \mathrm{Nerve}\,\mathcal{W} \to K_E$. Let $|\mu| : |\mathrm{Nerve}\,\mathcal{W}| \to \mathbb{R}^n$ denote the induced map of the corresponding polyhedra.

Now, for any $x \in f^{-1}(E)$ and $W_0, \dots, W_p \in \mathcal{W}$,

$$\nu(x) \in \langle W_0, \dots, W_p \rangle \implies |\mu| \circ \nu(x) \in \langle \mu(W_0), \dots, \mu(W_p) \rangle. \tag{4.2}$$

But $\nu(x) \in \langle W_0, \dots, W_p \rangle$ also implies that $x \in \bigcap_{i=0}^p W_i \subset \bigcap_{i=0}^p f^{-1}(\psi(\mathrm{st}_{K_E} \mu(W_i)))$, so $\psi^{-1} \circ f(x) \in \bigcap_{i=0}^p \mathrm{st}_{K_E} \mu(W_i)$, thus

$$\nu(x) \in \langle W_0, \dots, W_p \rangle \implies \psi^{-1} \circ f(x) \in \bigcap_{i=0}^{p} \mathrm{st}_{K_E} \mu(W_i). \tag{4.3}$$

Every point in $\bigcap_{i=0}^p \mathrm{st}_{K_E} \mu(W_i)$ is in a simplex having $\mu(W_0), \dots, \mu(W_p)$ for some of its vertices. Thus $|\mu| \circ \nu(x)$ is in a face of the open simplex that contains $\phi^{-1} \circ f(x)$. We may therefore use the linear structure in these simplices to define a homotopy $\{k'_t\}$ from $\psi^{-1} \circ (f | f^{-1}(E))$ to $|\mu| \circ \nu$ by

$$k'_t(x) = (1 - t)\psi^{-1} \circ f(x) + t|\mu| \circ \nu(x). \tag{4.4}$$

Then $k'_t(x)$ lies on the straight line segment joining a point in the unique open simplex containing $\phi^{-1} \circ f(x)$ to a point in one of its faces. Hence (since there are no $(n+1)$-simplices), if $k'_t(x) \in s$, we must also have $\psi^{-1} \circ f(x) \in s$. The contrapositive of this statement is

$$k'_t\left(f^{-1}(E) - f^{-1}(\psi(s))\right) \subset \mathbb{R}^n - s \quad \forall t \in I. \tag{4.5}$$

Now let $C$ be a closed neighborhood of $X - f^{-1}(E)$ disjoint from $\operatorname{Cl} f^{-1}(\psi(s))$, let $\beta : f^{-1}(E) \to I$ be a function that is 1 on $\operatorname{Cl} f^{-1}(\psi(s))$ and 0 on $C$, and define a homotopy $\{k_t : X \to Y\}$ by

$$k_t(x) = \begin{cases} \psi \circ k'_{\beta(x)t}(x) & \text{for } x \in f^{-1}(E), \\ f(x) & \text{for } x \in \operatorname{int} C. \end{cases} \tag{4.6}$$

The two formulas agree on the open set $(\operatorname{int} C) \cap f^{-1}(E)$, and $(\operatorname{int} C) \cup f^{-1}(E) \subset (X - f^{-1}(E)) \cup f^{-1}(E) = X$, so $\{k_t\}$ is well defined on all of $X$. Also the homotopy is constant off of $f^{-1}(E)$. Let $g = k_1$.

We now show that

$$g^{-1}(\psi(s)) = (|\mu| \circ \nu)^{-1}(s). \tag{4.7}$$

Suppose first that $x \in g^{-1}(\psi(s))$, so $g(x) \in \psi(s)$. If $x \in C$, then $g(x) = f(x)$, which implies that $f(x) \in \psi(s)$, which is impossible since $C$ and $f^{-1}(\psi(s))$ are disjoint. Therefore $x \in f^{-1}(E)$, so $g(x) = \psi \circ k'_{\beta(x)}(x)$. From (4.5), $k'_t$ maps $f^{-1}(E) - f^{-1}(\psi(s))$ into $\mathbb{R}^n - s$ for all $t$, and therefore $\psi \circ k'_{\beta(x)}$ maps $f^{-1}(E) - f^{-1}(\psi(s))$ into $E - \psi(s)$. Since $\psi \circ k'_{\beta(x)}(x) \in \psi(s)$, we cannot have $x \in f^{-1}(E) - f^{-1}(\psi(s))$, so $x \in f^{-1}(\psi(s))$. Therefore $\beta(x) = 1$ and $g(x) = \psi \circ k'_1(x) = \psi \circ |\mu| \circ \nu(x)$, so $\psi \circ |\mu| \circ \nu(x) \in \psi(s)$, which implies that $x \in (|\mu| \circ \nu)^{-1}(s)$.

Conversely, suppose that $x \in (|\mu| \circ \nu)^{-1}(s)$, so $k'_1(x) = |\mu| \circ \nu(x) \in s$. Since $k'_t$ maps $f^{-1}(E) - f^{-1}(s)$ into $\mathbb{R}^n - s$, this implies that $x \in f^{-1}(s)$ and therefore that $\beta(x) = 1$. Therefore $g(x) = k_1(x) = \psi \circ k'_{\beta(x)} = \psi \circ k'_1(x) = \psi \circ |\mu| \circ \nu(x) \in \psi(s)$, so $x \in g^{-1}(\psi(s))$. This proves (4.7).

Since $\mu$ is simplicial, $|\mu|^{-1}(s)$ is either empty or a disjoint union of open $n$-simplices $\langle W_0, \dots, W_n \rangle$. In the first case, we are done since then $g$ has no roots at $y_0$. In the second, $g^{-1}(\psi(s)) = (|\mu| \circ \nu)^{-1}(s)$ is the disjoint union of open sets $U$, where each $U \subset W_0 \cap \cdots \cap W_n$, for some $n+1$ sets $W_0, \dots, W_n \in \mathcal{W}$. Let $\mathcal{U}$ be the family of all these open sets $U$. Note that because $g^{-1}(y_0) \subset \bigsqcup_{U \in \mathcal{U}} U$, $g$ has no roots at $y_0$ in $\operatorname{Bd} \bigsqcup_{U \in \mathcal{U}} U$. Since the sets $U$ are open and disjoint, $\bigsqcup_{U \in \mathcal{U}} \operatorname{Bd} U \subset \operatorname{Bd} \bigsqcup_{U \in \mathcal{U}} U$, so $\operatorname{Bd} U$ is root-free for every $U \in \mathcal{U}$. Since each $U \in \mathcal{U}$ is a subset of $W_0 \cap \cdots \cap W_n$, for some sets $W_0, \dots, W_n \in \mathcal{W}$, then $\operatorname{Cl} U \subset \operatorname{Cl} W_0 \subset \operatorname{int} B$ for some $n$-ball $B$. This completes the first stage of the proof.

*Stage 2.* Again, let $\psi : \mathbb{R}^n \to E$ be a homeomorphism onto the Euclidean neighborhood $E \subset N$ of $y_0$. From the first stage we have a map $g$ homotopic to $f$ by a homotopy constant off of $f^{-1}(N)$, and a family $\mathcal{U}$ of disjoint open sets $U \subset g^{-1}(E)$ covering $g^{-1}(y_0)$, where, for each $U \in \mathcal{U}$, there is an $n$-ball $B$ with $\operatorname{Cl} U \subset \operatorname{int} B$, and $\operatorname{Bd} U$ contains no roots of $g$ at $y_0$.

