ON SOME BANACH SPACE CONSTANTS ARISING IN NONLINEAR FIXED POINT AND EIGENVALUE THEORY

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As is well known, in any infinite-dimensional Banach space one may find fixed point free self-maps of the unit ball, retractions of the unit ball onto its boundary, contractions of the unit sphere, and nonzero maps without positive eigenvalues and normalized eigenvectors. In this paper, we give upper and lower estimates, or even explicit formulas, for the minimal Lipschitz constant and measure of noncompactness of such maps.

1. A "folklore" theorem of nonlinear analysis

Given a Banach space X, we denote by $B_r(X) := \{x \in X : \|x\| \leq r\}$ the closed ball and by $S_r(X) := \{x \in X : \|x\| = r\}$ the sphere of radius $r > 0$ in X; in particular, we use the shortcut $B(X) := B_1(X)$ and $S(X) := S_1(X)$ for the unit ball and sphere. All maps considered in what follows are assumed to be continuous. By $\nu(x) := x/\|x\|$ we denote the radial retraction of $X \setminus \{0\}$ onto $S(X)$.

One of the most important results in nonlinear analysis is Brouwer's fixed point principle which states that every map $f : B(\mathbb{R}^N) \to B(\mathbb{R}^N)$ has a fixed point. Interestingly, this characterizes finite-dimensional Banach spaces, inasmuch as in each infinite-dimensional Banach space X one may find a fixed point free self-map of $B(X)$.

The existence of fixed point free self-maps is closely related to the existence of other "pathological" maps in infinite-dimensional Banach spaces, namely, retractions on balls and contractions on spheres. Recall that a set $S \subset X$ is a *retract* of a larger set $B \supset S$ if there exists a map $\rho : B \to S$ with $\rho(x) = x$ for $x \in S$; this means that one may extend the identity from S by continuity to B. Likewise, a set $S \subset X$ is called *contractible* if there exists a homotopy $h : [0,1] \times S \to S$ joining the identity with a constant map, that is, such that $h(0,x) = x$ and $h(1,x) \equiv x_0 \in S$. We summarize with the following Theorem 1.1; although this theorem seems to be known in topological nonlinear analysis, we sketch a brief proof which we will use in the sequel.

Theorem 1.1. *The following four statements are equivalent in a Banach space X:*

(a) *each map $f : B(X) \to B(X)$ has a fixed point,*

(b) *$S(X)$ is not a retract of $B(X)$,*

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(c) *$S(X)$ is not contractible,*
(d) *for each map $g : B(X) \to X \setminus \{0\}$, one may find $\lambda > 0$ and $e \in S(X)$ such that $g(e) = \lambda e$.*

Sketch of the proof. (a)⇒(b). If $\rho : B(X) \to S(X)$ is a retraction, the map $f : B(X) \to B(X)$ defined by

$$f(x) := -\rho(x) \tag{1.1}$$

is fixed point free.

(b)⇒(c). Given a homotopy $h : [0,1] \times S(X) \to S(X)$ with $h(0,x) = x$ and $h(1,x) \equiv x_0 \in S(X)$, for $0 < r < 1$ we set

$$\rho(x) := \begin{cases} x_0 & \text{for } \|x\| \le r, \\ h\left(\dfrac{1-\|x\|}{1-r}, \nu(x)\right) & \text{for } \|x\| > r. \end{cases} \tag{1.2}$$

Then, $\rho : B(X) \to S(X)$ is a retraction.

(c)⇒(d). Given $g : B(X) \to X \setminus \{0\}$, for $0 < r < 1$ we set

$$\sigma(x) := \begin{cases} -g\left(\dfrac{x}{r}\right) & \text{for } \|x\| \le r, \\ \dfrac{\|x\| - r}{1-r}x - \dfrac{1-\|x\|}{1-r}g(\nu(x)) & \text{for } \|x\| > r. \end{cases} \tag{1.3}$$

Then, there exists $z \in B(X)$ with $\sigma(z) = 0$, since otherwise $h(\tau,x) := \nu(\sigma((1-\tau)x))$ would be a homotopy on $S(X)$ satisfying $h(0,x) = x$ and $h(1,x) \equiv \nu(\sigma(0))$. Clearly, $r < \|z\| < 1$. Putting

$$\lambda := \frac{\|z\| - r}{1 - \|z\|}\|z\|, \qquad e := \nu(z), \tag{1.4}$$

one easily sees that $\lambda > 0$ and $e \in S(X)$ satisfy $g(e) = \lambda e$ as claimed.

(d)⇒(a). Given a fixed point free map $f : B(X) \to B(X)$, consider the map

$$g(x) := f(x) - x. \tag{1.5}$$

If $g(e) = \lambda e$ for some $e \in S(X)$, then we will certainly have $|\lambda + 1| = \|(\lambda+1)e\| = \|g(e) + e\| = \|f(e)\| \le 1$, hence $\lambda \le 0$. □

Although the above proof is complete, we still sketch another three implications.

(c)⇒(b). Given a retraction $\rho : B(X) \to S(X)$, consider the homotopy

$$h(\tau,x) := \rho((1-\tau)x). \tag{1.6}$$

Then, $h : [0,1] \times S(X) \to S(X)$ satisfies $h(0,x) = x$ and $h(1,x) \equiv \rho(0)$.

(c)⇒(a). Given a fixed point free map $f : B(X) \to B(X)$, consider the homotopy

$$h(\tau,x) := \begin{cases} \nu\left(x - \dfrac{\tau}{r} f(x)\right) & \text{for } 0 \le \tau < r, \\ \nu\left(\dfrac{1-\tau}{1-r}x - f\left(\dfrac{1-\tau}{1-r}x\right)\right) & \text{for } r \le \tau \le 1. \end{cases} \tag{1.7}$$

Then, $h : [0,1] \times S(X) \to S(X)$ satisfies $h(0,x) = x$ and $h(1,x) \equiv -\nu(f(0))$.

(a)⇒(d). Given $g : B(X) \to X \setminus \{0\}$, consider the map $f : B(X) \to B(X)$ defined by

$$f(x) := \begin{cases} g(x) + x & \text{for } \|g(x) + x\| \le 1, \\ \nu(g(x) + x) & \text{for } \|g(x) + x\| > 1. \end{cases} \tag{1.8}$$

Let e be a fixed point of f which exists by (a). If $\|g(e) + e\| \le 1$, then $g(e) = 0$, contradicting our assumption that $g(B(X)) \subseteq X \setminus \{0\}$. So, we must have $\|g(e) + e\| > 1$, hence $e \in S(X)$ and $g(e) = \lambda e$ with $\lambda = \|g(e) + e\| - 1 > 0$.

It is a striking fact that all four assertions of Theorem 1.1 are *true* if $\dim X < \infty$, but *false* if $\dim X = \infty$. This means that in any infinite-dimensional Banach space one may find not only fixed point free self-maps of the unit ball, but also retractions of the unit ball onto its boundary, contractions of the unit sphere, and nonzero maps without positive eigenvalues and normalized eigenvectors. The first examples of this type have been constructed in special spaces; for the reader's ease we recall two of them, the first one due to Kakutani [22] and the second is due to Leray [24].

