FIXED POINT THEORY ON EXTENSION-TYPE SPACES AND ESSENTIAL MAPS ON TOPOLOGICAL SPACES

DONAL O'REGAN

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We present several new fixed point results for admissible self-maps in extension-type spaces. We also discuss a continuation-type theorem for maps between topological spaces.

1. Introduction

In Section 2, we begin by presenting most of the up-to-date results in the literature [3, 5, 6, 7, 8, 12] concerning fixed point theory in extension-type spaces. These results are then used to obtain a number of new fixed point theorems, one concerning approximate neighborhood extension spaces and another concerning inward-type maps in extension-type spaces. Our first result was motivated by ideas in [12] whereas the second result is based on an argument of Ben-El-Mechaiekh and Kryszewski [9]. Also in Section 2 we present a new continuation theorem for maps defined between Hausdorff topological spaces, and our theorem improves results in [3].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose X and Y are topological spaces. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. We let

$$\mathscr{F}(\mathscr{X}) = \{Z : \operatorname{Fix} F \neq \varnothing \ \forall F \in \mathscr{X}(Z,Z)\}, \tag{1.1}$$

where $\operatorname{Fix} F$ denotes the set of fixed points of F.

The class $\mathscr{A}$ of maps is defined by the following properties:

(i) $\mathscr{A}$ contains the class $\mathscr{C}$ of single-valued continuous functions;
(ii) each $F \in \mathscr{A}_c$ is upper semicontinuous and closed valued;
(iii) $B^n \in \mathscr{F}(\mathscr{A}_c)$ for all $n \in \{1,2,\dots\}$; here $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.

Remark 1.1. The class $\mathscr{A}$ is essentially due to Ben-El-Mechaiekh and Deguire [7]. It includes the class of maps $\mathscr{U}$ of Park ($\mathscr{U}$ is the class of maps defined by (i), (iii), and (iv) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathscr{A}_c$ is compact

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valued, the classes $\mathcal{A}$ and $\mathcal{U}$ coincide and this is what occurs in Section 2 since our maps will be compact.

The following result can be found in [7, Proposition 2.2] (see also [11, page 286] for a special case).

Theorem 1.2. *The Hilbert cube I^∞ (subset of l^2 consisting of points $(x_1,x_2,\dots)$ with $|x_i| \leq 1/2^i$ for all i) and the Tychonoff cube T (Cartesian product of copies of the unit interval) are in $\mathcal{F}(\mathcal{A}_c)$.*

We next consider the class $\mathcal{U}_c^\kappa(X,Y)$ (resp., $\mathcal{A}_c^\kappa(X,Y)$) of maps $F : X \to 2^Y$ such that for each F and each nonempty compact subset K of X, there exists a map $G \in \mathcal{U}_c(K,Y)$ (resp., $G \in \mathcal{A}_c(K,Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

Theorem 1.3. *The Hilbert cube I^∞ and the Tychonoff cube T are in $\mathcal{F}(\mathcal{A}_c^\kappa)$ (resp., $\mathcal{F}(\mathcal{U}_c^\kappa)$).*

Proof. Let $F \in \mathcal{A}_c^\kappa(I^\infty,I^\infty)$. We must show that $\operatorname{Fix} F \neq \emptyset$. Now, by definition, there exists $G \in \mathcal{A}_c(I^\infty,I^\infty)$ with $G(x) \subseteq F(x)$ for all $x \in I^\infty$, so Theorem 1.2 guarantees that there exists $x \in I^\infty$ with $x \in Gx$. In particular, $x \in Fx$ so $\operatorname{Fix} F \neq \emptyset$. Thus $I^\infty \in \mathcal{F}(\mathcal{A}_c^\kappa)$. $\square$

Notice that $\mathcal{U}_c^\kappa$ is closed under compositions. To see this, let X, Y, and Z be topological spaces, $F_1 \in \mathcal{U}_c^\kappa(X,Y)$, $F_2 \in \mathcal{U}_c^\kappa(Y,Z)$, and K a nonempty compact subset of X. Now there exists $G_1 \in \mathcal{U}_c(K,Y)$ with $G_1(x) \subseteq F_1(x)$ for all $x \in K$. Also [4, page 464] guarantees that $G_1(K)$ is compact so there exists $G_2 \in \mathcal{U}_c^\kappa(G_1(K),Z)$ with $G_2(y) \subseteq F_2(y)$ for all $y \in G_1(K)$. As a result,

$$G_2G_1(x) \subseteq F_2G_1(x) \subseteq F_2F_1(x) \quad \forall x \in K \tag{1.2}$$

and $G_2G_1 \in \mathcal{U}_c(X,Z)$.