So let $U \in \mathcal{U}$, let $B$ be an $n$-ball with $\operatorname{Cl} U \subset \operatorname{int} B$, and let $(\phi : |K_B| \to B, K_B)$ be a triangulation of $B$. Let $C \subset \operatorname{int} B$ be a closed neighborhood of $\operatorname{Bd} U$ disjoint from $g^{-1}(y_0)$. Then $\phi^{-1}(C)$ and $\phi^{-1}(g^{-1}(y_0))$ are disjoint compact subsets of $|K_B|$ and therefore a positive distance $d > 0$ apart. We may assume, by subdividing $K_B$ if necessary, that the mesh of $K_B$ is less than $d$. Define subcomplexes $K$ and $L$ of $K_B$ by

$$\begin{aligned} K &= \{\sigma \in K_B \mid (\operatorname{st}_{K_B} \sigma) \cap \phi^{-1}(U \cup C) \neq \emptyset\}, \\ L &= \{\sigma \in K_B \mid (\operatorname{st}_{K_B} \sigma) \cap \phi^{-1}(C) \neq \emptyset\}. \end{aligned} \tag{4.8}$$

Clearly, $\phi^{-1}(U \cup C) \subset |K|$ and $\phi^{-1}(C) \subset |L|$, so $\phi^{-1}(\operatorname{Bd} U) \subset \operatorname{int}|L|$. Now, if $z \in |L|$, then $z$ is in the face of a simplex that meets $\phi^{-1}(C)$, and is therefore at a distance less than $d$ from $\phi^{-1}(C)$, so $z \notin (g \circ \phi)^{-1}(y_0)$. Therefore $|L| \cap (g \circ \phi)^{-1}(y_0) = \emptyset$. Thus $\psi^{-1} \circ g \circ \phi(|L|)$ is a compact set in $\mathbb{R}^n$ not containing $\psi^{-1}(y_0)$, so there is a positive distance $d' > 0$ between $\psi^{-1} \circ g \circ \phi(|L|)$ and $\psi^{-1}(y_0)$. Let $K'_E$ be a complex with mesh less than $d'$ such that $|K'_E| = \mathbb{R}^n$. We may assume that $\psi^{-1}(y_0)$ is in an open $n$-simplex $s'$ of $K'_E$, otherwise we could, as in Stage 1, modify $\psi$ by a translation so that it is. Then $\psi^{-1} \circ g \circ \phi(|L|) \cap s' = \emptyset$, so $\psi^{-1} \circ g \circ \phi$ defines a map $g' : (|K|,|L|) \to (\mathbb{R}^n, \mathbb{R}^n - s')$. By the simplicial approximation theorem, there are a subdivision $(K',L')$ of $(K,L)$, a simplicial approximation $k : (K',L') \to (K'_E, K'_E - s')$ to $g'$, and a homotopy $\{k'_t : (|K'|,|L'|) = (|K|,|L|) \to (\mathbb{R}^n, \mathbb{R}^n - s')\}$ from $g'$ to $|k|$. Since $\phi^{-1}(\operatorname{Bd} U) \subset \operatorname{int}|L|$, then the closed sets $\phi(|K| - \operatorname{int}|L|)$ and $\operatorname{Bd} U$ are disjoint, so there is a map $\beta : B \to I$ that is 1 on $\phi(|K| - \operatorname{int}|L|)$ and 0 on $\operatorname{Bd} U$. Define a homotopy $\{h_{Ut} : \operatorname{Cl} U \to Y\}$ by

$$h_{Ut}(x) = \psi \circ k'_{\beta(x)t} \circ \phi^{-1}(x) \quad \text{for } x \in \operatorname{Cl} U. \tag{4.9}$$

Then we assert the following:

(1) $h_{U0} = g|\operatorname{Cl} U$,
(2) $\{h_{Ut}\}$ is constant on $\operatorname{Bd} U$,
(3) $h_{U1}$ is a local homeomorphism at each $x \in h_{U1}^{-1}(y_0)$.

The first two assertions follow easily from the definitions, so we prove only the third. Let $x \in h_{U1}^{-1}(y_0)$. Then $\psi \circ k'_{\beta(x)} \circ \phi^{-1}(x) = h_{U1}(x) = y_0$, and therefore $k'_{\beta(x)} \circ \phi^{-1}(x) = \psi^{-1}(y_0) \in s'$. Since $k'_t(|L|) \subset \mathbb{R}^n - s'$ for all $t$, we must have $\phi^{-1}(x) \in |K| - |L| \subset |K| - \operatorname{int}|L|$, so $\beta(x) = 1$, and therefore $|k| \circ \phi^{-1}(x) = k'_1 \circ \phi^{-1}(x) = \psi^{-1}(y_0) \in s'$. Since $|k|$ is simplicial, this implies that $\phi^{-1}(x) \in \sigma$ for some open $n$-simplex $\sigma$ in $K'$, and $|k|$ takes $\sigma$ homeomorphically onto $s'$. This also implies that $\sigma \subset |K| - |L|$. Let $V = \phi^{-1}(\sigma) \cap U$. Then $V$ is a neighborhood of $x$, and we will show that $h_{U1}$ maps $V$ homeomorphically onto $h_{U1}(V)$. Now, for any $x'$ in $V$, we have $\phi^{-1}(x') \in \sigma \subset |K| - \operatorname{int}|L|$, so $\beta(x') = 1$. It follows that $h_{U1}|V = \psi \circ |k| \circ \phi^{-1}|V$. Moreover, since $\phi^{-1}(V) \subset \sigma$, we have $h_{U1}|V = \psi \circ |k| \circ \phi^{-1}|V = \psi \circ (|k|\,|\sigma) \circ (\phi^{-1}|V)$.

Since each of the maps $(\phi^{-1}|V)$, $(|k||\sigma)$, and $\psi$ is a homeomorphism onto its image, then so is $h_{U1}|V$. By invariance of domain, $(h|V)(V)$ is open in $E$ and therefore $Y$. This proves the third assertion.

Perform this construction for each $U \in \mathcal{U}$, and define a homotopy $\{h_t : X \to Y\}$ by

$$h_t(x) = \begin{cases} h_{Ut} & \text{if } x \in \operatorname{Cl} U \text{ for some } U \in \mathcal{U},\ t \in I, \\ g(x) & \text{if } x \in X - \bigsqcup_{U \in \mathcal{U}} U,\ t \in I. \end{cases} \tag{4.10}$$

Then $h_1$ is a local homeomorphism at each of its roots at $y_0$, and is homotopic to $g$ and therefore $f$ by a homotopy constant outside of $f^{-1}(N)$. □

For proper maps, we have the following corollary.

Corollary 4.2. *Let $f : X \to Y$ be a proper map from an n-manifold X into a space Y that is locally n-Euclidean at $y_0$, and let $N \subset Y$ be any neighborhood of $y_0$. Then f is properly homotopic to a map that is transverse to $y_0$ by a homotopy that is constant outside of $f^{-1}(N)$.*

*Proof.* We may assume that $N$ is compact, otherwise, we may replace $N$ by a compact neighborhood of $y_0$ contained in $N$. By the theorem, $f$ is homotopic to a map $g$ that is a local homeomorphism at each of its roots at $y_0$ by a homotopy that is constant outside of $f^{-1}(N)$. Since $f$ is proper, $f^{-1}(N)$ is compact, and since the homotopy from $f$ to $g$ is constant off of the compact set $f^{-1}(N)$, then it is a proper homotopy. So $f$ is properly homotopic to $g$, and therefore $g$ is proper. It follows from Theorem 2.6 that $g$ is transverse to $y_0$. □

## 5. Combining isolated roots

This section begins with a succession of lemmas that are needed to complete the proofs of Theorems 1.1 and 1.2. It ends with the proofs of Theorems 1.1 and 1.2. A proof of Theorem 1.1, for compact orientable triangulable manifolds, in [10] uses Whitney's lemma [8]. The proof of Theorem 1.1 for manifolds with boundary in [3] uses microbundle theory and a version of Whitney's lemma applicable to topological manifolds. The proof here, although somewhat longer, is more self-contained. It is centered on Lemma 5.2 below, the idea for which comes from Epstein [6, pages 378–380]. The proof of Theorem 1.2 is also centered on Lemma 5.2.