Example 1.2. In $X = \ell^2$, consider the map $f : B(\ell^2) \to B(\ell^2)$ defined by

$$f(x) = f(\xi_1, \xi_2, \xi_3, \dots) = \left(\sqrt{1 - \|x\|^2}, \xi_1, \xi_2, \dots\right) \quad \left(x = (\xi_n)_n\right). \tag{1.9}$$

It is easy to see that $f(x) \ne x$ for any $x \in B(\ell^2)$. By (1.5), this map gives rise to the operator

$$g(x) = g(\xi_1, \xi_2, \xi_3, \dots) = \left(\sqrt{1 - \|x\|^2} - \xi_1, \xi_1 - \xi_2, \xi_2 - \xi_3, \dots\right) \tag{1.10}$$

which clearly has no positive eigenvalues (actually, no eigenvalues at all) on $S(\ell^2)$.

Example 1.3. In $X = C[0,1]$, define for $0 \le \tau \le 1/2$ a family of maps $U(\tau) : S(C[0,1]) \to C[0,1]$ by

$$U(\tau)x(t) := \begin{cases} x\left(\dfrac{t}{1-\tau}\right) & \text{for } 0 \le t \le 1 - \tau, \\ x(1) + 4\tau(1 - x(1))(t - 1 + \tau) & \text{for } 1 - \tau \le t \le 1. \end{cases} \tag{1.11}$$

Then, the homotopy $h : [0,1] \times S(C[0,1]) \to S(C[0,1])$ defined by

$$h(\tau,x)(t) := \begin{cases} U(\tau)x(t) & \text{for } 0 \le \tau \le \dfrac{1}{2}, \\ (2\tau - 1)t + (2 - 2\tau)U\left(\dfrac{1}{2}\right)x(t) & \text{for } \dfrac{1}{2} \le \tau \le 1, \end{cases} \tag{1.12}$$

satisfies $h(0,x) = x$ and $h(1,x) \equiv x_0$, where $x_0(t) = t$. By (1.2) (with $r = 1/2$), this homotopy gives rise to the retraction

$$\rho(x) = \begin{cases} x_0 & \text{for } 0 \le \|x\| \le \dfrac{1}{2}, \\ (3 - 4\|x\|)x_0 + (4\|x\| - 2)U\left(\dfrac{1}{2}\right)x & \text{for } \dfrac{1}{2} \le \|x\| \le \dfrac{3}{4}, \\ U(2 - 2\|x\|)x & \text{for } \dfrac{3}{4} \le \|x\| \le 1, \end{cases} \tag{1.13}$$

of the ball $B(C[0,1])$ onto its boundary $S(C[0,1])$.

2. Lipschitz conditions and measures of noncompactness

Given two metric spaces M and N and some (in general, nonlinear) operator $F : M \to N$, we denote by

$$\operatorname{Lip}(F) = \inf\{k > 0 : d(F(x), F(y)) \le kd(x,y)\ (x,y \in M)\} \tag{2.1}$$

its (minimal) *Lipschitz constant*. Recall that a nonnegative set function ϕ defined on the bounded subsets of a normed space X is called *measure of noncompactness* if it satisfies the following requirements ($A, B \subset X$ bounded, $K \subset X$ compact, $\lambda > 0$):

(i) $\phi(A \cup B) = \max\{\phi(A), \phi(B)\}$ (set additivity);
(ii) $\phi(\lambda A) = \lambda\phi(A)$ (homogeneity);
(iii) $\phi(A + K) = \phi(A)$ (compact perturbations);
(iv) $\phi([0,1] \cdot A) = \phi(A)$ (absorption invariance).

We point out that in the literature it is usually required that $\phi(\overline{\text{co}}A) = \phi(A)$, that is, ϕ is invariant with respect to the convex closure of a set A; however, since in our calculations we only need to consider convex closures of sets of the form $A \cup \{0\}$, absorption invariance suffices for our purposes.

The most important examples are the *Kuratowski measure of noncompactness* (or *set measure of noncompactness*)

$$\alpha(M) = \inf\{\varepsilon > 0 : M \text{ may be covered by finitely many sets of diameter} \le \varepsilon\}, \tag{2.2}$$

the *Istrăţescu measure of noncompactness* (or *lattice measure of noncompactness*)

$$\beta(M) = \sup\left\{\varepsilon > 0 : \exists \text{ a sequence } (x_n)_n \text{ in } M \text{ with } \|x_m - x_n\| \ge \varepsilon \text{ for } m \ne n\right\}, \tag{2.3}$$

and the *Hausdorff measure of noncompactness* (or *ball measure of noncompactness*)

$$\gamma(M) = \inf\{\varepsilon > 0 : \exists \text{ a finite } \varepsilon\text{-net for } M \text{ in } X\}. \tag{2.4}$$

These measures of noncompactness are mutually equivalent in the sense that

$$\gamma(M) \le \beta(M) \le \alpha(M) \le 2\gamma(M) \tag{2.5}$$

for any bounded set $M \subset X$. Given $M \subseteq X$, an operator $F : M \to Y$, and a measure of noncompactness ϕ on X and Y, the characteristic

$$\phi(F) = \inf\{k > 0 : \phi(F(A)) \le k\phi(A) \text{ for bounded } A \subseteq M\} \tag{2.6}$$

is called the *ϕ-norm* of F. It follows directly from the definitions that $\phi(F) \le \operatorname{Lip}(F)$ in case $\phi = \alpha$ or $\phi = \beta$. Moreover, if L is linear, then clearly $\operatorname{Lip}(L) = \|L\|$, and so $\alpha(L) \le \|L\|$ and $\beta(L) \le \|L\|$. A detailed account of the theory and applications of measures of noncompactness may be found in the monographs [1, 2].

In view of conditions (a) and (b) of Theorem 1.1, the two characteristics

$$L(X) = \inf\{k > 0 : \exists \text{ a fixed point free map } f : B(X) \longrightarrow B(X) \text{ with } \operatorname{Lip}(f) \le k\}, \tag{2.7}$$

$$R(X) = \inf\{k > 0 : \exists \text{ a retraction } \rho : B(X) \longrightarrow S(X) \text{ with } \operatorname{Lip}(\rho) \le k\} \tag{2.8}$$

have found a considerable interest in the literature; we call (2.7) the *Lipschitz constant* and (2.8) the *retraction constant* of the space X. Surprisingly, for the characteristic (2.7), one has $L(X) = 1$ in each infinite-dimensional Banach space X. Clearly, $L(X) \ge 1$, by the classical Banach-Caccioppoli fixed point theorem. On the other hand, it was proved in [26] that $L(X) < \infty$ in every infinite-dimensional space X. Now, if $f : B(X) \to B(X)$ satisfies $\operatorname{Lip}(f) > 1$, without loss of generality, then following [8] we fix $\varepsilon \in (0, \operatorname{Lip}(f) - 1)$ and consider the map $f_\varepsilon : B(X) \to B(X)$ defined by

$$f_\varepsilon(x) := x + \varepsilon \frac{f(x) - x}{\operatorname{Lip}(f) - 1}. \tag{2.9}$$

A straightforward computation shows then that every fixed point of f_ε is also a fixed point of f, and that $\operatorname{Lip}(f_\varepsilon) \le 1 + \varepsilon$, hence $L(X) \le 1 + \varepsilon$. On the other hand, calculating or estimating the characteristic (2.8) is highly nontrivial and requires rather sophisticated individual constructions in each space X (see [3, 4, 5, 6, 7, 11, 13, 16, 17, 19, 23, 25, 28, 29, 30, 35]). To cite a few examples, one knows that $R(X) \ge 3$ in any Banach space, while $4.5 \le R(X) \le 31.45\ldots$ if X is Hilbert. Moreover, the special upper estimates

$$R(\ell^1) < 31.64\ldots, \quad R(c_0) < 35.18\ldots, \quad R(L^1[0,1]) \le 9.43\ldots, \quad R(C[0,1]) \le 23.31\ldots, \tag{2.10}$$

are known; a survey of such estimates and related problems may be found in the book [19] or, more recently, in [18].