For a subset K of a topological space X, we denote by $\operatorname{Cov}_X(K)$ the set of all coverings of K by open sets of X (usually we write $\operatorname{Cov}(K) = \operatorname{Cov}_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in \operatorname{Cov}(X)$, a point $x \in X$ is said to be an α-fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps $F,G : X \to 2^Y$ and $\alpha \in \operatorname{Cov}(Y)$, F and G are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$, and $w \in G(x) \cap U_x$.

The following results can be found in [5, Lemmas 1.2 and 4.7].

Theorem 1.4. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq \operatorname{Cov}_X(\overline{F(X)})$ such that F has an α-fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Theorem 1.5. *Let T be a Tychonoff cube contained in a Hausdorff topological vector space. Then T is a retract of* $\operatorname{span}(T)$.

Remark 1.6. From Theorem 1.4 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values, it suffices [6, page 298] to prove the existence of approximate fixed points (since open covers of a compact set A

admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [14, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular) [10, page 431] (see also [10, page 434]). Note in Theorem 1.4 if F is compact valued, then the assumption that X is regular can be removed. For convenience in this paper we will apply Theorem 1.4 only when the space is uniform.

2. Extension-type spaces

We begin this section by recalling some results we established in [3]. By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in \mathrm{ES}(Q)$) if for all $X \in Q$ and all $K \subseteq X$ closed in X, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

Using (i) the fact that every compact space is homeomorphic to a closed subset of the Tychonoff cube and (ii) Theorem 1.3, we established the following result in [3].

Theorem 2.1. *Let $X \in \mathrm{ES}(\text{compact})$ and $F \in \mathfrak{U}_c^{\kappa}(X,X)$ a compact map. Then F has a fixed point.*

Remark 2.2. If $X \in \mathrm{AR}$ (an absolute retract as defined in [11]), then of course $X \in \mathrm{ES}(\text{compact})$.

A space Y is an *approximate extension space* for Q (written $Y \in \mathrm{AES}(Q)$) if for all $\alpha \in \mathrm{Cov}(Y)$, all $X \in Q$, all $K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f : X \to Y$ such that $f|_K$ is α-close to f_0.

Theorem 2.3. *Let $X \in \mathrm{AES}(\text{compact})$ be a uniform space and $F \in \mathfrak{U}_c^{\kappa}(X,X)$ a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Remark 2.4. This result was established in [3]. However, we excluded some assumptions (X uniform and F upper semicontinuous with closed values) so the proof in [3] has to be adjusted slightly.

Proof. Let $\alpha \in \mathrm{Cov}_X(K)$ where $K = \overline{F(X)}$. From Theorem 1.4 (see Remark 1.6), it suffices to show that F has an α-fixed point. We know (see [13]) that K can be embedded as a closed subset K^* of T; let $s : K \to K^*$ be a homeomorphism. Also let $i : K \hookrightarrow X$ and $j : K^* \hookrightarrow T$ be inclusions. Next let $\alpha' = \alpha \cup \{X \backslash K\}$ and note that α' is an open covering of X. Let the continuous map $h : T \to X$ be such that $h|_{K^*}$ and s^{-1} are α'-close (guaranteed since $X \in \mathrm{AES}(\text{compact})$). Then it follows immediately from the definition (note that $\alpha' = \alpha \cup \{X \backslash K\}$) that $hs : K \to X$ and $i : K \to X$ are α-close. Let $G = jsFh$ and notice that $G \in \mathfrak{U}_c^{\kappa}(T,T)$. Now Theorem 1.3 guarantees that there exists $x \in T$ with $x \in Gx$. Let $y = h(x)$, and so, from the above, we have $y \in hjsF(y)$, that is, $y = hjs(q)$ for some $q \in F(y)$. Now since hs and i are α-close, there exists $U \in \alpha$ with $hs(q) \in U$ and $i(q) \in U$, that is, $q \in U$ and $y = hjs(q) = hs(q) \in U$ since $s(q) \in K^*$. Thus $q \in U$ and $y \in U$, so $y \in U$ and $F(y) \cap U \neq \emptyset$ since $q \in F(y)$. As a result, F has an α-fixed point. □

Definition 2.5. Let V be a uniform space. Then V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in \mathrm{Cov}_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha : K \to V$ such that

(i) π_α and $i : K \hookrightarrow V$ are α-close;
(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in \mathrm{AES(compact)}$.