Lemma 5.1. *Suppose $n > 2$ and Y is locally n-Euclidean at $y_0 \in Y$.*

(1) *Any path in Y with endpoints in $Y - y_0$ is fixed-endpoint-homotopic in Y to a path in $Y - y_0$.*

(2) *Any two paths in $Y - y_0$ that are fixed-endpoint-homotopic in Y are fixed-endpoint-homotopic in $Y - y_0$.*

*Proof.* We may assume that $Y$ is path-connected, otherwise replace $Y$ by the path component containing $y_0$. Then $Y - y_0$ is also path-connected. To see this, let $y_1, y_2 \in Y - y_0$; we will find a path in $Y - y_0$ from $y_1$ to $y_2$. Let $A_1$ be a path in $Y$ from $y_1$ to $y_2$. If $A_1$ is also in $Y - y_0$, then we are done. Otherwise $A_1$ passes through $y_0$. Let $B$ be an $n$-ball with $y_0 \in \operatorname{int} B$. Then $A_1^{-1}(B) \subset I$ is compact and therefore has a minimum $t_{\min}$ and maximum $t_{\max}$. Because $y_0 \in \operatorname{int} B$, it is easy to see that $t_{\min} < t_{\max}$. Since $n > 2$, there is a path $A_2$ in $B - y_0$ from $A_1(t_{\min})$ to $A_1(t_{\max})$. Connect $y_1$ to $y_2$ by the path $A_3$ defined by

$$A_3(t) = \begin{cases} A_1(t) & \text{for } 0 \le t \le t_{\min}, \\ A_2\left(\dfrac{t - t_{\min}}{t_{\max} - t_{\min}}\right) & \text{for } t_{\min} \le t \le t_{\max}, \\ A_1(t) & \text{for } t_{\max} \le t \le 1. \end{cases} \tag{5.1}$$

Let $E$ be a Euclidean neighborhood of $y_0$ and let $y \in E - y_0$. Since $n > 2$, then both $E$ and $E \cap (Y - y_0) = E - y_0$ are simply connected. Therefore an application of van Kampen's theorem [9, pages 211 and 217] to the pair $Y - y_0$ and $E$ shows that the inclusion $i : Y - y_0 \subset E \cup (Y - y_0) = Y$ induces a fundamental group isomorphism $i_\# : \pi(Y - y_0, y) \approx\!\!> \pi(Y, y)$.

To prove the first statement, let $A$ be a path in $Y$ with endpoints in $Y - y_0$. Let $A_1$ be a path in $Y - y_0$ from $y$ to $A(0)$, and $A_2$ a path in $Y - y_0$ from $A(1)$ to $y$. Then $(A_1A)A_2$ is a loop in $Y$ at $y$, so, since $i_\#$ is surjective, $[(A_1A)A_2] = [A_3]$ for some loop $A_3$ in $Y - y_0$ at $y$. Then $(A_1^{-1}A_3)A_2^{-1}$ is a path in $Y - y_0$ and $[A] = [(A_1^{-1}A_3)A_2^{-1}]$.

To prove the second statement, let $A$ and $A'$ be paths in $Y - y_0$ that are fixed-endpoint-homotopic in $Y$. Let $A_1$ be a path in $Y - y_0$ from $y$ to $A(0) = A'(0)$ and let $A_2$ be a path in $Y - y_0$ from $A(1) = A'(1)$ back to $y$. Then $A_1(AA_2)$ and $A_1(A'A_2)$ are loops in $Y - y_0$ at $y$ that are fixed-endpoint-homotopic in $Y$. Since $i_\#$ is injective, then they are fixed-endpoint-homotopic in $Y - y_0$, and therefore $A$ and $A'$ are fixed-endpoint-homotopic in $Y - y_0$. □

Lemma 5.2. *Suppose $n > 2$ and $f : X \to Y$ is a map from a connected $n$-manifold $X$ into a well-connected space $Y$ that is locally $n$-Euclidean at $y_0$. Suppose also that $x_0$ and $x_1$ are two isolated roots of $f$ at $y_0$ that are Nielsen-related by a path $A$ in $X$ from $x_0$ to $x_1$, that $N \subset X$ is a neighborhood of $A$ containing no roots of $f$ other than $x_0$ and $x_1$, and that $E$ is a Euclidean neighborhood of $y_0$. Then there are an $n$-ball $B \subset N$, a map $g : X \to Y$, and a homotopy $\{h_t\}$ from $f$ to $g$ with the following properties:*

(1) *$\{h_t\}$ is constant on a neighborhood of $f^{-1}(y_0)$ and constant off of $N$,*
(2) *$h_t^{-1}(y_0) = f^{-1}(y_0)$ for all $t \in I$,*
(3) *$g = h_1$ maps the pair $(B, \mathrm{Bd}B)$ into the pair $(E, E - y_0)$,*
(4) *any path in $B$ from $x_0$ to $x_1$ is fixed-endpoint-homotopic in $N$ to $A$.*

*Proof.* By taking a smaller neighborhood if necessary, we may assume $N$ connected and open, and therefore a connected $n$-manifold. By [1, Lemma 5.30] there is an $n$-ball $C \subset N$ such that $x_0, x_1 \in \mathrm{int}\, C$ and any path in $C$ from $x_0$ to $x_1$ is fixed-endpoint-homotopic in $N$ to $A$. Let $\phi : \mathbf{B}^n \to C$ be a homeomorphism and set $x_0' = \phi^{-1}(x_0)$ and $x_1' = \phi^{-1}(x_1)$. The picture in Figure 5.1 will be helpful for subsequent constructions.

In this picture, $B' \subset \mathrm{int}\,\mathbf{B}^n$ is a Euclidean ball concentric with $\mathbf{B}^n$ that also has $x_0'$ and $x_1'$ in its interior. (By "Euclidean ball" we mean a ball of the form $\{z \in \mathbb{R}^n \mid \|z - z_0\| \le \epsilon\}$, not just a homeomorph of $\mathbf{B}^n$.) The sets $C_0', C_1' \subset \mathrm{int}\, B'$ are disjoint Euclidean balls centered at $x_0'$ and $x_1'$ such that $f \circ \phi(C_0') \subset E$ and $f \circ \phi(C_1') \subset E$, $\ell$ is the straight line segment from $x_0'$ to $x_1'$, the points where $\ell$ intersects $\mathrm{Bd}\, C_0'$ and $\mathrm{Bd}\, C_1'$ are labeled $z_0'$ and $z_1'$, and $a'$ is the arc from $z_0'$ to $z_1'$ parameterized by $a'(t) = (1-t)z_0' + tz_1'$.

We now construct a deformation retraction

$$\{r_t' : \mathbf{B}^n - (\mathrm{int}\, C_0' \cup \mathrm{int}\, C_1') \longrightarrow \mathbf{B}^n - (\mathrm{int}\, C_0' \cup \mathrm{int}\, C_1')\} \tag{5.2}$$

of $\mathbf{B}^n - (\mathrm{int}\, C_0' \cup \mathrm{int}\, C_1')$ onto $\mathrm{Bd}\, C_0' \cup \mathrm{Bd}\, C_1' \cup a'(I)$. First define $r_1'(x)$, for any $x \in \mathbf{B}^n - (\mathrm{int}\, C_0' \cup \mathrm{int}\, C_1')$, to be the unique point where the line segment joining $x$ to the closest point on $\ell$ intersects $\mathrm{Bd}\, C_0' \cup \mathrm{Bd}\, C_1' \cup a'(I)$. Then, for any $t \in I$, let $r_t'(x) = (1-t)x + tr_1'(x)$.