In view of Theorem 1.1, it seems interesting to introduce yet another two characteristics, namely,

$$\begin{gathered} E(X) = \inf\{k > 0 : \exists\, g : B(X) \longrightarrow X \setminus \{0\} \text{ with } \operatorname{Lip}(g) \le k, \\ g(e) \ne \lambda e \ \forall \lambda > 0,\ e \in S(X)\} \end{gathered} \tag{2.11}$$

which we call the *eigenvalue constant* of X, and

$$\begin{gathered} H(X) = \inf\{k > 0 : \exists\, h : [0,1] \times S(X) \longrightarrow S(X) \text{ with } \operatorname{Lip}(h) \le k, \\ h(0,x) = x,\ h(1,x) \equiv \text{const}\}, \end{gathered} \tag{2.12}$$

which we call the *contraction constant* of X. Here, by $\mathrm{Lip}(h)$ we mean the smallest $k > 0$ such that

$$||h(\tau,x) - h(\tau,y)|| \leq k\|x-y\| \quad (0 \leq \tau \leq 1,\ x,y \in S(X)). \tag{2.13}$$

Observe that, similarly as for the constant (2.7), the calculation of (2.11) is trivial, because $E(X) = 0$ in every infinite-dimensional space X. In fact, according to [26] we may choose first some fixed point free Lipschitz map $f : B(X) \to B(X)$, and then define a Lipschitz continuous map $g : B(X) \to X \setminus \{0\}$ without positive eigenvalues on $S(X)$ as in (1.5). This shows that $E(X) < \infty$. Now, it suffices to observe that the eigenvalue equation $g(e) = \lambda e$ is invariant under rescaling, that is, the map εg has, for any $\varepsilon > 0$, no positive eigenvalues on $S(X)$. But $\mathrm{Lip}(\varepsilon g) = \varepsilon \mathrm{Lip}(g)$, and so $E(X)$ may be made arbitrarily small.

If we define a homotopy h through a given Lipschitz continuous retraction $\rho : B(X) \to S(X)$ like in (1.6), then an easy calculation shows that (2.13) holds for h with $k = \mathrm{Lip}(\rho)$, and so $H(X) \leq R(X)$.

The main problem we are now interested in consists in finding (possibly sharp) estimates for $\phi(F)$, where F is one of the maps f, ρ, h, and g arising in Theorem 1.1, and ϕ is some measure of noncompactness (e.g., $\phi \in \{\alpha,\beta,\gamma\}$). To this end, for a normed space X we introduce the characteristics

$$L_\phi(X) = \inf\{k>0 : \exists \text{ a fixed point free map } f : B(X) \longrightarrow B(X) \text{ with } \phi(f) \leq k\}, \tag{2.14}$$

$$R_\phi(X) = \inf\{k > 0 : \exists \text{ a retraction } \rho : B(X) \longrightarrow S(X) \text{ with } \phi(\rho) \leq k\}, \tag{2.15}$$

$$\begin{aligned} H_\phi(X) = \inf\{k > 0 : \exists\, h : [0,1] \times S(X) \longrightarrow S(X) \text{ with } \phi(h) \leq k, \\ h(0,x) = x,\ h(1,x) \equiv \text{const}\}, \end{aligned} \tag{2.16}$$

where

$$\phi(h) = \inf\{k > 0 : \phi(h([0,1] \times A)) \leq k\phi(A) \text{ for } A \subseteq S(X)\}, \tag{2.17}$$

$$\begin{aligned} E_\phi(X) = \inf\{k > 0 : \exists\, g : B(X) \longrightarrow X \setminus \{0\} \text{ with } \phi(g) \leq k, \\ g(e) \neq \lambda e\ \forall \lambda > 0,\ e \in S(X)\}. \end{aligned} \tag{2.18}$$

From Darbo's fixed point principle [9] it follows that $L_\phi(X) \geq 1$ for every infinite-dimensional Banach space X and $\phi \in \{\alpha,\beta,\gamma\}$. On the other hand, $L_\phi(X) \leq L(X)$, and so $L_\phi(X) = 1$ in every space X, by what we have observed before. Similarly, $R_\phi(X) \leq R(X)$, because $\phi(F) \leq \mathrm{Lip}(F)$ for any map F.

We point out that the paper [32] is concerned with characterizing some classes of spaces X in which the infimum $L_\phi(X) = 1$ is actually *attained*, that is, there exists a fixed point free ϕ-nonexpansive self-map of $B(X)$. This is a nontrivial problem to which we will come back later (see the remarks after Theorem 3.3).

3. Some estimates and equalities

In [33], it was shown that $H_\alpha(X), R_\alpha(X), H_\gamma(X), R_\gamma(X) \leq 6$ and $H_\beta(X), R_\beta(X) \leq 4 + \beta(B(X))$. Moreover, $H_\phi(X), R_\phi(X) \leq 4$ for separable or reflexive spaces. It has also been proved in [33] that all spaces X containing an isometric copy of ℓ^p with $p \leq (2 - \log 3/\log 2)^{-1} = 2.41\ldots$ even satisfy $H_\phi(X), R_\phi(X) \leq 3$. A comparison of the characteristics (2.14)–(2.18) is provided by the following theorem.

Theorem 3.1. *The relations*

$$1 = L_\phi(X) \leq R_\phi(X) = H_\phi(X), \quad E_\phi(X) = 0 \quad (\phi \in \{\alpha,\beta,\gamma\}) \tag{3.1}$$

hold in every infinite-dimensional Banach space X.

Proof. The fact that $L_\phi(X) = 1$ and $E_\phi(X) = 0$ is a trivial consequence of the estimate $\phi(F) \leq \mathrm{Lip}(F)$ and our discussion above. The proof of the implication (a)⇒(b) in Theorem 1.1 shows that always $L_\phi(X) \leq R_\phi(X)$. Now, if we define a retraction ρ through a homotopy h as in (1.2), then for $M \subseteq B(X) \setminus B_r(X)$ we have $r\nu(M) \subseteq [0,1] \cdot M$, and so $\phi(\nu(M)) \leq (1/r)\phi(M)$, hence $\phi(\rho(M)) \leq (1/r)\phi(h)\phi(M)$. We conclude that $\phi(\rho) \leq \phi(h)/r$, and since $r < 1$ was arbitrary this proves that $R_\phi(X) \leq H_\phi(X)$. Conversely, if we define a homotopy h through a retraction ρ as in (1.6), then clearly $\phi(h([0,1] \times M)) \leq \phi(\rho)\phi(M)$ for each $M \subseteq S(X)$, and so we obtain $H_\phi(X) \leq R_\phi(X)$. □

Later (see Theorem 4.2), we will discuss a class of spaces in which the estimate in (3.1) also turns into equality.

The equality $E(X) = 0$ which we have obtained before for the characteristic (2.11) shows that in every Banach space X one may find "arbitrarily small" operators without zeros on $B(X)$ and positive eigenvalues on $S(X)$. Observe, however, that the infimum in (2.11) is *not* a minimum, since $\mathrm{Lip}(g) = 0$ means that g is constant, say $g(x) \equiv y_0 \neq 0$, and then g has the positive eigenvalue $\lambda = \|y_0\|$ with normalized eigenvector $e = y_0/\|y_0\|$.