Theorem 2.6. *Let V be a uniform space and Schauder admissible and $F \in \mathcal{U}_c^\kappa(V,V)$ a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Proof. Let $K = \overline{F(X)}$ and let $\alpha \in \mathrm{Cov}_V(K)$. From Theorem 1.4 (see Remark 1.6), it suffices to show that F has an α-fixed point. There exists $\pi_\alpha : K \to V$ (as described in Definition 2.5) and a subset $C \subseteq V$ with $C \in \mathrm{AES(compact)}$ such that (here $F_\alpha = \pi_\alpha F$)

$$F_\alpha(V) = \pi_\alpha F(V) \subseteq C. \tag{2.1}$$

Notice that $F_\alpha \in \mathcal{U}_c^\kappa(C,C)$ is a compact upper semicontinuous map with closed (in fact compact) values. So Theorem 2.3 guarantees that there exists $x \in C$ with $x \in \pi_\alpha F(x)$, that is, $x = \pi_\alpha q$ for some $q \in F(x)$. Now Definition 2.5(i) guarantees that there exists $U \in \alpha$ with $\pi_\alpha(q) \in U$ and $i(q) \in U$, that is, $x \in U$ and $q \in U$. Thus $x \in U$ and $F(x) \cap U \neq \emptyset$ since $q \in F(x)$, so F has an α-fixed point. □

A space Y is a *neighborhood extension space* for Q (written $Y \in \mathrm{NES}(Q)$) if for all $X \in Q$, all $K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a continuous extension $f : U \to Y$ of f_0 over a neighborhood U of K in X.

Let $X \in \mathrm{NES}(Q)$ and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact map. Now let K, K^*, s, and i be as in the proof of Theorem 2.3. Let U be an open neighborhood of K^* in T and let $h_U : U \to X$ be a continuous extension of $is^{-1} : K^* \to X$ on U (guaranteed since $X \in \mathrm{NES(compact)}$). Let $j_U : K^* \hookrightarrow U$ be the natural embedding, so $h_U j_U = is^{-1}$. Now consider $\mathrm{span}(T)$ in a Hausdorff locally convex topological vector space containing T. Now Theorem 1.5 guarantees that there exists a retraction $r : \mathrm{span}(T) \to T$. Let $i^* : U \hookrightarrow r^{-1}(U)$ be an inclusion and consider $G = i^* j_U s F h_U r$. Notice that $G \in \mathcal{U}_c^\kappa(r^{-1}(U), r^{-1}(U))$. We now *assume* that

$$G \in \mathcal{U}_c^\kappa(r^{-1}(U), r^{-1}(U)) \text{ has a fixed point.} \tag{2.2}$$

Now there exists $x \in r^{-1}(U)$ with $x \in Gx$. Let $y = h_U r(x)$, so $y \in h_U r i^* j_U s F(y)$, that is, $y = h_U r i^* j_U s(q)$ for some $q \in F(y)$. Since $h_U(z) = is^{-1}(z)$ for $z \in K^*$, we have

$$h_U r i^* j_U s(q) = (h_U r i^* j_U) s(q) = i(q), \tag{2.3}$$

so $y \in F(y)$.

Theorem 2.7. *Let $X \in \mathrm{NES(compact)}$ and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact map. Also assume that (2.2) holds with K, K^*, s, i, i^*, j_U, h_U, and r as described above. Then F has a fixed point.*

Remark 2.8. Theorem 2.7 was also established in [3]. Note that if F is admissible in the sense of Gorniewicz and the Lefschetz set $\Lambda(F) \neq \{0\}$, then we know [11] that (2.2) holds. Note that if $X \in \mathrm{ANR}$ (see [11]), then of course $X \in \mathrm{NES(compact)}$.