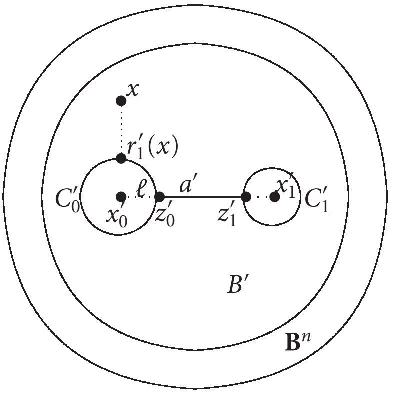



Figure 5.1. The construction in $\mathbf{B}^n$.

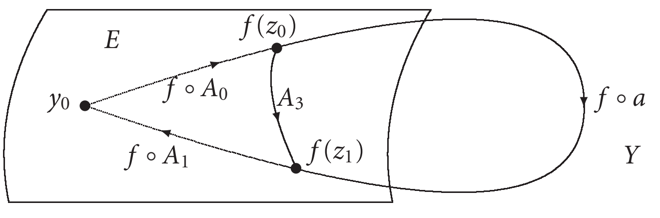



Figure 5.2. The picture in $Y$.

Use $\phi$ to copy this construction into $C$ by letting $B = \phi(B')$, $C_0 = \phi(C_0')$, $C_1 = \phi(C_1')$, $z_0 = \phi(z_0')$, $z_1 = \phi(z_1')$, $a = \phi \circ a'$, and $\{r_t\} = \{\phi \circ r_t' \circ \phi^{-1}\}$. So $\{r_t\}$ is a deformation retraction of $C - (\operatorname{int} C_0 \cup \operatorname{int} C_1)$ onto $\operatorname{Bd} C_0 \cup \operatorname{Bd} C_1 \cup a(I)$, and $a$ is an arc from $z_0$ to $z_1$.

Now let $A_0$ be a path in $C_0$ from $x_0$ to $z_0 = a(0)$, $A_1$ a path in $C_1$ from $z_1 = a(1)$ to $x_1$, and $A_3$ a path in $E - y_0$ from $f(z_0)$ to $f(z_1)$. Then we have the picture in $Y$ shown in Figure 5.2.

Since $(A_0 a)A_1$ is a path in $C$ from $x_0$ to $x_1$, then $[(A_0 a)A_1] = [A]$, and therefore $[f \circ A_0][f \circ a][f \circ A_1] = [f \circ A] = [y_0]$. But $((f \circ A_0)A_3)f \circ A_1$ is a loop in the simply connected space $E$, so we also have $[f \circ A_0][A_3][f \circ A_1] = [y_0]$ which implies that $[f \circ a] = [A_3]$. Since $n > 2$ and the paths $f \circ a$ and $A_3$ are in $Y - y_0$, then by statement (2) of Lemma 5.1 they are not only fixed-endpoint-homotopic in $Y$, but are also fixed-endpoint-homotopic in $Y - y_0$. Thus, there is a map $H : I \times I \to Y - y_0$ such that for every $(s,t) \in I \times I$,

$$\begin{aligned} H(s,0) &= f(z_0), \qquad & H(s,1) &= f(z_1), \\ H(0,t) &= f \circ a(t), \qquad & H(1,t) &= A_3(t) \in E - y_0. \end{aligned} \tag{5.3}$$

Define a homotopy $\{h'_s : C - (\operatorname{int} C_0 \cup \operatorname{int} C_1) \to Y - y_0\}$ by

$$h'_s(x) = \begin{cases} f \circ r_{2s}(x) & \text{for } 0 \le s \le \dfrac{1}{2},\ x \in C - (\operatorname{int} C_0 \cup \operatorname{int} C_1), \\ f \circ r_1(x) & \text{for } \dfrac{1}{2} \le s \le 1,\ x \in r_1^{-1}(\operatorname{Bd} C_1 \cup \operatorname{Bd} C_2), \\ H(2s-1, a^{-1}(r_1(x))) & \text{for } \dfrac{1}{2} \le s \le 1,\ x \in r_1^{-1}(a(I)). \end{cases} \tag{5.4}$$

In the last formula, $a^{-1}(r_1(x))$ is meant to denote the value of $t$ for which $a(t) = r_1(x)$. This makes sense since $x \in r_1^{-1}(a(I))$, and therefore $a^{-1}(r_1(x))$ is a continuous function of $x \in r_1^{-1}(a(I))$. The last two formulas agree on the overlap of their domains, $(x,s) \in r_1^{-1}(\{z_0,z_1\}) \times [1/2,1]$, and this set is closed in $X \times I$. The first formula agrees with the last two when $s = 1/2$, and the set $X \times 1/2$ is also closed in $X \times I$. Thus $h'_s(x)$ is a well-defined continuous function of $(x,s)$.

Now let $\beta : X \to I$ be a map such that $\beta(x) = 1$ for $x \in B$ and $\beta(x) = 0$ for $x \in X - \operatorname{int} C$. Define a homotopy $\{h_t : X \to Y\}$ by

$$h_t(x) = \begin{cases} h'_{\beta(x)t}(x) & \text{for } x \in C - (\operatorname{int} C_0 \cup \operatorname{int} C_1),\ t \in I, \\ f(x) & \text{for } x \in (X - \operatorname{int} C) \cup C_0 \cup C_1. \end{cases} \tag{5.5}$$

The two formulas have the closed set $(x,t) \in (\operatorname{Bd} C \cup \operatorname{Bd} C_0 \cup \operatorname{Bd} C_1) \times I$ for common domain and are easily seen to agree there. Thus $h_t$ is well defined and is continuous in $(x,t)$. Let $g = h_1$. We now verify assertions (1), (2), (3), and (4) of the lemma.

By its definition, $\{h_t\}$ is constant on $X - (C - (\operatorname{int} C_0 \cup \operatorname{int} C_1))$, which is a neighborhood of $f^{-1}(y_0)$. Also $C - (\operatorname{int} C_0 \cup \operatorname{int} C_1) \subset N$, so $\{h_t\}$ is constant off of $N$. This proves assertion (1).

For all $s \in I$, neither $f \circ r_s$ nor $H$ has any roots at $y_0$, so the map $h_s$ has no roots in $C - (\operatorname{int} C_0 \cup \operatorname{int} C_1)$. Moreover, as we have seen, $\{h_t\}$ is constant on $f^{-1}(y_0)$. Therefore, $h_s^{-1}(y_0) = f^{-1}(y_0)$ for all $s \in I$. This verifies assertion (2).

From the definition of $\{h'_s\}$, we see that $h'_1(C - (\operatorname{int} C_0 \cup \operatorname{int} C_1)) = f(\operatorname{Bd} C_0 \cup \operatorname{Bd} C_1) \cup A(3) \subset E - y_0$. Since $\beta(x) = 1$ for $x \in B$, this implies that $h_1(B - (\operatorname{int} C_0 \cup \operatorname{int} C_1)) \subset E - y_0$. Also, $h_1(C_0 \cup C_1) = f(C_0 \cup C_1) \subset E$. Thus $g = h_1$ maps the pair $(B, \operatorname{Bd} B)$ into $(E, E - y_0)$, which verifies assertion (3).