On the other hand, the equality $E_\phi(X) = 0$ for the characteristic (2.18) shows that in every Banach space X, one may find such operators which are "arbitrarily close to being compact". As we will show later (see Theorem 3.3), in this case the infimum in (2.18) *is* a minimum, that is, the operator g may always be chosen as a compact map. The operator g from (1.10) is not optimal in this sense, since $g(e_k) = e_{k+1} - e_k$, where $(e_k)_k$ is the canonical basis in ℓ^2, and thus $\phi(g) \geq 1$. In the following Example 3.2, we give a *compact* operator in ℓ^2 without positive eigenvalues. This example has been our motivation for proving the general result contained in the subsequent Theorem 3.3.

Example 3.2. In $X = \ell^2$, consider the linear multiplication operator

$$L(\xi_1,\xi_2,\xi_3,\ldots) = (\mu_1\xi_1,\mu_2\xi_2,\mu_3\xi_3,\ldots), \tag{3.2}$$

where $m = (\mu_1,\mu_2,\mu_3,\ldots)$ is some fixed element in $S(X)$ with $0 < \mu_n < 1$ for all n. Since $\mu_n \to 0$ as $n \to \infty$, the operator (3.2) is compact on ℓ^2. Define $g : \ell^2 \to \ell^2 \setminus \{0\}$ by $g(x) := R(x) - L(x)$, where R is the nonlinear operator defined by $R(x) = (1 - \|x\|)m$. Being the sum of a one-dimensional nonlinear and a compact linear operator, g is certainly compact.

Suppose that $g(x) = \lambda x$ for some $\lambda > 0$ and $x \in S(\ell^2)$. Writing this out in components means that $-\mu_k\xi_k = -\mu_k\xi_k + (1 - \|x\|)\mu_k = \lambda\xi_k$ for all k, hence $\lambda = -\mu_k$ for some k, contradicting our assumptions $\lambda > 0$ and $\mu_k > 0$.

Recall that, given $M \subseteq X$, an operator $F : M \to Y$, and a measure of noncompactness ϕ on X and Y, the characteristic

$$\underline{\phi}(F) = \sup\{k > 0 : \phi(F(A)) \geq k\phi(A) \ (A \subseteq M)\} \tag{3.3}$$

is called the *lower ϕ-norm* of F. This characteristic is closely related to *properness*. In fact, from $\underline{\phi}(F) > 0$ it obviously follows that F is proper on closed bounded sets, that is, the preimage $F^{-1}(N)$ of any compact set $N \subset Y$ is compact. The converse is not true: for example, the operator $F : X \to X$ defined on an infinite-dimensional space X by $F(x) := \|x\|x$ is a homeomorphism with inverse $F^{-1}(y) = y/\sqrt{\|y\|}$ for $y \neq 0$ and $F^{-1}(0) = 0$, hence proper, but obviously satisfies $\underline{\phi}(F) = 0$.

Theorem 3.3. *Let X be an infinite-dimensional Banach space and $\varepsilon > 0$. Then, the following is true:*

(a) *there exists a compact map $g : B(X) \to B_\varepsilon(X) \setminus \{0\}$ such that $g(x) \neq \lambda x$ for all $x \in S(X)$ and $\lambda > 0$,*
(b) *there exists a fixed point free map $f : B(X) \to B(X)$ with $\phi(f) = 1$ and $\underline{\phi}(f) \geq 1 - \varepsilon$ for any measure of noncompactness ϕ.*

If X contains a complemented infinite-dimensional subspace with a Schauder basis, it may be arranged in addition that $\mathrm{Lip}(g) \leq \varepsilon$ *and* $\mathrm{Lip}(f) \leq 2 + \varepsilon$.

Proof. To prove (a), we imitate the construction of Example 3.2 in a more general setting. By a theorem of Banach (see, e.g., [27]), we find an infinite-dimensional closed subspace $X_0 \subseteq X$ with a Schauder basis $(e_n)_n$, $\|e_n\| = 1$. If we even find such a space complemented, let $P : X \to X_0$ be a bounded projection. In general, the set $B(X_0) = X_0 \cap B(X)$ is separable, convex, and complete, and so by [31] we may extend the identity map I on $B(X_0)$ to a continuous map $P : B(X) \to B(X_0)$. In both cases, we have $P(x) = x$ for $x \in B(X_0)$ and $P(B(X)) \subseteq B_C(X_0)$ for some $C \geq 1$.

Let $c_n \in X_0^*$ be the coordinate functions with respect to the basis $(e_n)_n$, and choose $\mu_n > 0$ with

$$\sum_{k=1}^{\infty} \mu_k \|c_k\| < \frac{\varepsilon}{2C}. \tag{3.4}$$

Now, we set $g := R - L$, where

$$R(x) := (1 - \|P(x)\|) \sum_{k=1}^{\infty} \mu_k e_k, \qquad L(x) := \sum_{k=1}^{\infty} \mu_k c_k(P(x)) e_k. \tag{3.5}$$

Since

$$L_n(x) := \sum_{k=1}^{n} \mu_k c_k(P(x)) e_k \longrightarrow L(x) \quad (n \longrightarrow \infty) \tag{3.6}$$

uniformly on $B(X)$, and since $L_n(B(X))$ and $R(B(X))$ are bounded subsets of finite-dimensional spaces, it follows that $g(B(X))$ is precompact. Clearly,

$$||R(x)||,||L(x)|| \leq C\frac{\varepsilon}{2C} = \frac{\varepsilon}{2} \tag{3.7}$$

for $x \in B(X)$, and if P is linear, we have also

$$\mathrm{Lip}(R), \mathrm{Lip}(L) \leq \frac{\|P\|\varepsilon}{2C} \leq \frac{\varepsilon}{2}. \tag{3.8}$$

This implies that $g(B(X)) \subseteq B_\varepsilon(X)$ and, if the subspace X_0 is complemented, then also $\mathrm{Lip}(g) \leq \varepsilon$.

We show now that $g(x) \neq 0$ for all $x \in B(X)$. In fact, $g(x) = 0$ implies that $L(x) = R(x) \in X_0$ and so, since $(e_n)_n$ is a basis, that $\mu_n c_n(P(x)) = (1 - \|P(x)\|)\mu_n$ for all n. In view of $\mu_n > 0$, this means that $c_n(P(x)) = 1 - \|P(x)\|$, which shows that $c_n(P(x))$ is actually independent of n. Since $P(x) \in X_0$, this is only possible if $P(x) = 0$ which contradicts the equality $c_n(P(x)) = 1 - \|P(x)\|$. So, we have shown that $g(B(X)) \subseteq B_\varepsilon(X) \setminus \{0\}$.

We still have to prove that the equation $g(x) = \lambda x$ has no solution with $\lambda > 0$ and $\|x\| = 1$. Assume by contradiction that we find such a solution $(\lambda, x) \in (0,\infty) \times S(X)$. Since $g(x) \in X_0$ and $\|x\| = 1$, we must have $P(x) = x \in X_0$, say

$$x = \sum_{k=1}^{\infty} \xi_k e_k. \tag{3.9}$$

But the relation $\|x\| = 1$ also implies that $R(x) = 0$, and so the equality $g(x) = \lambda x$ becomes $\lambda x + L(x) = 0$. Writing this in coordinates with respect to the basis $(e_n)_n$, we obtain, in view of $c_n(P(x)) = c_n(x) = \xi_n$, that $\lambda\xi_n + \mu_n\xi_n = 0$. But from $\lambda + \mu_n > 0$, we conclude that $\xi_n = 0$ for all n, that is, $x = 0$, contradicting $\|x\| = 1$.