A space Y is an *approximate neighborhood extension space* for Q (written $Y \in \mathrm{ANES}(Q)$) if for all $\alpha \in \mathrm{Cov}(Y)$, all $X \in Q$, all $K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α.

Let $X \in \mathrm{ANES}(\text{compact})$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also let $\alpha \in \mathrm{Cov}_X(K)$ where $K = \overline{F(X)}$. To show that F has a fixed point, it suffices (Theorem 1.4 and Remark 1.6) to show that F has an α-fixed point. Let $\alpha' = \alpha \cup \{X \backslash K\}$ and let K^*, s, and i be as in the proof of Theorem 2.3. Since $X \in \mathrm{ANES}(\text{compact})$, there exists an open neighborhood U_α of K^* in T and $f_\alpha : U_\alpha \to X$ a continuous function such that $f_\alpha|_{K^*}$ and s^{-1} are α'-close and as a result $f_\alpha s : K \to X$ and $i : K \to X$ are α-close. Let $j_{U_\alpha} : K^* \hookrightarrow U_\alpha$ be the natural imbedding. We know (see [5, page 426]) that $U_\alpha \in \mathrm{NES}(\text{compact})$. Also notice that $G_\alpha = j_{U_\alpha} sF f_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha)$ is a compact upper semicontinuous map with closed values. We now *assume* that

$$G_\alpha = j_{U_\alpha} sF f_\alpha \in \mathcal{U}_c^\kappa(U_\alpha, U_\alpha) \text{ has a fixed point for each } \alpha \in \mathrm{Cov}_X\big(\overline{F(X)}\big). \tag{2.4}$$

We still have $\alpha \in \mathrm{Cov}_X(K)$ fixed and we let x be a fixed point of G_α. Now let $y_\alpha = f_\alpha(x)$, so $y = f_\alpha j_{U_\alpha} sF(y)$, that is, $y = f_\alpha j_{U_\alpha} s(q)$ for some $q \in F(y)$. Now since $f_\alpha s$ and i are α-close, there exists $U \in \alpha$ with $f_\alpha s(q) \in U$ and $i(q) \in U$, that is, $q \in U$ and $y = f_\alpha j_{U_\alpha} s(q) = f_\alpha s(q) \in U$ since $s(q) \in K^*$. Thus $q \in U$ and $y \in U$, so

$$y \in U, \quad F(y) \cap U \neq \varnothing \quad \text{since } q \in F(y). \tag{2.5}$$

Theorem 2.9. *Let $X \in \mathrm{ANES}(\text{compact})$ be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also assume that (2.4) holds with K, s, U_α, j_{U_α}, and f_α as described above. Then F has a fixed point.*

Next we present continuation results for multimaps. Let Y be a completely regular topological space and U an open subset of Y. We consider a subclass $\mathfrak{D}$ of $\mathcal{U}_c^\kappa$. This subclass must have the following property: for subsets X_1, X_2, and X_3 of Hausdorff topological spaces, if $F \in \mathfrak{D}(X_2,X_3)$ is compact and $f \in \mathscr{C}(X_1,X_2)$, then $F \circ f \in \mathfrak{D}(X_1,X_3)$.

Definition 2.10. The map $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ if $F \in \mathfrak{D}(\overline{U}, Y)$ with F compact and $x \notin Fx$ for $x \in \partial U$; here $\overline{U}$ (resp., ∂U) denotes the closure (resp., the boundary) of U in Y.

Definition 2.11. A map $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathfrak{D}_{\partial U}(\overline{U}, Y)$ if for every $G \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ with $G|_{\partial U} = F|_{\partial U}$, there exists $x \in U$ with $x \in Gx$.

Theorem 2.12 (homotopy invariance). *Let Y and U be as above. Suppose $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathfrak{D}_{\partial U}(\overline{U}, Y)$ and $H \in \mathfrak{D}(\overline{U} \times [0,1], Y)$ is a closed compact map with $H(x,0) = F(x)$ for $x \in \overline{U}$. Also assume that*

$$x \notin H_t(x) \quad \text{for any } x \in \partial U,\ t \in (0,1]\ \big(H_t(\cdot) = H(\cdot,t)\big). \tag{2.6}$$

Then H_1 has a fixed point in U.