Any path in $B$ from $x_0$ to $x_1$ is also a path in $C$ from $x_0$ to $x_1$ and thus, by the construction of $C$, must be fixed-endpoint-homotopic to $A$. This verifies assertion (4). □

Lemma 5.3. *Suppose $n \ge 1$, $f : X \to Y$ is a map from an $n$-manifold $X$ into a space $Y$ that is locally $n$-Euclidean at $y_0$, $B$ is an $n$-ball in $X$ such that $f(B) \subset E$ for some $n$-Euclidean neighborhood $E$ of $y_0$, and* $\operatorname{Bd} B$ *contains no roots of $f$ at $y_0$. Then there is a homotopy $\{h_t : X \to Y\}$ such that*

(1) $h_0 = f$,
(2) *$\{h_t\}$ is constant at $f$ outside of* $\operatorname{int} B$,
(3) *$B$ contains exactly one root of $h_1$ at $y_0$.*

*Proof.* Let $\phi : \mathbf{B}^n \to B$ and $\psi : \mathbb{R}^n \to E$ be homeomorphisms with $\psi(0) = y_0$. Define $h_1 : X \to Y$ by

$$h_1(x) = \begin{cases} \psi\left(\|\phi^{-1}(x)\|\psi^{-1} \circ f \circ \phi\left(\dfrac{\phi^{-1}(x)}{\|\phi^{-1}(x)\|}\right)\right) & \text{for } x \in B,\ \phi^{-1}(x) \neq 0, \\ y_0 & \text{for } x \in B,\ \phi^{-1}(x) = 0, \\ f(x) & \text{for } x \notin B. \end{cases} \tag{5.6}$$

Then it is easy to see that $h_1$ is continuous, $f$ and $h_1$ agree outside of $\operatorname{int} B$, and $\phi(0)$ is the only root of $h_1$ in $B$ at $y_0$. Define a homotopy $\{h_t : X \to Y\}$ from $f$ to $h_1$ by

$$h_t(x) = \begin{cases} \psi((1-t)\psi^{-1} \circ f(x) + t\psi^{-1} \circ h_1(x)) & \text{for } x \in B,\ t \in I, \\ f(x) & \text{for } x \notin B,\ t \in I. \end{cases} \tag{5.7}$$

Since $f(x) = h_1(x)$ for $x \in \operatorname{Bd} B$, then $h_t$ is a well-defined homotopy from $f$ to $h_1$. The homotopy is clearly constant at $f$ outside of $\operatorname{int} B$. □

LEMMA 5.4. *Suppose $n > 2$ and $k : (\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n) \to (\mathbb{R}^n, \mathbb{R}^n - 0)$ is a map whose induced homomorphism $k_n : H_n(\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n; \mathbb{Z}) \to H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is trivial. Then there is a homotopy $\{\ell_t : (\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n) \to (\mathbb{R}^n, \mathbb{R}^n - 0)\}$ such that*

(1) $\ell_0 = k$,
(2) $\{\ell_t\}$ *is constant on* $\operatorname{Bd}\mathbf{B}^n$,
(3) $\ell_1(\mathbf{B}^n) \subset \mathbb{R}^n - 0$.

*Proof.* Choose a base point $b_0 \in \operatorname{Bd}\mathbf{B}^n$, let $e_0 = k(b_0) \in \mathbb{R}^n - 0$, and consider the commutative diagram

$$\begin{array}{ccc} \pi_n(\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n, b_0) & \xrightarrow{k_{\pi n}} & \pi_n(\mathbb{R}^n, \mathbb{R}^n - 0, e_0) \\ \downarrow{\scriptstyle \text{Hurewicz}} & & \downarrow{\scriptstyle \text{Hurewicz}} \\ H_n(\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n; \mathbb{Z}) & \xrightarrow[k_n]{} & H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \end{array} \tag{5.8}$$

where $k_{\pi n}$ and $k_n$ are induced by $k$. Since $H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) = 0$ for $p < n$ and $\mathbb{R}^n - 0$ is simply connected for $n > 2$, then the right-hand Hurewicz homomorphism is an isomorphism. Since $k_n$ is trivial, it follows that $k_{\pi n}$ is also trivial.

The identity $i : (\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n, b_0) \to (\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n, b_0)$ represents an element $[i] \in \pi_n(\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n, b_0)$, whose image under $k_{\pi n}$ is $[k \circ i] = [k] \in \pi_n(\mathbb{R}^n, \mathbb{R}^n - 0, e_0)$. Since $k_{\pi n}$ is trivial, then $[k] = [e_0]$, so there is a homotopy $\{h_t : (\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n, b_0) \to (\mathbb{R}^n, \mathbb{R}^n - 0, e_0)\}$ such that $h_0 = k$ and $h_1(\mathbf{B}^n) = e_0$. We will use the homotopy $\{h_t\}$ to construct $\{\ell_t\}$.

By Theorem 2.1 the set $C = \bigcup_{t \in I} h_t^{-1}(0)$ is compact and therefore closed in $B$. Since $h_t(\operatorname{Bd}\mathbf{B}^n) \subset \mathbb{R}^n - 0$ for all $t$, then $C$ and $\operatorname{Bd}\mathbf{B}^n$ are disjoint, so there is a map $\beta : \mathbf{B}^n \to I$ that is 0 on $\operatorname{Bd}\mathbf{B}^n$ and 1 on $C$. Define the homotopy $\{\ell_t : (\mathbf{B}^n, \operatorname{Bd}\mathbf{B}^n) \to (\mathbb{R}^n, \mathbb{R}^n - 0)\}$ by $\ell_t(x) = h_{\beta(x)t}$ for all $(x,t) \in \mathbf{B}^n \times I$. Clearly, $\{\ell_t\}$ satisfies properties (1) and (2) of the lemma. Suppose, contrary to (3), that $\ell_1(x) = 0$ for some $x \in \mathbf{B}^n$. Then, by the definition of $\ell$, we have $h_{1\beta(x)}(x) = 0$, so, by the definition of $C$, we have $x \in C$. Therefore $\beta(x) = 1$, so $\ell_1(x) = h_1(x) = e_0 \neq 0$. This contradiction proves (3). $\square$

*Remark 5.5.* The conclusion of Lemma 5.4 is true even when $n = 1,2$. However, we will only need it for $n > 2$.

LEMMA 5.6. *Suppose $n > 2$ and $f : X \to Y$ is a proper map of a connected orientable $n$-manifold $X$ into a well-connected space $Y$. Suppose that $E \subset Y$ is an $n$-dimensional Euclidean neighborhood of $y_0 \in Y$, and $B \subset X$ is an $n$-ball such that $f(B) \subset E$, $f(\operatorname{Bd} B) \subset E - y_0$, and $\lambda(f, \operatorname{int} B) = 0$, where $\lambda$ is the integer root index for $X$, $Y$, $y_0$ relative to some orientation $s_X$ of $X$ and local orientation $\mu$ of $Y$ at $y_0$. Then there is a homotopy $\{h_t : X \to Y\}$ such that $h_0 = f$, $\{h_t\}$ is constant off of $B$, and $h_1(B) \subset E - y_0$.*

Note that because $f$ is proper and $\{h_t\}$ is constant off of the compact set $B$, then $\{h_t\}$ is proper.

*Proof.* We use the following diagram in which $x$ is an arbitrary point in $\operatorname{int} B$, $N$ is a neighborhood of $B$ such that $N - \operatorname{int} B$ is root-free, the maps $f'$ and $f''$ are defined by $f$, and all other maps are inclusions:

$$
\begin{array}{ccccccc}
(X, X - B) & \overset{i}{\subset} & (X, X - \operatorname{int} B) & \overset{e}{\supset} & (N, N - \operatorname{int} B) & \xrightarrow{f'} & (Y, Y - y_0) \\
 & & j \cap & & \cup & & \cup \\
 & & (X, X - x) & & (B, \operatorname{Bd} B) & \xrightarrow{f''} & (E, E - y_0)
\end{array}
\tag{5.9}
$$

Let $o_B \in H_n(X, X - B; \mathbb{Z})$ be the fundamental class around $B$ (using the orientation $s_X$). Then $(j \circ i)_n(o_B) = s_X(x) \in H_n(X, X - x; \mathbb{Z})$. Since $s_X(x)$ generates the infinite cyclic group $H_n(X, X - x; \mathbb{Z})$, then $(j \circ i)_n$ is onto. But $B$ is connected, so according to [4, Corollary 3.4, page 260] $H_n(X, X - B; \mathbb{Z})$ is also infinite cyclic. It follows that $(j \circ i)_n$ is an isomorphism and that $o_X$ generates $H_n(X, X - B; \mathbb{Z})$. Now $X - \operatorname{int} B$ is a deformation retract of $X - x$, so $j_n : H_n(X, X - \operatorname{int} B; \mathbb{Z}) \to H_n(X, X - x; \mathbb{Z})$ is an isomorphism. Thus $i_n : H_n(X, X - B; \mathbb{Z}) \to H_n(X, X - \operatorname{int} B; \mathbb{Z})$ is also an isomorphism. Hence $H_n(N, N - \operatorname{int} B; \mathbb{Z})$ is infinite cyclic and generated by $e_n^{-1} \circ i_n(o_X)$. Since $f_n'(e_n^{-1} \circ i_n(o_X)) = \lambda(f, \operatorname{int} B)\mu = 0\mu = 0$, it follows that $f_n' = 0$. The inclusion $(E, E - y_0) \subset (Y, Y - y_0)$ is an excision and therefore induces homology isomorphisms, so, since $f_n' = 0$, we also have $f_n'' = 0$.

Since $f_n'' = 0$, we may use Lemma 5.4 to construct a homotopy $\{h_t' : (B, \operatorname{Bd} B) \to (E, E - y_0)\}$ such that $h_0' = f''$, $\{h_t'\}$ is constant on $\operatorname{Bd} B$, and $h_0'$ has no roots at $y_0$. Define the desired homotopy $\{h_t : X \to Y\}$ by

$$
h_t(x) = \begin{cases} h_t'(x) & \text{for } (x,t) \in B \times I, \\ f(x) & \text{for } (x,t) \notin B \times I. \end{cases}
\tag{5.10}
$$

$\square$

We are now ready to complete the proofs of Theorems 1.1 and 1.2.

*Proof of Theorem 1.1.* Assume that $f : X \to Y$ is a proper map of a connected $n$-manifold $X$ into a well-connected space $Y$ that is $n$-Euclidean at $y_0$. By Theorem 3.28 every map properly homotopic to $f$ and transverse to $y_0$ has at least $\mathscr{A}(f, y_0)$ roots. To prove the rest of the theorem, assume that $n > 2$; we will show that there is a map properly homotopic to $f$ and transverse to $y_0$ that has no more than $\mathscr{A}(f, y_0)$ roots.

By Corollary 4.2 there is at least one map properly homotopic to $f$ and transverse to $y_0$. Every such map has a finite number of roots at $y_0$, so there must be a map $f_{\min}$ properly homotopic to $f$ and transverse to $y_0$ that has, among all such maps, the fewest number of roots. Call such a map *minimal*. We need to show that $\operatorname{card} f_{\min}^{-1}(y_0) \leq \mathscr{A}(f, y_0)$. To do this, we will assume to the contrary that $\operatorname{card} f_{\min}^{-1}(y_0) > \mathscr{A}(f, y_0)$ and show that $f_{\min}$ is not minimal. This contradiction will prove the theorem.

Since $\sum_{\alpha \in f_{\min}^{-1}(y_0)/N} \operatorname{card} \alpha = \operatorname{card} f_{\min}^{-1}(y_0) > \mathscr{A}(f_{\min}, y_0) = \sum_{\alpha \in f_{\min}^{-1}(y_0)/N} \operatorname{mult}(f_{\min}, \alpha, y_0)$, there must be a root class $\alpha$ such that $\operatorname{card} \alpha > \operatorname{mult}(f_{\min}, \alpha, y_0)$. To show that $f_{\min}$ is not minimal we consider three cases: $X$ is orientable, $X$ is nonorientable but $f_{\min}$ is orientable, and $f_{\min}$ is nonorientable.

*Case 1* ($X$ is orientable). Since $f_{\min}$ is transverse to $y_0$, $f_{\min}$ is a local homeomorphism at each root $x \in \alpha$, and so it is easy to see, using Theorem 3.14, that $\lambda(f_{\min}, x) = \pm 1$ for each $x \in \alpha$. It follows that since $\operatorname{card} \alpha > \operatorname{mult}(f_{\min}, \alpha, y_0) = |\sum_{x \in \alpha} \lambda(f_{\min}, x)|$, there must be two roots, $x_0$ and $x_1$ say, in $\alpha$ such that $\lambda(f_{\min}, x_0) + \lambda(f_{\min}, x_1) = 0$. We will find a homotopy of $f_{\min}$ that eliminates these two roots.

Let $E$ be a Euclidean neighborhood of $y_0$ and let $A$ be a path in $X$ from $x_0$ to $x_1$ such that $[f_{\min} \circ A] = [y_0]$. Since $f_{\min}$ has only a finite number of roots, we may apply statement (1) of Lemma 5.1, with $X$ in place of $Y$, a finite number of times to find a path fixed-endpoint-homotopic to $A$ that avoids all roots other than $x_0$ and $x_1$. So we assume that $A$ already avoids all roots of $f$ other than $x_0$ and $x_1$. Then $A(I)$ has a compact neighborhood $N$ that is disjoint from the closed set $f_{\min}^{-1}(y_0) - \{x_0, x_1\}$. Thus we may apply Lemma 5.2 with $f_{\min}$ in place of $f$ to find an $n$-ball $B \subset N$, a map $g : X \to Y$, and a homotopy $\{h_t\}$ from $f_{\min}$ to $g$ with the properties enumerated in Lemma 5.2. Since $\{h_t\}$ is constant off of the compact set $N$, it is a proper homotopy, and since $\{h_t\}$ is constant on a neighborhood of $f^{-1}(y_0) = g^{-1}(y_0)$, then $g$ is still a local homeomorphism at each of its roots. Since for every $t \in I$, $h_t$ has no roots on $\operatorname{Bd} B$, we have, by the homotopy and additivity properties of the index, $\lambda(g, \operatorname{int} B) = \lambda(f_{\min}, \operatorname{int} B) = \lambda(f_{\min}, x_0) + \lambda(f_{\min}, x_1) = 0$. Now apply Lemma 5.6 with $g$ in place of $f$ to find another homotopy $\{h'_t\}$ that is constant off of $B$ such that $h'_0 = g$ and $h'_1$ has no roots at $y_0$ in $B$. Then $h'_1$ agrees with $g$ on $X - B$ and has no roots in $B$, so it has two fewer roots than $f_{\min}$ does. It is also properly homotopic to $f_{\min}$ and a local homeomorphism at each of its roots and therefore, since it is proper, transverse to $y_0$. Thus $f_{\min}$ is not minimal, and the proof is complete in the $X$ orientable case.

*Case 2* ($X$ is nonorientable, $f_{\min}$ is orientable). Let $\widetilde{p} : \widetilde{X} \to X$ be the orientation covering for $X$, and $\widetilde{\alpha}$ a root class of $f_{\min} \circ \widetilde{p}$ at $y_0$ such that $\widetilde{p}^{-1}(\alpha) = \widetilde{\alpha} \sqcup (-\widetilde{\alpha})$. Then $\widetilde{p}$ takes $\widetilde{\alpha}$ bijectively onto $\alpha$, so $\operatorname{card} \widetilde{\alpha} = \operatorname{card} \alpha > \operatorname{mult}(f_{\min}, \alpha, y_0) = |\lambda(f_{\min} \circ \widetilde{p}, \widetilde{\alpha})|$. Since $\widetilde{p}$ is a covering and $f_{\min}$ is transverse to $y_0$, it follows that $f_{\min} \circ \widetilde{p}$ is a local homeomorphism at each $\widetilde{x} \in \widetilde{\alpha}$. Then, arguing as in Case 1, there are two roots, $\widetilde{x}_0$ and $\widetilde{x}_1$ say, in $\widetilde{\alpha}$ such that $\lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_0) + \lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_1) = 0$. Let $\widetilde{A}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ such that $[f_{\min} \circ \widetilde{p} \circ \widetilde{A}] = [y_0]$. Let $A = \widetilde{p} \circ \widetilde{A}$, $x_0 = \widetilde{p}(\widetilde{x}_0)$, and $x_1 = \widetilde{p}(\widetilde{x}_1)$, so $A$ is a path in $X$ from $x_0$ to $x_1$ such that $[f_{\min} \circ A] = [y_0]$. Since $n > 2$, we may assume that $\widetilde{A}$ avoids all roots of $f_{\min} \circ \widetilde{p}$ other than $\widetilde{x}_0$ and $\widetilde{x}_1$, and therefore $A$ avoids all roots of $f_{\min}$ other than $x_0$

and $x_1$. Let $N$ be a compact neighborhood of $A(I)$ containing no roots of $f_{\min}$ other than $x_0$ and $x_1$, and apply Lemma 5.2 with $f_{\min}$ in place of $f$ to find an $n$-ball $B \subset N$, a map $g : X \to Y$, and a homotopy $\{h_t\}$ from $f_{\min}$ to $g$ with the properties enumerated in Lemma 5.2. Since $B$ is simply connected, then it is evenly covered. Let $\widetilde{B}$ be the component of $\widetilde{p}^{-1}(B)$ containing $\widetilde{x}_0$. Then $\widetilde{x}_1 \in \widetilde{B}$ as well. To see this, let $A'$ be a path in $B$ from $x_0$ to $x_1$. Then $[A'] = [A]$. Since $(\widetilde{p}|\widetilde{B})^{-1} \circ A'$ and $\widetilde{A}$ are lifts of $A'$ and $A$ that both begin at $\widetilde{x}_0$, it follows that $[(\widetilde{p}|\widetilde{B})^{-1} \circ A'] = [\widetilde{A}]$, and therefore that $\widetilde{x}_1 = \widetilde{A}(1) = (\widetilde{p}|\widetilde{B})^{-1} \circ A'(1) \in \widetilde{B}$. Now apply Lemma 5.6 with $\widetilde{B}$ in place of $B$ and $g \circ \widetilde{p}$ in place of $f$ to find a homotopy $\{\widetilde{h}_t : \widetilde{X} \to Y\}$ that begins at $g \circ \widetilde{p}$ that is constant off of $\widetilde{B}$, and such that $\widetilde{h}_1$ has no roots in $\widetilde{B}$. Define a homotopy $\{h_t : X \to Y\}$ beginning at $g$ by

$$h_t(x) = \begin{cases} \widetilde{h}_t \circ (\widetilde{p}|\widetilde{B})^{-1}(x) & \text{for } (x,t) \in B \times I, \\ g(x) & \text{for } (x,t) \notin B \times I. \end{cases} \tag{5.11}$$

Then it is straightforward that $h_1$ is properly homotopic to $f_{\min}$, transverse to $y_0$, and has two fewer roots at $y_0$ than $f_{\min}$. Thus $f_{\min}$ is not minimal, and this completes the proof for Case 2.

*Case 3* ($f_{\min}$ nonorientable). In this case $\text{mult}(f_{\min}, \alpha, y_0) = |\lambda_2(f, \alpha)|$, so $\text{mult}(f_{\min}, \alpha, y_0)$ is either 0 or 1. Since $f_{\min}$ is a local homeomorphism at each root, then $\lambda_2(x) = [1] \in \mathbb{Z}/2\mathbb{Z}$ for each $x \in \alpha$. Thus, if $\text{mult}(f_{\min}, \alpha, y_0) = 0$, then $\alpha$ has an even number of roots greater than 0. On the other hand, since $\operatorname{card} \alpha > \text{mult}(f_{\min}, \alpha, y_0)$, if $\text{mult}(f_{\min}, \alpha, y_0) = 1$, then $\alpha$ contains two or more roots. Therefore, in either case, we may find two distinct roots $x_0, x_1 \in \alpha$. Let $\widetilde{p} : \widetilde{X} \to X$ be the orientation covering of $X$. Let $\widetilde{x}_0 \in \widetilde{p}^{-1}(x_0)$ and $\widetilde{x}_1 \in \widetilde{p}^{-1}(x_1)$. Then, since $f_{\min}$ is nonorientable, Theorem 3.18 implies that all four points $\widetilde{x}_0, -\widetilde{x}_0$, $\widetilde{x}_1$, and $-\widetilde{x}_0$ are Nielsen related roots of $f_{\min} \circ \widetilde{p}$ at $y_0$. Also $f_{\min} \circ \widetilde{p}$ is a local homeomorphism at each root, so the integer root index at each of these roots is $\pm 1$. Since $\widetilde{x} \mapsto -\widetilde{x}$ is an orientation-reversing homeomorphism of $\widetilde{X}$, then $\lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_1) = -\lambda(f_{\min} \circ \widetilde{p}, -\widetilde{x}_1)$. Hence either $\lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_0) + \lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_1) = 0$ or $\lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_0) + \lambda(f_{\min} \circ \widetilde{p}, -\widetilde{x}_1) = 0$. Assume, without loss of generality, that $\lambda(f_{\min} \circ \widetilde{p}, \widetilde{x}_0) + \lambda(f \circ \widetilde{p}, \widetilde{x}_1) = 0$ (otherwise we would replace $\widetilde{x}_1$ by $-\widetilde{x}_1$). The proof now proceeds exactly as in Case 2 above. □

For nonorientable $f$, Theorem 1.1 has the following corollary.

Corollary 5.7. *Suppose $n > 2$ and $f : X \to Y$ is a nonorientable proper map from a connected nonorientable $n$-manifold $X$ into a well-connected space $Y$ that is locally $n$-Euclidean at $y_0$. Then*

(1) $\text{PNR}(f, y_0) = \mathcal{A}(f, y_0)$,
(2) *a Nielsen root class $\alpha$ of $f$ at $y_0$ is properly essential if and only if* $\text{mult}(f, \alpha, y_0) \neq 0$.

*Proof.* Let $S_{\neq 0}$ be the set of all Nielsen root classes of $f$ at $y_0$ that have nonzero multiplicity, and let $S_{\text{ess}}$ be the set of all properly essential Nielsen root classes of $f$ at $y_0$. We first prove

$$\text{PNR}(f, y_0) \leq \mathcal{A}(f, y_0) = \operatorname{card} S_{\neq 0} \leq \operatorname{card} S_{\text{ess}} = \text{PNR}(f, y_0). \tag{5.12}$$

Now, $\text{PNR}(f, y_0)$ is a proper homotopy invariant lower bound on the number of roots of $f$, but according to Theorem 1.1 there is a map properly homotopic to $f$ that has $\mathcal{A}(f, y_0)$ roots at $y_0$. This justifies the first inequality. Since $f$ is nonorientable, each of its root classes has multiplicity 0 or 1, and since the absolute degree is the sum of these multiplicities, we have the first equality above. Since every Nielsen class with nonzero multiplicity is essential, then $S_{\neq 0} \subset S_{\text{ess}}$. This justifies the second inequality. The last equality is the definition of $\text{PNR}(f, y_0)$.

The first assertion follows directly from (5.12). Also, from (5.12), we have $\operatorname{card} S_{\neq 0} = \operatorname{card} S_{\text{ess}}$, and since $S_{\neq 0} \subset S_{\text{ess}}$ and the sets are finite, this proves $S_{\neq 0} = S_{\text{ess}}$, which is the second assertion. □

*Proof of Theorem 1.2.* Again assume that $f : X \to Y$ is a proper map of a connected $n$-manifold $X$ into a well-connected space $Y$ that is $n$-Euclidean at $y_0$. We have already seen that every map properly homotopic to $f$ has at least $\text{PNR}(f, y_0)$ roots at $y_0$ (Theorem 3.2) and every Nielsen root class of $f$ at $y_0$ with nonzero multiplicity is properly essential (Corollary 3.24). It remains to show that if $n > 2$, then

(1) there is a map properly homotopic to $f$ that has exactly $\text{PNR}(f, y_0)$ roots at $y_0$,
(2) a root class of $f$ is properly essential only if it has nonzero multiplicity.

For nonorientable maps, both of these assertions follow from Theorem 1.1 and Corollary 5.7, so we need to consider only orientable maps.

Call a map *minimal* if no other map properly homotopic to $f$ has fewer roots. Then, since there are maps properly homotopic to $f$ with only a finite number of roots, we know that there is a minimal map $f_{\min}$ and it has only a finite number of roots.

We first show that every root class of $f_{\min}$ has only one element. Suppose to the contrary that a root class $\alpha$ has two distinct roots $x_0, x_1 \in \alpha$. Let $A$ be a path in $X$ from $x_0$ to $x_1$ such that $[f \circ A] = [y_0]$. Since $n > 2$ and $f_{\min}^{-1}(y_0)$ is finite, we may apply statement (1) of Lemma 5.1 a finite number of times to ensure that $A$ does not pass through any roots of $f$ other than $x_0$ and $x_1$. There is then a compact neighborhood $N$ of $A(I)$ containing no roots of $f$ other than $x_0$ and $x_1$. Now apply Lemma 5.2 with $f_{\min}$ in place of $f$ to find an $n$-ball $B \subset N$, a map $g : X \to Y$, and a homotopy $\{h_t\}$ from $f$ to $g$ with the properties enumerated in Lemma 5.2. Since $\{h_t\}$ is constant off of the compact set $N$, it is a proper homotopy. Now apply Lemma 5.3 with $g$ in place of $f$ to obtain a homotopy $\{h_t\}$ beginning at $g$, and constant off of $B$ such that $h_1$ has only one root in $B$. Then $h_1$ is properly homotopic to $f_{\min}$ but has fewer roots—contradicting the minimality of $f_{\min}$. It follows that every root class of $f$ has only one element.

We now show that each root class of $f_{\min}$ has nonzero multiplicity. Let $\alpha = \{x\}$ be a root class of $f_{\min}$ and suppose, contrary to what we want to show, that $\text{mult}(f_{\min}, \alpha, y_0) = 0$. Let $E$ be a Euclidean neighborhood of $y_0$. Then $x$ has an $n$-ball neighborhood $B$ such that $f_{\min}(B) \subset E$ and $f_{\min}(\operatorname{Bd} B) \subset E - y_0$. We consider two cases, $X$ orientable, and $X$ nonorientable but $f$ orientable. (The case for nonorientable $f$ has already been covered.)
*Case 1* ($X$ orientable). Since $|\lambda(f_{\min}, \alpha)| = \text{mult}(f_{\min}, \alpha, y_0) = 0$, by additivity we have $\lambda(f_{\min}, \operatorname{int} B) = \lambda(f_{\min}, \alpha) = 0$. Thus we may apply Lemma 5.6 with $f_{\min}$ in place of $f$ to find a homotopy $\{h_t\}$ that is constant off of $B$ such that $h_0 = f_{\min}$ and $h_1$ has no roots at $y_0$ in $B$. Then $h_1$ agrees with $f_{\min}$ on $X - B$ and has no roots in $B$, so it has fewer roots than

$f_{\min}$ does. It is also properly homotopic to $f_{\min}$ since $f_{\min}$ is proper and $\{h_t\}$ is constant off of the compact set $B$. This contradicts the minimality of $f_{\min}$ and thereby shows that we must have $\operatorname{mult}(f_{\min},\alpha,y_0)\neq 0$.

*Case 2* ($X$ nonorientable and $f_{\min}$ orientable). Let $\widetilde{p}:\widetilde{X}\to X$ be the orientation covering of $X$. Since $B$ is simply connected, it is evenly covered by $\widetilde{p}$, so there is an $n$-ball $\widetilde{B}\subset\widetilde{X}$ such that $\widetilde{p}$ maps $\widetilde{B}$ and $-\widetilde{B}$ homeomorphically onto $B$. Let $\widetilde{\alpha}=(\widetilde{p}|\widetilde{B})^{-1}(\alpha)$, so $\widetilde{\alpha}=\{(\widetilde{p}|\widetilde{B})^{-1}(x)\}$ and $-\widetilde{\alpha}$ are the two Nielsen root classes of $f_{\min}\circ\widetilde{p}$ that $\widetilde{p}$ maps onto $\alpha$. Then $|\lambda(f_{\min}\circ\widetilde{p},\widetilde{\alpha})|=\operatorname{mult}(f_{\min},\alpha,y_0)=0$, so by additivity we have $\lambda(f_{\min}\circ\widetilde{p},\operatorname{int}\widetilde{B})=\lambda(f_{\min}\circ\widetilde{p},\alpha)=0$. Thus we may apply Lemma 5.6 with $f_{\min}\circ\widetilde{p}$ in place of $f$ and $\widetilde{B}$ in place of $B$ to find a homotopy $\{\widetilde{h}_t\}$ that is constant off of $\widetilde{B}$ such that $\widetilde{h}_0=f_{\min}\circ\widetilde{p}$ and $\widetilde{h}_1$ has no roots at $y_0$ in $B$. Define $\{h_t:X\to Y\}$ by

$$h_t(x)=\begin{cases}\widetilde{h}_t\circ(\widetilde{p}|\widetilde{B})^{-1}(x) & \text{for } (x,t)\in B\times I,\\ f_{\min}(x) & \text{for } (x,t)\notin B\times I.\end{cases}\tag{5.13}$$

Then $h_1$ is properly homotopic to $f_{\min}$, has the same roots as $f_{\min}$ outside of $B$, but has no roots in $B$. This contradicts the minimality of $f_{\min}$ and completes the proof that $\operatorname{mult}(f_{\min},\alpha,y_0)\neq 0$ for every Nielsen root class of $f_{\min}$.

Since each root class of $f_{\min}$ has nonzero multiplicity, then each root class is properly essential. Thus $f_{\min}$ has only $\operatorname{PNR}(f_{\min},y_0)=\operatorname{PNR}(f,y_0)$ root classes. Since each root class contains only one root, then $f_{\min}$ has only $\operatorname{PNR}(f_{\min},y_0)=\operatorname{PNR}(f,y_0)$ roots. This proves the first assertion.

Now let $S_{\neq 0}(f)$ be the set of root classes of $f$ that have nonzero multiplicity, let $S_{\text{ess}}(f)$ be the set of essential root classes of $f$, and similarly for $f_{\min}$. Then

$$\begin{aligned}\operatorname{PNR}(f,y_0)&=\operatorname{card} f_{\min}^{-1}(y_0)=\operatorname{card} S_{\neq 0}(f_{\min})\\ &=\operatorname{card} S_{\neq 0}(f)\leq \operatorname{card} S_{\text{ess}}(f)=\operatorname{PNR}(f,y_0).\end{aligned}\tag{5.14}$$

Here, the first two equalities are what we have just proved, the third follows from Corollary 3.23, the inequality follows from Corollary 3.24, which implies that $S_{\neq 0}(f)\subset S_{\text{ess}}(f)$, and the last equality is the definition of PNR. Thus the two finite sets $S_{\neq 0}(f)\subset S_{\text{ess}}(f)$ have the same cardinality and must therefore be equal. This proves the second assertion. □

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