To prove (b), let $\rho : B_{1+\varepsilon}(X) \to B(X)$ be the radial retraction of the ball $B_{1+\varepsilon}(X)$ onto the unit ball in X. Then, $\mathrm{Lip}(\rho) \leq 2$ and $\phi(\rho(M)) \leq \phi(M)$ for all $M \subseteq B_{1+\varepsilon}(X)$, hence $\phi(\rho) \leq 1$. Let $g : B(X) \to B_\varepsilon(X)$ be the map whose existence was proved in (a). We put

$$f(x) := \rho(x + g(x)) \quad (x \in B(X)). \tag{3.10}$$

It is easy to see that $\phi(f(M)) \leq \phi(M)$ for all $M \subseteq B(X)$, and $\phi(f(B(X))) = \phi(B(X))$, which means that $\phi(f) = 1$. If $\mathrm{Lip}(g) \leq \varepsilon$, we have also $\mathrm{Lip}(f) \leq 2(1+\varepsilon)$. Moreover, we claim that the map (3.10) has no fixed points in $B(X)$. Indeed, suppose that $x = f(x) = \rho(x + g(x))$ for some $x \in B(X)$. Then, the fact that $g(x) \neq 0$ implies that $x + g(x) \neq x = \rho(x + g(x))$, and from the definition of ρ it follows that $r := \|x + g(x)\| > 1$. But then $\|x\| = \|f(x)\| = 1$ and $x = f(x) = (1/r)(x + g(x))$, and thus $g(x) = (r-1)x$ with $r - 1 > 0$, contradicting our choice of g.

It remains to show that $\underline{\phi}(f) \geq 1 - \varepsilon$. The radial retraction $\rho : B_{1+\varepsilon}(X) \to B(X)$ satisfies $\underline{\phi}(\rho) \geq 1/(1+\varepsilon)$, because

$$\rho^{-1}(M) \subseteq [0,1] \cdot (1+\varepsilon)M, \tag{3.11}$$

hence $\phi(\rho^{-1}(M)) \le (1+\varepsilon)\phi(M)$, for every $M \subseteq B(X)$. So, given $A \subseteq B_{1+\varepsilon}(X)$, by considering $M := \rho(A)$ we see that $\phi(\rho(A)) \ge (1/(1+\varepsilon))\phi(A)$. Since g is compact, from (3.10) we immediately deduce that

$$\underline{\phi}(f) = \underline{\phi}(\rho) \ge \frac{1}{1+\varepsilon} \tag{3.12}$$

as claimed. The proof is complete. □

We make some remarks on Theorem 3.3. Although the above construction works in any (infinite-dimensional) Banach space, the completeness of X (at least that of X_0) is essential. Moreover, in such spaces uniform limits of finite-dimensional operators must have a precompact range, but it is not clear whether or not they have a relatively compact range. The construction of fixed point free maps in [32] does not have this flaw. Moreover, the maps considered in [32] have even stronger compactness properties, because they send "most" sets (except those of full measure of noncompactness) into relatively compact sets.

4. Connections with Banach space geometry

The operator g constructed in the proof of Theorem 3.3(a) may be used to show that $R_\phi(X) = 1$ in many spaces. To be more specific, we recall some definitions from Banach space geometry. Recall that a space X with (Schauder) basis $(e_n)_n$ is said to have a *monotone norm* (with respect to $(e_n)_n$) if

$$|\xi_k| \le |\eta_k| \ \forall k \in \{1,2,\dots,n\} \Longrightarrow \left\|\sum_{k=1}^{n} \xi_k e_k\right\| \le \left\|\sum_{k=1}^{n} \eta_k e_k\right\| \tag{4.1}$$

for all n. In view of the continuity of the norm, it is equivalent to require

$$|\xi_k| \le |\eta_k| \ \forall k \in \mathbb{N} \Longrightarrow \left\|\sum_{k=1}^{\infty} \xi_k e_k\right\| \le \left\|\sum_{k=1}^{\infty} \eta_k e_k\right\| \tag{4.2}$$

for all sequences $(\xi_k)_k$ and $(\eta_k)_k$ for which the two series on the right-hand side of (4.2) converge.

A basis $(e_n)_n$ in X is called *unconditional* if any rearrangement of $(e_n)_n$ is also a basis. Banach spaces with an unconditional basis have some remarkable properties: for example, they are either reflexive, or they contain an isomorphic copy of ℓ^1 or c_0. So, there are many Banach spaces with a Schauder basis but without an unconditional basis. In fact, no space with the so-called *Daugavet property* has an unconditional basis [20, 34]. Moreover, no space with the Daugavet property embeds into a space with an unconditional basis [21]. In particular, $C[0,1]$ and $L_1[0,1]$ (and all spaces into which they embed) do *not* possess an unconditional basis.

The following proposition relates spaces with unconditional bases and spaces with monotone norm and seems to be of independent interest.

Proposition 4.1. *Let X be a Banach space with basis $(e_n)_n$. Then, this basis is unconditional if and only if X has an equivalent norm which is monotone with respect to the basis $(e_n)_n$.*

Proof. Assume first that X has an equivalent norm $\|\cdot\|$ which is monotone with respect to the basis $(e_n)_n$. Let $(\eta_n)_n$ be such that $\sum_{k=1}^{\infty}\eta_k e_k$ converges, and assume that $|\xi_k| \le |\eta_k|$ for all k. Applying (4.1) with $\xi_k = \eta_k := 0$ for $k < m \le n$, we obtain

$$\left\|\sum_{k=m}^{n}\xi_k e_k\right\| \le \left\|\sum_{k=m}^{n}\eta_k e_k\right\| \quad (m \le n), \tag{4.3}$$

and so the Cauchy criterion implies the convergence of $\sum_{k=1}^{\infty}\xi_k e_k$.

Conversely, suppose that the basis $(e_n)_n$ is unconditional. Let $c_n \in X^*$ be the corresponding coordinate functionals, and define $A_n : \ell^\infty \times X \to X$ by

$$A_n\big((\mu_k)_k, x\big) := \sum_{k=1}^{n}\mu_k c_k(x) e_k. \tag{4.4}$$

Since the basis $(e_n)_n$ is unconditional, by assumption, we have

$$\sup_n \|A_n(m,x)\| < \infty \quad (m \in \ell^\infty,\ x \in X), \tag{4.5}$$

and so the uniform boundedness principle implies that

$$\begin{aligned} \|x\|^* :&= \sup_n \sup_{|\eta_k| \le |c_k(x)|}\left\|\sum_{k=1}^{n}\eta_k e_k\right\| = \sup_{\|m\|_{\ell^\infty}\le 1}\sup_n \|A_n(m,x)\| \\ &= \sup_{\|m\|_{\ell^\infty}\le 1}\sup_n \|A_n\|\|(m,x)\| \le C\|x\| \quad (x \in X) \end{aligned} \tag{4.6}$$

with some finite constant C. This, together with the obvious estimate $\|x\| \le \|x\|^*$, implies that the two norms $\|\cdot\|$ and $\|\cdot\|^*$ are equivalent. Clearly, $\|\cdot\|^*$ is a norm which satisfies the monotonicity condition (4.1), and so the proof is complete. □

THEOREM 4.2. *Let X be an infinite-dimensional Banach space whose norm is monotone with respect to some basis $(e_n)_n$. Then, the equality*

$$R_\gamma(X) = 1 \tag{4.7}$$

holds.