Proof. Let

$$B = \{x \in \overline{U} : x \in H_t(x) \text{ for some } t \in [0,1]\}. \tag{2.7}$$

When $t = 0$, $H_t = F$, and since $F \in \mathfrak{D}_{\partial U}(\overline{U}, Y)$ is essential in $\mathfrak{D}_{\partial U}(\overline{U}, Y)$, there exists $x \in U$ with $x \in Fx$. Thus $B \neq \varnothing$ and note that B is closed, in fact compact (recall that H is a closed, compact map). Notice also that (2.6) implies $B \cap \partial U = \varnothing$. Thus, since Y is completely regular, there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Define a map R by $R(x) = H(x,\mu(x))$ for $x \in \overline{U}$. Let $j : \overline{U} \to \overline{U} \times [0,1]$ be given by $j(x) = (x,\mu(x))$. Note that j is continuous, so $R = H \circ j \in \mathfrak{D}(\overline{U},Y)$ (see the description of the class $\mathfrak{D}$ before Definition 2.10). In addition, R is compact, and for $x \in \partial U$, we have $R(x) = H_0(x) = F(x)$. As a result, $R \in \mathfrak{D}_{\partial U}(\overline{U},Y)$ with $R|_{\partial U} = F|_{\partial U}$. Now since F is essential in $\mathfrak{D}_{\partial U}(\overline{U},Y)$, there exists $x \in U$ with $x \in R(x)$, that is, $x \in H_{\mu(x)}(x)$. Thus $x \in B$ and so $\mu(x) = 1$. Consequently, $x \in H_1(x)$. □

Next we give an example of an essential map.

Theorem 2.13 (normalization). *Let Y and U be as above with $0 \in U$. Suppose the following conditions are satisfied:*

$$\text{for any map } \theta \in \mathfrak{D}_{\partial U}(\overline{U},Y) \text{ with } \theta|_{\partial U} = \{0\}, \text{ the map } J \text{ is in } \mathcal{U}_c^{\kappa}(Y,Y);$$
$$J(x) = \begin{cases} \theta(x), & x \in \overline{U}, \\ \{0\}, & x \in Y \backslash \overline{U}, \end{cases} \tag{2.8}$$

and

$$J \in \mathcal{U}_c^{\kappa}(Y,Y) \text{ has a fixed point.} \tag{2.9}$$

Then the zero map is essential in $\mathfrak{D}_{\partial U}(\overline{U},Y)$.

Remark 2.14. Note that examples of spaces Y for (2.9) to be true can be found in Theorems 2.1, 2.3, 2.6, 2.7, and 2.9 (notice that J is compact).

Proof of Theorem 2.13. Let $\theta \in \mathfrak{D}_{\partial U}(\overline{U},Y)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Define a map J as in (2.8). From (2.8) and (2.9), we know that there exists $x \in Y$ with $x \in J(x)$. Now if $x \notin U$, we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$. □

Remark 2.15. Other homotopy and essential map results in a topological vector space setting can be found in [1, 2].

To conclude this paper, we discuss inward-type maps for a general class of admissible maps. The proof presented involves minor modifications of an argument due to Ben-El-Mechaiekh and Kryszewski [9]. Let Y be a normed space and $X \subseteq Y$, and consider a subclass $\mathcal{R}(X,Y)$ of $\mathcal{U}_c^{\kappa}(X,Y)$. This subclass must have the following properties: (i) if $X \subseteq Z \subseteq Y$ and if $I : X \hookrightarrow Z$ is an inclusion, $t > 0$, and $F \in \mathcal{R}(X,Y)$ with $(I + tF)(X) \subseteq Z$, then $I + tF \in \mathcal{U}_c^{\kappa}(X,Z)$, and (ii) each $F \in \mathcal{R}(X,Y)$ is upper semicontinuous and compact valued.

In our next result we assume that Ω is a compact $\mathcal{L}$-retract [9], that is,

(A) Ω is a compact neighborhood retract of a normed space $E = (E, \|\cdot\|)$ and there exist $\beta > 0$, $r : B(\Omega,\beta) \to \Omega$ a retraction, and $L > 0$ such that $\|r(x) - x\| \le L d(x;\Omega)$ for $x \in B(\Omega,\beta)$.