Proof. Consider the map $g : B(X) \to X \setminus \{0\}$ from Theorem 3.3(a), that is, $g(x) = R(x) - L(x)$ with R and L as in (3.5). We already know that g is compact and $g(x) \ne \lambda x$ for $\lambda > 0$ and all $x \in S(X)$. Define $\sigma : B(X) \to X$ as in (1.3). Then, $\sigma(x) \ne 0$ on $B(X)$. Indeed, the assumption $\sigma(z) = 0$ leads to $g(e) = \lambda e$, with λ and e defined as in (1.4), a contradiction. So, the map $\rho(x) := \nu(\sigma(x))$ is a retraction from $B(X)$ onto $S(X)$.

Since g is compact, for any $M \subseteq B(X)$ the set $\sigma(M \cap B_r(X))$ is precompact, and so also the set $\rho(M \cap B_r(X))$. Consequently,

$$\gamma(\rho(M)) = \gamma\big(\rho(M \cap B_r(X)) \cup \rho(M \setminus B_r(X))\big) = \gamma\big(\rho(M \setminus B_r(X))\big). \tag{4.8}$$

For $x \in M \setminus B_r(X)$, we have

$$\sigma(x) = \frac{\|x\| - r}{1 - r} x + \frac{1 - \|x\|}{1 - r} L(\nu(x)). \tag{4.9}$$

Putting

$$h(t) := \frac{t - r}{1 - r} t \quad (0 \le t \le 1), \tag{4.10}$$

by the monotonicity property (4.1) of the norm in X, we conclude that $\|\sigma(x)\| \ge h(\|x\|)$.

Now we distinguish two cases. We assume first that there is a sequence $(x_n)_n$ in $M \setminus B_r(X)$ with $\sigma(x_n) \to 0$ as $n \to \infty$. In view of $\|\sigma(x)\| \ge h(\|x\|)$ and the definition of h, we obtain then $\|x_n\| \to r$. Moreover, the definition of σ implies $L(x_n) \to 0$ as $n \to \infty$. Denoting by P_k the canonical projection of X onto the linear hull of $\{e_1, \dots, e_k\}$, we have $P_k x_n \to 0$, as $n \to \infty$, hence

$$\sup_n ||(I - P_k)x_n|| \ge \limsup_{n\to\infty} ||(I - P_k)x_n|| = r \quad (k = 1, 2, 3, \dots). \tag{4.11}$$

This implies that $\gamma(\{x_1, x_2, x_3, \dots\}) \ge r$, and so $\gamma(M) \ge r \ge r\gamma(\rho(M))$. Assume now that there is no sequence $(x_n)_n$ as above. Then we find a constant $c > 0$ (possibly depending on r and M) such that

$$K := \left\{ \frac{1 - \|x\|}{||\sigma(x)||(1 - r)} L(x) : x \in M \setminus B_r(X) \right\} \subseteq [0, 1] \cdot c \cdot L(M \setminus B_r(X)). \tag{4.12}$$

Being L a compact operator, it follows that K is contained in a compact set. For $x \in M \setminus B_r(X)$, we have

$$\rho(x) = \frac{\sigma(x)}{||\sigma(x)||} \in \frac{\|x\| - r}{||\sigma(x)||(1 - r)} x + K = \frac{h(\|x\|) r}{||\sigma(x)|| \, \|x\|} \cdot \frac{x}{r} + K, \tag{4.13}$$

and thus

$$\rho(M \setminus B_r(X)) \subseteq [0, 1] \cdot \frac{M}{r} + K. \tag{4.14}$$

In all cases, we conclude that

$$\gamma(\rho(M)) \le \frac{1}{r} \gamma(M). \tag{4.15}$$

Since $r \in (0, 1)$ is arbitrary, we see that $R_\gamma(X) \le 1$ as claimed. □

The proof of Theorem 4.2 shows that an analogous estimate of the form $R_\phi(X) \le C(\phi)\phi(B(X))$ holds for any measure of noncompactness ϕ on X with the property that

$$\inf_k \sup_{x \in A} ||(I - P_k)x|| \le C(\phi)\phi(A) \quad (A \subset X \text{ bounded}) \tag{4.16}$$

for some $C(\phi) > 0$. Some estimates, or even explicit formulas, for the minimal constant $C(\phi)$ in some important Banach spaces may be found in [2, Chapter 2].

In view of the above proposition, one might think that it suffices to require in Theorem 4.2 that the basis $(e_n)_n$ be unconditional, by passing then, if necessary, to an equivalent norm which is monotone with respect to this basis. Unfortunately, in this case the unit sphere will change, and so the constant $R_\phi(X)$ will usually change as well. In this connection, the following question arises: given two equivalent norms $\|\cdot\|$ and $\|\cdot\|^*$ on X with corresponding unit spheres $S(X)$ and $S^*(X)$, do there exist a constant $c > 0$ and a homeomorphism $\omega : S(X) \to S^*(X)$ such that $\phi(\omega(M)) = c\phi(M)$ for all $M \subseteq S(X)$? If the answer is affirmative, then Theorem 4.2 holds true if the basis $(e_n)_n$ in X is merely unconditional. We do not know, however, whether or not such a homeomorphism may be found in every space X.

We briefly recall an application of Theorem 4.2 to a long-standing open problem in nonlinear spectral theory which was solved quite recently by Furi [12]. A map $f : B(X) \to X$ is called 0-*epi* [15] if $f(x) \neq 0$ on $S(X)$ and, given any compact map $g : B(X) \to X$ which vanishes on $S(X)$, one may find a solution $x \in B(X)$ of the coincidence equation $f(x) = g(x)$. More generally, f is called k-*epi* ($k > 0$) if this solvability result still holds true for noncompact right-hand sides g satisfying $\alpha(g) \leq k$. In this terminology, Schauder's fixed point theorem asserts that the identity operator is 0-epi, and Darbo's fixed point theorem asserts that the identity operator is k-epi for $k < 1$. It was an open question for some time to find a Banach space X and a map which is 0-epi on $B(X)$, but not k-epi for any positive k. This problem was solved quite recently by Furi [12] by means of an explicit retraction $\rho : B(C[0,1]) \to S(C[0,1])$ with $\alpha(\rho) \leq 1 + \varepsilon$. In fact, the homeomorphism $f : C[0,1] \to C[0,1]$, defined by $f(x) := \|x\|x$, is obviously 0-epi, by Schauder's fixed point theorem. However, it is not k-epi on $B(C[0,1])$ for any positive k, as may be seen by considering the noncompact right-hand side

$$g(x) := \begin{cases} \|x\|x - \dfrac{1}{n}\rho(nx) & \text{for } \|x\| \leq \dfrac{1}{n}, \\ 0 & \text{for } \|x\| > \dfrac{1}{n}, \end{cases} \tag{4.17}$$

for sufficiently large $n \in \mathbb{N}$. Theorem 4.2 shows that such a construction is possible not only in the space $C[0,1]$, but in any infinite-dimensional space X with monotone norm.

5. Asymptotically regular maps

Sometimes it is interesting to find maps without fixed points or eigenvalues which have some additional properties. One particularly important class in metric fixed point theory is that of *asymptotically regular maps* f, that is, those satisfying

$$\lim_{n\to\infty} d\big(f^n(x), f^{n-1}(x)\big) = 0. \tag{5.1}$$

It turns out that the fixed point free map f we constructed in the proof of Theorem 3.3(b) may be chosen asymptotically regular.

Theorem 5.1. *Let X be an infinite-dimensional Banach space whose norm is monotone with respect to some basis $(e_n)_n$, and let $\varepsilon > 0$. Then, there exists an asymptotically regular* *fixed point free map* $f : B(X) \to B(X)$ *satisfying* $\operatorname{Lip}(f) \le 1+\varepsilon$ *and* $\phi(f(M)) = \phi(M)$ *for each* $M \subseteq B(X)$ *and* $\phi \in \{\alpha,\beta,\gamma\}$.

Proof. Define f as in the proof of Theorem 3.3 (with $P(x) = x$ and $C = 2$). We claim that, in view of the monotonicity of the norm in X with respect to the basis $(e_n)_n$, the formula (3.10) may be replaced by the simpler formula

$$f(x) = x+g(x). \tag{5.2}$$

In fact, for $x = \sum_{n=1}^{\infty} \xi_n e_n \in B(X)$, we have

$$x+g(x) = R(x)+\sum_{n=1}^{\infty}(1-\mu_n)\xi_n e_n, \tag{5.3}$$

and so the monotonicity of the norm implies, in view of $0 \le \mu_n \le \varepsilon \le 1$, that

$$\|x+g(x)\| \le \|R(x)\|+\left\|\sum_{n=1}^{\infty}\xi_n e_n\right\| \le \varepsilon(1-\|x\|)+\|x\| = \varepsilon+(1-\varepsilon)\|x\|. \tag{5.4}$$

In particular, $\|x+g(x)\| \le \varepsilon+(1-\varepsilon) \le 1$, and so $f(x) = \rho(x+g(x)) = x+g(x)$. This proves (5.2).

We have already seen that f has no fixed points. Moreover, (5.2) implies, in view of the compactness of g, that $\phi(f(M)) = \phi(M)$, and $\operatorname{Lip}(f) \le 1+\operatorname{Lip}(g) \le 1+\varepsilon$.

It remains to show that f is asymptotically regular. From (5.2) it follows that $g(x) = f(x)-x$, and so $g(f^n(x)) = f^{n+1}(x)-f^n(x)$. Since g is compact, this implies that the set $\{f^{n+1}(x)-f^n(x) : n = 1,2,\dots\} \subseteq g(B(X))$ is precompact for every x. Now, it suffices to show that every subsequence of $(f^{n+1}(x)-f^n(x))_n$ contains in turn a subsequence converging to 0. Since we have seen that each subsequence contains a convergent subsequence, we only have to show that the corresponding limit cannot be different from 0. In other words, we must prove that $c_i(f^{n+1}(x))-c_i(f^n(x)) \to 0$, as $n \to \infty$, where $c_i(y)$ denotes the ith coordinate of y as before.

We claim that

$$\lim_{n\to\infty}\|f^n(x)\| = 1 \tag{5.5}$$

for every $x \in B(X)$. Indeed, one may easily show by induction that

$$\begin{aligned} c_i(f^n(x)) &= (1-\mu_i)c_i(f^{n-1}(x))+(1-\|f^{n-1}(x)\|)\mu_i \\ &= (1-\mu_i)^n c_i(x)+\sum_{j=1}^{n}(1-\|f^{j-1}(x)\|)(1-\mu_i)^{n-j}\mu_i. \end{aligned} \tag{5.6}$$

For $\varepsilon \in (0,1)$ we denote by $b(\varepsilon;n)$ the set of all indices $j \in \{1,2,\dots,n\}$ such that $\|f^{j-1}(x)\| < 1-\varepsilon$. Let

$$\beta(\varepsilon,i,n) := \sum_{j\in b(\varepsilon;n)}(1-\mu_i)^{n-j}\mu_1. \tag{5.7}$$

Now, we prove (5.5) by contradiction. If (5.5) is not true, we may find an infinite sequence of numbers $(n_k)_k$ (which may depend on ε) such that $\|f^{n_k}(x)\| < 1-\varepsilon$ for all k. By definition of (5.7), we have

$$\beta(\varepsilon,i,n_{k+1}) = \beta(\varepsilon,i,n_k)\left(1-\mu_i\right)^{n_{k+1}-n_k} + \mu_i. \tag{5.8}$$

Now, we distinguish two cases. Suppose first that the sequence $(n_{k+1}-n_k)_k$ is bounded. Passing to a subsequence, if necessary, we may then suppose that

$$\lim_{k\to\infty}\left(n_{k+1}-n_k\right) =: c. \tag{5.9}$$

Since the sequence $(\beta(\varepsilon,i,n_k))_k$ is bounded, we may also assume, without loss of generality, that the limit

$$\beta(\varepsilon,i) := \lim_{k\to\infty}\beta(\varepsilon,i,n_k) \tag{5.10}$$

exists. Letting k in (5.8) tend to infinity yields $\beta(\varepsilon,i) = \beta(\varepsilon,i)(1-\mu_i)^c + \mu_i$, hence

$$\beta(i,\varepsilon) = \frac{\mu_i}{1-\left(1-\mu_i\right)^c}. \tag{5.11}$$

By L'Hospital's rule we see that

$$\lim_{i\to\infty}\frac{\mu_i}{1-\left(1-\mu_i\right)^c} = \lim_{t\to 0}\frac{t}{1-(1-t)^c} = \lim_{t\to 0}\frac{1}{c(1-t)^{c-1}} = \frac{1}{c}. \tag{5.12}$$

On the other hand, from (5.6) it follows that

$$c_i\left(f^n(x)\right) \geq \left(1-\mu_i\right)^n c_i(x) + \varepsilon\beta(\varepsilon,i,n), \tag{5.13}$$

contradicting the fact that $\|f^n(x)\| \leq 1$ for all n.

Suppose now that the sequence $(n_{k+1}-n_k)_k$ is unbounded, and so

$$\lim_{k\to\infty}\left(n_{k+1}-n_k\right) = \infty. \tag{5.14}$$

Consequently, for some fixed $\varepsilon > 0$, we have then

$$\left|\left|f^{n_k(\varepsilon)}(x)\right|\right| < 1-\varepsilon \tag{5.15}$$

for an infinite sequence of indices $(n_k(\varepsilon))_k$ depending on ε. By (5.14) (with ε replaced by $\varepsilon/3$) we find $k_0 \in \mathbb{N}$ such that $n_{k+1}(\varepsilon/3) - n_k(\varepsilon/3) > 3$ for $k \geq k_0$. Taking into account the definition of f, we conclude that

$$\begin{aligned}
&\big|\|f^{n_k}(x)\| - \|f^{n_k-1}(x)\|\big| \\
&\quad \leq \|f^{n_k}(x) - f^{n_k-1}(x)\| \\
&\quad = \big|L(f^{n_k-1}(x)) + (1 - \|f^{n_k-1}(x)\|)f(0) - L(f^{n_k-2}(x)) - (1 - \|f^{n_k-2}(x)\|)f(0)\big| \\
&\quad = \big|L(f^{n_k-1}(x) - f^{n_k-2}(x)) + (\|f^{n_k-2}(x)\| - \|f^{n_k-1}(x)\|)f(0)\big| \\
&\quad \leq 2\big|\|f^{n_k-1}(x)\| - \|f^{n_k-2}(x)\|\big|.
\end{aligned} \tag{5.16}$$

Therefore, if we assume that $1 - \|f^{n_k-1}(x)\| \leq \varepsilon/3$ and $1 - \|f^{n_k-2}(x)\| \leq \varepsilon/3$, then $1 - \|f^{n_k}(x)\| \leq \varepsilon$. But this contradicts the estimate (5.15), and so we arrived in both cases at a contradiction. This shows that our assumption was false, that is, (5.5) is true. Consequently, combining (5.5) and (5.6), we conclude that

$$\lim_{n\to\infty} c_i(f^n(x)) = 0 \tag{5.17}$$

for every i, and so the proof of the asymptotic regularity of f is complete. □

6. The minimal displacement

Given a normed space X and a map $f : B(X) \to X$, recall that the *minimal displacement* of f on $B(X)$ is defined by

$$\eta(f) := \inf_{\|x\|\leq 1} \|x - f(x)\|. \tag{6.1}$$

Clearly, $\eta(f) > 0$ implies that f has no fixed point, but the converse is true in general only in finite dimensions. For instance, in Kakutani's example (1.9) we have $\eta(f) = 0$.

We point out that, by the classical Birkhoff-Kellogg theorem (see, e.g., [10]) for compact maps, the operator g constructed in Theorem 3.3(a) must satisfy

$$\inf_{\|x\|\leq 1} \|g(x)\| = 0. \tag{6.2}$$

From this it follows in turn that the fixed point free operator f from Theorem 3.3(b) satisfies $\eta(f) = 0$. This is not accidental. In fact, in [14] the following remarkable connection between the minimal displacement (6.1) and the α-norm $\alpha(f)$ of f is given, which may be proved quite easily, even for $\phi \in \{\alpha, \beta, \gamma\}$:

Theorem 6.1. *Let X be a Banach space and suppose that $f : B(X) \to B(X)$ satisfies $\phi(f) < \infty$. Then,*

$$\eta(f) \leq \max\left\{1 - \frac{1}{\phi(f)}, 0\right\}. \tag{6.3}$$

In particular, $\phi(f) \leq 1$ implies $\eta(f) = 0$.

Proof. If $\phi(f) < 1$, then f has a fixed point by Darbo's fixed point theorem [9]. Thus, assume that $\phi(f) \geq 1$ and choose some $\varepsilon > 0$ with $\varepsilon\phi(f) < 1$. Then, $\varepsilon f : B(X) \to B_\varepsilon(X) \subseteq B(X)$ is condensing and thus has a fixed point $x = \varepsilon f(x)$. So, we obtain

$$\|x - f(x)\| = \|\varepsilon f(x) - f(x)\| = (1-\varepsilon)\|f(x)\| \leq 1 - \varepsilon, \tag{6.4}$$

hence $\eta(f) \leq 1 - \varepsilon$. Since $\varepsilon \in (0, 1/\phi(f))$ was arbitrary, we conclude that $\eta(f) \leq 1 - 1/\phi(f)$ as claimed. □

Taking into account the relation (6.3), it seems reasonable to introduce the characteristic

$$\begin{aligned} \tilde{L}_\phi(X) = \inf\bigg\{\frac{k}{k\delta + 1} : k \geq 1,\ \delta \geq 0, \exists \text{ a map } f : B(X) \longrightarrow B(X) \\ \text{with } \eta(f) \geq \delta,\ \phi(f) \leq k\bigg\}. \end{aligned} \tag{6.5}$$

Clearly, for $\delta = 0$ (6.5) simply reduces to the characteristic (2.14). On the other hand, for $\delta > 0$ the estimate (6.3) shows then that $\tilde{L}_\phi(X) \geq 1$ in every Banach space X. Conversely, in [33] it was shown that, given any infinite-dimensional space X, $k > 1$, and $\varepsilon > 0$, one may find $f : B(X) \to B(X)$ with $\phi(f) \leq k$ and

$$\eta(f) \geq \frac{1}{2} - \frac{1}{k} - \varepsilon. \tag{6.6}$$

This gives the upper estimate $\tilde{L}_\phi(X) \leq 2$. Moreover, in spaces X with the so-called "separable retraction property" (e.g., reflexive or separable spaces), the constant $1/2$ in (6.6) may be replaced by 1 for $\phi = \gamma$, and so one even has $\tilde{L}_\gamma(X) = 1$. A similar result holds for spaces X which contain an isometric copy of ℓ^p or c_0; in this case, one may also for $\phi = \alpha$ and $\phi = \beta$ replace the constant $1/2$ in (6.6) at least by $2^{(1-p)/p}$ and obtain $\tilde{L}_\alpha(X), \tilde{L}_\beta(X) \leq 2^{1-1/p}$.

However, we can do much better. From all maps occurring in our definitions, the retraction $\rho : B(X) \to S(X)$ is the most "powerful" map. In fact, each such retraction can be used to construct a continuous map $f : B(X) \to B(X)$ with minimal displacement $\eta(f) = \delta < 1$ as close to 1 as we want, by putting

$$f(x) := \begin{cases} -\rho\left(\dfrac{x}{r}\right) & \text{if } \|x\| \leq r, \\ -\nu(x) & \text{if } \|x\| > r, \end{cases} \tag{6.7}$$

where $r := 1 - \delta$. This map satisfies $\phi(f) \leq \phi(\rho)/r = \phi(\rho)/(1-\delta)$, because

$$\begin{aligned} \phi(f(M)) &= \max\{\phi(f(M \cap B_r(X))), \phi(f(M \setminus B_r(X)))\} \\ &\leq \max\left\{\phi\left(\rho\left(\frac{1}{r}M\right)\right), \phi\left([0,1] \cdot \frac{1}{r}M\right)\right\}. \end{aligned} \tag{6.8}$$

Moreover, if ρ is Lipschitz continuous, then also f is Lipschitz continuous. More precisely,

$$\operatorname{Lip}(f) \le \max\left\{\frac{\operatorname{Lip}(\rho)}{r}, \frac{2}{r}\right\} = \frac{\operatorname{Lip}(\rho)}{r} = \frac{\operatorname{Lip}(\rho)}{1-\delta}, \tag{6.9}$$

since $\operatorname{Lip}(\rho) \ge 3$, as mentioned in the introduction. In fact, in case $\|x\| < r < \|y\|$, let $z \in S_r(X)$ be a convex combination of x and y and observe that

$$\|f(x) - f(y)\| \le \frac{\operatorname{Lip}(\rho)}{r}\left(\|x - z\| + \|z - y\|\right) = \frac{\operatorname{Lip}(\rho)}{r}\|x - y\|. \tag{6.10}$$

We already used several times the fact that in each infinite-dimensional normed space X there is a Lipschitz continuous retraction ρ of the unit ball onto its boundary. Using the shortcut $k := \operatorname{Lip}(f)$ and $c := \operatorname{Lip}(\rho)$ we have, in particular,

$$\frac{k}{k\delta + 1} = \frac{1}{\delta + (1-\delta)/c} \longrightarrow 1 \quad (\delta \longrightarrow 1^-), \tag{6.11}$$

and so we get the surprising consequence that $\tilde{L}_\phi(X) = 1$ in *every* infinite-dimensional normed space, even if we would have replaced $\phi(f)$ by $\operatorname{Lip}(f)$ in the definition (6.5) of $\tilde{L}_\phi(X)$.

Note that the above calculation means in a sense that the estimate (6.3) in Theorem 6.1 becomes "arbitrarily sharp" in each space if $\eta(f)$ is sufficiently close to 1, even if we replace $\phi(f)$ by $\operatorname{Lip}(f)$.

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