As a result,

$$\exists \eta > 0, \quad \eta < \frac{\beta}{2} \quad \text{with } \|r(x) - x\| < \eta \ \forall x \in B(\Omega, \eta). \tag{2.10}$$

Theorem 2.16. *Let $E = (E, \|\cdot\|)$ be a normed space and Ω as in assumption (A), and assume either (i) Ω is Schauder admissible or (ii) (2.2) holds with $X = \Omega$. In addition, suppose $F \in \mathcal{R}(\Omega, E)$ with*

$$F(x) \subseteq C_\Omega(x) \quad \forall x \in \Omega. \tag{2.11}$$

Then there exists $x \in \Omega$ with $0 \in Fx$.

Remark 2.17. Here C_Ω is the Clarke tangent cone, that is,

$$C_\Omega(x) = \{v \in E : c(x, v) = 0\}, \tag{2.12}$$

where

$$c(x, y) = \limsup_{\substack{y \to x,\, y \in \Omega \\ t \downarrow 0}} \frac{d(x + tv; \Omega)}{t}. \tag{2.13}$$

Remark 2.18. If Ω is a compact neighborhood retract, then of course $\Omega \in \text{NES(compact)}$.

Remark 2.19. The proof is basically due to Ben-El-Mechaiekh and Kryszewski [9] and is based on [9, Lemma 5.1] (this lemma is a modification of a standard argument in the literature using partitions of unity).

Proof. Now [9, Lemma 5.1] (choose $\Psi(x) = \{x \in E : c(x, v) < \delta\}$ ($\delta > 0$ appropriately chosen), $\Phi(x) = co(F(x))$ and apply the argument in [9, page 4176]) implies that there exists $M > 0$ such that for each $x \in K$ and each $y \in Fx$, we have $\|y\| \le M$. Choose $\tau > 0$ with $M\tau < \eta$ (here η is as in (2.10)) and a sequence $(t_n)_{n \in N}$ in $(0, \tau]$ with $t_n \downarrow 0$; here $N = \{1, 2, \ldots\}$. Define a sequence of maps ψ_n, $n \in N$, by

$$\psi_n(x) = r(x + t_n F(x)) \quad \text{for } x \in \Omega; \tag{2.14}$$

note that $d(x + t_n y; \Omega) < \eta$ for $x \in \Omega$ and $y \in F(x)$ since $M\tau < \eta$. Fix $n \in N$ and notice that $\psi_n \in \mathcal{U}_c^\kappa(\Omega, \Omega)$ is a compact map (note that Ω is compact and ψ_n is upper semicontinuous with compact values). Now Theorem 2.6 or Theorem 2.7 guarantees that there exists $x_n \in \Omega$ and $y_n \in Fx_n$ with

$$x_n = r(x_n + t_n y_n). \tag{2.15}$$

Also notice from (2.15) and assumption (A) (note that $M\tau < \eta < \beta/2 < \beta$) that

$$t_n \|y_n\| = \|x_n + t_n y_n - r(x_n + t_n y_n)\| \le L d(x_n + t_n y_n; \Omega). \tag{2.16}$$

Now Ω is compact so $F(\Omega)$ is compact, and as a result, there exists a subsequence S of N with $(x_n, y_n) \in \text{Graph} F$ and $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ in S. Of course, since F is upper semicontinuous, we have $\overline{y} \in F(\overline{x})$. Also from (2.11), we have $F(\overline{x}) \subseteq C_\Omega(\overline{x})$ and as a result, $\overline{y} \in F(\overline{x}) \subseteq C_\Omega(\overline{x})$, so $c(\overline{x}, \overline{y}) = 0$. Note also that

$$d(x_n + t_n y_n; \Omega) \le d(x_n + t_n \overline{y}; \Omega) + t_n \|y_n - \overline{y}\| \tag{2.17}$$

and this together with (2.16) yields

$$\|\overline{y}\| = \limsup_{n\to\infty} \|y_n\| \le \limsup \left(\frac{Ld(x_n + t_n \overline{y}; \Omega)}{t_n} + \|y_n - \overline{y}\| \right) = c(\overline{x}, \overline{y}) = 0, \tag{2.18}$$

so $0 \in F(\overline{x})$. □

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Donal O'Regan: Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie