QUADRATIC OPTIMIZATION OF FIXED POINTS FOR A FAMILY OF NONEXPANSIVE MAPPINGS IN HILBERT SPACE

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Given a finite family of nonexpansive self-mappings of a Hilbert space, a particular quadratic functional, and a strongly positive selfadjoint bounded linear operator, Yamada et al. defined an iteration scheme which converges to the unique minimizer of the quadratic functional over the common fixed point set of the mappings. In order to obtain their result, they needed to assume that the maps satisfy a commutative type condition. In this paper, we establish their conclusion without the assumption of any type of commutativity.

Finding an optimal point in the intersection F of the fixed point sets of a family of nonexpansive maps is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive maps. (See, e.g., [3, 4].) The problem of finding an optimal point that minimizes a given cost function $\Theta : \mathcal{H} \to \mathbb{R}$ over F is of wide interdisciplinary interest and practical importance. (See, e.g., [2, 4, 5, 7, 14].) A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including the set-theoretic signal estimation. (See, e.g., [5, 6, 10, 14].) The best approximation problem of finding the projection $P_F(a)$ (in the norm induced by the inner product of $\mathcal{H}$) from any given point a in $\mathcal{H}$ is the simplest case of our problem. Some papers dealing with this best approximation problem are [2, 9, 11].

Let $\mathcal{H}$ be a Hilbert space, C a closed convex subset of $\mathcal{H}$, and T_i, where $i = 1, 2, \ldots, N$, a finite family of nonexpansive self-maps of C, with $F := \cap_{i=1}^{n} \operatorname{Fix}(T_i) \neq \varnothing$. Define a quadratic function $\Theta : \mathcal{H} \to \mathbb{R}$ by

$$\Theta(u) := \frac{1}{2}\langle Au, u\rangle - \langle b, u\rangle \quad \forall u \in \mathcal{H}, \tag{1}$$

where $b \in \mathcal{H}$ and A is a selfadjoint strongly positive operator. We will also assume that $B := I - A$ satisfies $\|B\| < 1$, although this is not restrictive, since μA is strongly positive

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with $\|I-\mu A\|<1$ for any $\mu \in (0, 2/\|A\|)$, and minimizing $\widetilde{\Theta}(u) := (1/2)\langle \mu A u, u\rangle - \langle \mu b, u\rangle$ over F is equivalent to the original minimization problem.

Yamada et al. [13] show that there exists a unique minimizer u^* of Θ over C if and only if u^* satisfies

$$\langle Au^* - b, u - u^*\rangle \geq 0 \quad \forall u \in C. \tag{2}$$

In their solution of this problem, Yamada et al. [13] add the restriction that the T_i satisfy

$$\operatorname{Fix}(T_N \cdots T_1) = \operatorname{Fix}(T_1 T_N \cdots T_3 T_2) = \operatorname{Fix}(T_{N-1} T_{N-2} \cdots T_1 T_N). \tag{3}$$

There are many nonexpansive maps, with a common fixed point set, that do not satisfy (3). For example, if $X = [0,1]$ and T_1 and T_2 are defined by $T_1 x = x/2 + 1/4$ and $T_2 x = 3x/4$, then $\operatorname{Fix}(T_1, T_2) = \{2/5\}$, whereas $\operatorname{Fix}(T_2, T_1) = \{3/10\}$.

In our solution, we are able to remove restriction (3). We will take advantage of the modified Wittmann iteration scheme developed by Atsushiba and Takahashi [1].

Let $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nN} \in (0,1]$, $n = 1, 2, \dots$. Given the mappings $T_1, T_2, \dots, T_N$, one can define, for each n, new mappings $U_1, \dots, U_N$ by

$$\begin{aligned}
U_{n1} &= \alpha_{n1} T_1 + (1 - \alpha_{n1}) I, \\
U_{n2} &= \alpha_{n2} T_2 U_{n1} + (1 - \alpha_{n2}) I, \\
&\vdots \\
U_{n,N-1} &= \alpha_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \alpha_{n,N-1}) I, \\
W_n := U_{nN} &= \alpha_{nN} T_N U_{n,N-1} + (1 - \alpha_{nN}) I.
\end{aligned} \tag{4}$$

From [1, Lemmas 3.1 and 3.2], if the T_i are nonexpansive, so are the U_{ni}, and both sets of functions have the same fixed point set.

The iteration scheme we will use is the following. Let $b \in C$ and choose any $u_0 \in C$. Define $\{u_n\}$ by

$$u_{n+1} = \lambda_n b + (I - \lambda_n A) W_n u_n, \tag{5}$$

where the W_n are the self-maps of C generated by (4).

Theorem 1. *Let $T_i : \mathcal{H} \to \mathcal{H}$ $(i = 1, \dots, N)$ be nonexpansive with nonempty common fixed point set $F \neq \emptyset$. Assume that $\{\lambda_n\}$ and $\{\alpha_{ni}\}$ satisfy*

(i) $0 \leq \lambda_n \leq 1$,
(ii) $\lim \lambda_n = 0$,
(iii) $\sum_{n \geq 1} \lambda_n = \infty$,

(iv) $\sum_{n\geq 1}|\lambda_n - \lambda_{n-1}| < \infty$,
(v) $\sum_{n\geq 1}|\alpha_{ni} - \alpha_{n-1,i}| < \infty$ *for each* $i = 1,2,\dots,N$.

Then, for any point $u_0 \in \mathcal{H}$*, the sequence* $\{u_n\}$ *generated by (5) converges strongly to the unique minimizer* u^* *of the function* Θ *of (1) over* F.

Proof. From [15], u^* exists and is unique. We will first assume that

$$u_0 \in C_{u^*} := \left\{x \in \mathcal{H} \mid ||x - u^*|| \leq \frac{||b - Au^*||}{1 - \|B\|}\right\}, \tag{6}$$

where A and B are as previously defined.

For any $x \in \mathcal{H}$ and $0 \leq \lambda \leq 1$, define

$$T_\lambda(x) = \lambda b + (I - \lambda A)W(x). \tag{7}$$

Then, for any $y \in \mathcal{H}$, since W is nonexpansive,

$$||T_\lambda(x) - T_\lambda(y)|| = ||(I - \lambda A)(W(x) - W(y))|| \leq [1 - \lambda(1 - \|B\|)]\|x - y\|. \tag{8}$$

Also, since $u^* \in F$,

$$||T_\lambda(u^*) - u^*|| = \lambda||b - Au^*||. \tag{9}$$

Thus,

$$\begin{aligned}||T_\lambda(x) - u^*|| &\leq ||T_\lambda(x) - T_\lambda(u^*)|| + ||T_\lambda(u^*) - u^*|| \\ &\leq [1 - \lambda(1 - \|B\|)]||x - u^*|| + \lambda(1 - \|B\|)\frac{||b - Au^*||}{1 - \|B\|} \\ &\leq \frac{||x - Au^*||}{1 - \|B\|}.\end{aligned} \tag{10}$$

If, in (7), we make the substitution $\lambda = \lambda_n$, $T_\lambda(x) = u_{n+1}$, and $W(x) = W_n u_n$, then it follows from (9) and (10) that u_n and $W_n u_n$ belong to C_{u^*} for each n. Thus, $\{u_n\}$ and $\{W_n u_n\}$ are bounded. Since $\|B\| < 1$, $\{BW_n u_n\}$ is also bounded.

Let K denote the diameter of C_{u^*}.

We may write (5) in the form

$$\begin{aligned}u_{n+1} &= \lambda_n b + (I - \lambda_n(I - B))W_n u_n \\ &= \lambda_n b + (I - \lambda_n)W_n u_n + \lambda_n B W_n u_n.\end{aligned} \tag{11}$$

We will first show that

$$\lim ||u_{n+1} - u_n|| = 0. \tag{12}$$

Using (11), since each W_n is nonexpansive and $\|B\| < 1$,

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\quad = \|\lambda_n b + (1-\lambda_n) W_n u_n + \lambda_n B W_n u_n - \lambda_{n-1} b \\
&\qquad - (1-\lambda_{n-1}) W_{n-1} u_{n-1} - \lambda_{n-1} B W_{n-1} u_{n-1}\| \\
&\quad \le |\lambda_n - \lambda_{n-1}| \|b\| + (1-\lambda_n)\|W_n u_n - W_{n-1} u_n\| \\
&\qquad + |\lambda_n - \lambda_{n-1}| \|W_{n-1} u_{n-1}\| + (1-\lambda_n)\|W_{n-1} u_n - W_{n-1} u_{n-1}\| \\
&\qquad + \lambda_n \|B\| \|W_n u_n - W_{n-1} u_n\| + \lambda_n \|B\| \|W_{n-1} u_n - W_{n-1} u_{n-1}\| \\
&\qquad + |\lambda_n - \lambda_{n-1}| \|B W_{n-1} u_{n-1}\| \\
&\quad \le 3 |\lambda_n - \lambda_{n-1}| K + (1-\lambda_n + \lambda_n \|B\|) \\
&\qquad \times \big[\|W_n u_n - W_{n-1} u_n\| + (1-\lambda_n + \lambda_n \|B\|) \|W_{n-1} u_n - W_{n-1} u_{n-1}\|\big].
\end{aligned} \tag{13}$$

From (4), since T_N and $U_{n-1,N-1}$ are nonexpansive,

$$\begin{aligned}
&\|W_n u_n - W_{n-1} u_n\| \\
&\quad = \|\alpha_{nN} T_N U_{n,N-1} u_n + (1-\alpha_{nN}) u_n - \alpha_{n-1,N} T_N U_{n-1,N-1} u_n - (1-\alpha_{n-1,N}) u_n\| \\
&\quad \le |\alpha_{nN} - \alpha_{n-1,N}| \|u_n\| + \|\alpha_{nN} T_N U_{n,N-1} u_n - \alpha_{n-1,N} T_N U_{n-1,N-1} u_n\| \\
&\quad \le |\alpha_{nN} - \alpha_{n-1,N}| \|u_n\| + \|\alpha_{nN} (T_N U_{n,N-1} u_n - T_N U_{n-1,N-1} u_n)\| \\
&\qquad + |\alpha_{nN} - \alpha_{n-1,N}| \|T_N U_{n-1,N-1} u_n\| \\
&\quad \le |\alpha_{nN} - \alpha_{n-1,N}| \|u_n\| + \alpha_{nN} \|U_{n,N-1} u_n - U_{n-1,N-1} u_n\| + |\alpha_{nN} - \alpha_{n-1,N}| K \\
&\quad \le 2K |\alpha_{nN} - \alpha_{n-1,N}| + \alpha_{nN} \|U_{n,N-1} u_n - U_{n-1,N-1} u_n\|.
\end{aligned} \tag{14}$$

Again, from (4),

$$\begin{aligned}
&\|U_{n,N-1} u_n - U_{n-1,N-1} u_n\| \\
&\quad = \|\alpha_{n,N-1} T_{N-1} U_{n,N-2} u_n + (1-\alpha_{n,N-1}) u_n \\
&\qquad - \alpha_{n-1,N-1} T_{N-1} U_{n-1,N-2} u_n - (1-\alpha_{n-1,N-1} u_n)\| \\
&\quad \le |\alpha_{n,N-1} - \alpha_{n-1,N-1}| \|u_n\| \\
&\qquad + \|\alpha_{n,N-1} T_{N-1} U_{n,N-2} u_n - \alpha_{n-1,N-1} T_{N-1} U_{n-1,N-2} u_n\| \\
&\quad \le |\alpha_{n,N-1} - \alpha_{n-1,N-1}| \|u_n\| \\
&\qquad + \alpha_{n,N-1} \|T_{N-1} U_{n,N-2} u_n - T_{N-1} U_{n-1,N-2} u_n\| \\
&\qquad + |\alpha_{n,N-1} - \alpha_{n-1,N-1}| K \\
&\quad \le 2K |\alpha_{n,N-1} - \alpha_{n-1,N-1}| + \alpha_{n,N-1} \|U_{n,N-2} u_n - U_{n-1,N-2} u_n\| \\
&\quad \le 2K |\alpha_{n,N-1} - \alpha_{n-1,N-1}| + \|U_{n,N-2} u_n - U_{n-1,N-2} u_n\|.
\end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned}
&\left|\left|U_{n,N-1}u_n - U_{n-1,N-1}u_n\right|\right| \\
&\quad \le 2K\left|\alpha_{n,N-1} - \alpha_{n-1,N-1}\right| + 2K\left|\alpha_{n,N-2} - \alpha_{n-1,N-2}\right| \\
&\qquad + \left|\left|U_{n,N-3}u_n - U_{n-1,N-3}u_n\right|\right| \\
&\quad \vdots \\
&\quad \le 2K\sum_{i=2}^{N-1}\left|\alpha_{ni} - \alpha_{n-1,i}\right| + \left|\left|U_{n1}u_n - U_{n-1,1}u_n\right|\right| \\
&\quad = \left|\left|\alpha_{n1}T_1u_n + (1-\alpha_{n1})u_n - \alpha_{n-1,1}T_1u_n - (1-\alpha_{n-1,1})u_n\right|\right| \\
&\qquad + 2K\sum_{i=2}^{N-1}\left|\alpha_{ni} - \alpha_{n-1,i}\right| \\
&\quad \le \left|\alpha_{n1} - \alpha_{n-1,1}\right|\left|\left|u_n\right|\right| + \left|\left|\alpha_{n1}T_1u_n - \alpha_{n-1,1}T_1u_n\right|\right| \\
&\qquad + 2K\sum_{i=2}^{N-1}\left|\alpha_{ni} - \alpha_{n-1,i}\right| \\
&\quad \le 2K\sum_{i=1}^{N-1}\left|\alpha_{ni} - \alpha_{n-1,i}\right|.
\end{aligned} \tag{16}$$

Substituting (16) into (14),

$$\begin{aligned}
\left|\left|W_nu_n - W_{n-1}u_n\right|\right| &\le 2K\left|\alpha_{nN} - \alpha_{n-1,N}\right| + 2\alpha_{nN}K\sum_{i=1}^{N-1}\left|\alpha_{ni} - \alpha_{n-1,i}\right| \\
&\le 2K\sum_{i=1}^{N}\left|\alpha_{ni} - \alpha_{n-1,i}\right|.
\end{aligned} \tag{17}$$

Using (17) in (13),

$$\begin{aligned}
\left|\left|u_{n+1} - u_n\right|\right| \le{}& \left(1 - \lambda_n(1 - \|B\|)\right)\left|\left|u_n - u_{n-1}\right|\right| + 3K\left|\lambda_n - \lambda_{n-1}\right| \\
&+ 2\left(1 - \lambda_n(1 - \|B\|)\right)K\sum_{i=1}^{N}\left|\alpha_{ni} - \alpha_{n-1,i}\right|.
\end{aligned} \tag{18}$$

Thus, since $0 < 1 - \lambda_n(1 - \|B\|) < 1$ for all n,

$$\begin{aligned}
\left|\left|u_{n+m+1} - u_{n+m}\right|\right| \le{}& \prod_{i=m}^{n+m}\left(1 - \lambda_i(1 - \|B\|)\left|\left|u_{i+1} - u_i\right|\right|\right) \\
&+ 3K\left(\sum_{i=m}^{n+m}\left|\lambda_i - \lambda_{i-1}\right| + 2K\sum_{i=m}^{n+m}\sum_{j=1}^{N}\left|\alpha_{ij} - \alpha_{i-1,j}\right|\right).
\end{aligned} \tag{19}$$

From (iii), since the product diverges to zero,

$$\begin{aligned}\limsup_n \|u_{n+1}-u_n\| &= \limsup_n \|u_{n+m+1}-u_{m+n}\| \\ &\le 2K\sum_{i=m}^{\infty}|\lambda_i-\lambda_{i-1}| + 2K\sum_{i=m}^{\infty}\sum_{j=1}^{N}|\alpha_{ij}-\alpha_{i-1,j}|.\end{aligned} \tag{20}$$

Therefore, taking the $\limsup_m$ of both sides and using (iv) and (v),

$$\limsup_n \|u_{n+1}-u_n\| = 0, \tag{21}$$

and (12) is satisfied.

Now, for any nonexpansive self-map T of C_{u^*}, define $G_t : C_{u^*} \to C_{u^*}$ by

$$G_t(x) = tb + (1-t)TG_t(x) + tBTG_t(x) \tag{22}$$

for each $t \in (0,1]$. Using an argument similar to the proof of [8, Theorem 12.2, page 45], we will now show that if T has a fixed point, then, for each x in C_{u^*}, the strong $\text{limit}_{t\to 0} G_t(x)$ exists and is a fixed point of T.

Define $y(t) = G_t(x)$ and let w be a fixed point of T:

$$y(t) - w = t(b-w) + (1-t)\big(Ty(t)-w\big) + tBTy(t). \tag{23}$$

Since T is nonexpansive,

$$\begin{aligned}\|y(t)-w\| &\le t\|b-w\| + (1-t)\|Ty(t)-w\| + t\|B\|\|Ty(t)\| \\ &\le t\|b-w\| + (1-t)\|y(t)-w\| + t\|B\|\|Ty(t)\|, \\ t\|y(t)-w\| &\le t\|b-w\| + t\|B\|\|Ty(t)-w\| + t\|B\|\,\|w\|,\end{aligned} \tag{24}$$

or

$$\|y(t)-w\| \le \|b-w\| + \|B\|\|y(t)-w\| + \|B\|\,\|w\|, \tag{25}$$

which, since $\|B\| < 1$, yields

$$\|y(t)-w\| \le \frac{1}{1-\|B\|}\big[\|b-w\| + \|B\|\,\|w\|\big], \tag{26}$$

and $y(t)$ remains bounded as $t \to 0$.

Also,

$$\|BTy(t)\| < \|Ty(t)\| \le \|Ty(t)-Tw\| + \|w\| \le \|y(t)-w\| + \|w\|, \tag{27}$$

and both $BTy(t)$ and $Ty(t)$ remain bounded as $t \to 0$.

Hence,

$$\|y(t)-Ty(t)\| = t\|b - Ty(t) + BTy(t)\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0. \tag{28}$$

Define $y_n = y(t_n)$ and let $t_n \to 0$. Let μ_n be a Banach limit and $f : C_{u^*} \to \mathbb{R}^+$ defined by

$$f(z) = \mu_n\left\{||y_n - z||^2\right\}. \tag{29}$$

Since f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$. Since $\mathcal{H}$ is reflexive, f attains it infimum over C_{u^*}.

Let M be the set of minimizers of f over C_{u^*}. If $u \in C_{u^*}$, then

$$f(Tu) = \mu_n\left\{||y_n - Tu||^2\right\} = \mu_n\left\{||Ty_n - Tu||^2\right\} \le \mu_n\left\{||y_n - u||^2\right\} = f(u). \tag{30}$$

Therefore, M is invariant under T. Since it is also bounded, closed, and convex, it must contain a fixed point of T. Denote this fixed point by v. Then,

$$\begin{aligned}\langle y_n - Ty_n, y_n - v\rangle &= \langle y_n - v, y_n - v\rangle + \langle v - Ty_n, y_n - v\rangle \\ &= ||y_n - v||^2 + \langle Tv - Ty_n, y_n - v\rangle.\end{aligned} \tag{31}$$

But

$$|\langle Tv - Ty_n, y_n - v\rangle| \le ||Tv - Ty_n||\,||y_n - v|| \le ||y_n - v||^2, \tag{32}$$

so that

$$\langle y_n - Ty_n, y_n - v\rangle \ge 0. \tag{33}$$

Since

$$\begin{aligned} y_n &= t_n b + (1 - t_n)Ty_n + t_n BTy_n, \\ y_n - b &= (1 - t_n)(Ty_n - b) + t_n BTy_n \\ &= (1 - t_n)(Ty_n - y_n) + (1 - t_n)(y_n - b) + t_n BTy_n, \end{aligned} \tag{34}$$

thus,

$$t(y_n - b) = (1 - t_n)(Ty_n - y_n) + t_n BTy_n \tag{35}$$

or

$$y_n - b - Bv = \frac{1 - t_n}{t_n}(Ty_n - y_n) + BTy_n - Bv. \tag{36}$$

Therefore, from (33),

$$\begin{aligned} &\langle y_n - b - Bv, y_n - v\rangle \\ &\quad = \frac{1 - t_n}{t_n}\langle Ty_n - y_n, y_n - v\rangle + \langle BTy_n - Bv, y_n - v\rangle \\ &\quad \le \langle BTy_n - Bv, y_n - v\rangle. \end{aligned} \tag{37}$$

For any $z \in C_{u^*}$,

$$\begin{aligned} ||y_n - v||^2 &= ||y_n - (1-t_n)v - t_n z + t_n(z-v) - t_n b - t_n Bv + t_n(b+Bv)||^2 \\ &\geq ||y_n - (1-t_n)v - t_n b - t_n Bv||^2 \\ &\quad + 2t_n\langle z - v + b + Bv, y_n - (1-t_n)v - t_n z - t_n b - t_n Bv\rangle. \end{aligned} \tag{38}$$

Let $\epsilon > 0$ be given. Since $\mathscr{H}$ is uniformly smooth, there exists a $t_0 > 0$ such that, for all $t_n \leq t_0$,

$$\left|\langle z - v + b + Bv, (y_n - v) - (y_n - (1-t_n)v - t_n z - t_n b - t_n Bv)\rangle\right| < \epsilon. \tag{39}$$

Thus, from (38),

$$\begin{aligned} &\langle z - v + b + Bv, y_n - v\rangle \\ &\quad < \epsilon + \langle z - v + b + Bv, y_n - (1-t_n)v - t_n z - t_n b - t_n B\rangle \\ &\quad < \epsilon + \frac{1}{2t}\Big[||y_n - v||^2 - ||y_n - (1-t_n)v - t_n b - t_n Bv||^2\Big]. \end{aligned} \tag{40}$$

Since the Gateaux derivative exists in $\mathscr{H}$, we obtain

$$\mu_n\{\langle z - v + b + Bv, y_n - v\rangle\} \leq 0. \tag{41}$$

Setting $z = \theta$ in (41) and adding (37) and (41) yields

$$\mu_n\{\langle y_n - v, y_n - v\rangle\} \leq \mu_n\{\langle BTy_n - Bv, y_n - v\rangle\} \tag{42}$$

or

$$\begin{aligned} \mu_n\Big\{||y_n - v||^2\Big\} &\leq \mu_n\{||BTy_n - Bv||\,||y_n - v||\} \\ &\leq \mu_n\{\|B\|\,||Ty_n - Tv||\,||y_n - v||\} \\ &\leq \|B\|\mu_n\Big\{||y_n - v||^2\Big\}. \end{aligned} \tag{43}$$

Therefore, $\mu_n\|y_n - v\|^2 = 0$. Thus, there is a subsequence of $\{y_n\}$ converging strongly to v. Suppose that $\lim_{k\to\infty} y(t_{n_k}) = v_1$ and $\lim_{k\to\infty} y(t_{m_k}) = v_2$. From (37), we have

$$\begin{aligned} \langle v_1 - b - Bv_2, v_1 - v_2\rangle &\leq \langle BTv_1 - Bv_2, v_1 - v_2\rangle, \\ \langle v_2 - b - Bv_1, v_2 - v_1\rangle &\leq \langle BTv_2 - Bv_1, v_2 - v_1\rangle. \end{aligned} \tag{44}$$

Adding these inequalitites, we obtain

$$\langle v_1 - BTv_1 + BTv_2 - v_2, v_1 - v_2\rangle \leq 0 \tag{45}$$

or

$$\langle v_1 - v_2, v_1 - v_2\rangle \leq \langle BTv_1 - BTv_2, v_1 - v_2\rangle; \tag{46}$$

that is,

$$\begin{aligned} \|v_1 - v_2\|^2 &\le \|BTv_2 - BTv_1\| \|v_1 - v_2\| \\ &\le \|B\| \|Tv_2 - Tv_1\| \|v_1 - v_2\| \\ &\le \|B\| \|v_2 - v_1\|^2, \end{aligned} \tag{47}$$

which, since $\|B\| < 1$, implies that $v_1 = v_2$, and thus $\lim y_n = v$.

Now, setting $z = \theta$ in (41), we obtain

$$\mu_n \langle b - (I - B)v, y_n - v \rangle \le 0 \tag{48}$$

or

$$\mu_n \langle b - Av, y_n - v \rangle \le 0, \tag{49}$$

which, from (2), implies that $v = u^*$.

Let u_{nk} denote the unique element of C_{u^*} such that

$$u_{nk} = \frac{1}{k} b + \left(1 - \frac{1}{k}\right) W_n u_{nk} + \frac{1}{k} B W_n u_{nk}. \tag{50}$$

From what we have just proved, $\lim_k u_{nk} \to u^*$. Using (11),

$$\begin{aligned} &\|u_{n+1} - W_{n+1} u_{n+1,k}\| \\ &\quad = \|\lambda_n b + (1 - \lambda_n + \lambda_n B) W_n u_n - W_{n+1} u_{n+1,k}\| \\ &\quad \le \lambda_n \|b - W_{n+1} u_{n+1,k}\| + (1 - \lambda_n) \|W_n u_n - W_{n+1} u_{n+1,k}\| \\ &\qquad + \lambda_n \|B\| \|W_n u_n - W_{n+1} u_{n+1,k}\| + \lambda_n \|B W_{n+1} u_{n+1,k}\| \\ &\quad < 3K\lambda_n + (1 - \lambda_n + \lambda_n \|B\|) [\|W_n u_n - W_n u_{nk}\| + \|W_n u_{nk} - W_n u_{n+1,k}\| \\ &\qquad\qquad + \|W_n u_{n+1,k} - W_{n+1} u_{n+1,k}\|] \\ &\quad \le 3K\lambda_n + (1 - \lambda_n + \lambda_n \|B\|) [\|u_n - u_{nk}\| + \|u_{nk} - u_{n+1,k}\| \\ &\qquad\qquad + \|W_n u_{n+1,k} - W_{n+1} u_{n+1,k}\|]. \end{aligned} \tag{51}$$

As in (17),

$$\|W_n u_{n+1,k} - W_{n+1} u_{n+1,k}\| \le 2K \sum_{i=1}^{N} |\alpha_{n+1,i} - \alpha_{ni}|. \tag{52}$$

From the definition of u_{nk},

$$\begin{aligned} u_{nk} &= \frac{1}{k} b + \left(1 - \frac{1}{k}\right) W_n u_{nk} + \frac{1}{k} B W_n u_{nk}, \\ u_{n+1,k} &= \frac{1}{k} b + \left(1 - \frac{1}{k}\right) W_{n+1} u_{n+1,k} + \frac{1}{k} B W_{n+1} u_{n+1,k}, \\ u_{n+1,k} - u_{nk} &= \left(1 - \frac{1}{k}\right) (W_{n+1} u_{n+1,k} - W_n u_{nk}) + \frac{1}{k} B (W_{n+1} u_{n+1,k} - W_n u_{nk}). \end{aligned} \tag{53}$$

Therefore, since W_{n+1} is nonexpansive,

$$\begin{aligned}
||u_{n+1,k} - u_{nk}|| &\le \left(1 - \frac{1}{k} + \frac{1}{k}\|B\|\right)||W_{n+1}u_{n+1,k} - W_n u_{nk}|| \\
&\le \left(1 - \frac{1}{k} + \frac{1}{k}\|B\|\right)\left[||W_{n+1}u_{n+1,k} - W_{n+1}u_{nk}|| + ||W_{n+1}u_{nk} - W_n u_{nk}||\right] \\
&\le \left(1 - \frac{1}{k} + \frac{1}{k}\|B\|\right)\left[||u_{n+1,k} - u_{nk}|| + ||W_{n+1}u_{nk} - W_n u_{nk}||\right].
\end{aligned} \tag{54}$$

Thus, using (17),

$$\frac{(1 - \|B\|)}{k}||u_{n+1,k} - u_{nk}|| \le \frac{(k - 1 + \|B\|)}{k}2K\sum_{i=1}^{N}|\alpha_{n+1,i} - \alpha_{n,i}| \tag{55}$$

or

$$||u_{n+1,k} - u_{nk}|| \le \frac{(k - 1 + \|B\|)}{1 - \|B\|}2K\sum_{i=1}^{N}|\alpha_{n+1,i} - \alpha_{n,i}|. \tag{56}$$

Substituting (56) and (52) into (51) yields

$$\begin{aligned}
&||u_{n+1} - W_{n+1}u_{n+1,k}|| \\
&\quad \le 3K\lambda_n + (1 - \lambda_n + \lambda_n\|B\|)||u_n - u_{nk}|| + \left(\frac{k}{1 - \|B\|}\right)2K\sum_{i=1}^{N}|\alpha_{n+1,i} - \alpha_{n,i}|.
\end{aligned} \tag{57}$$

Thus, using (iii) and (v), we have

$$\mu_n\left\{||u_n - W_n u_{nk}||^2\right\} = \mu_n\left\{||u_{n+1} - W_{n+1}u_{n+1,k}||^2\right\} \le \mu_n\left\{||u_n - u_{nk}||^2\right\}. \tag{58}$$

From (53),

$$u_{nk} - u_n = \frac{1}{k}(b - u_n) + \left(1 - \frac{1}{k}\right)(W_n u_{nk} - u_n) + \frac{1}{k}BW_n u_{nk}. \tag{59}$$

Hence,

$$\left(1 - \frac{1}{k}\right)(u_n - W_n u_{nk}) = u_n - u_{nk} - \frac{1}{k}(u_n - b) + \frac{1}{k}BW_n u_{nk}, \tag{60}$$

$$\begin{aligned}
\left(1 - \frac{1}{k}\right)^2||u_n - W_n u_{nk}||^2 &\ge ||u_n - u_{nk}||^2 - \frac{2}{k}\langle u_n - b - BW_n u_{nk}, u_n - u_{nk}\rangle \\
&= \left(1 - \frac{2}{k}\right)||u_n - u_{nk}||^2 - \frac{2}{k}\langle u_{nk} - b - BW_n u_{nk}, u_n - u_{nk}\rangle.
\end{aligned} \tag{61}$$

Therefore, using (58) and (61),

$$\begin{aligned}\left(1-\frac{1}{k}\right)^2\mu_n\|u_n-u_{nk}\|^2 \geq \left(1-\frac{1}{k}\right)^2\mu_n\|u_n-W_nu_{nk}\|^2 \\ \geq \left(1-\frac{2}{k}\right)\mu_n\|u_n-u_{nk}\|^2 \\ -\frac{2}{k}\mu_n\langle u_{nk}-b-BW_nu_{nk},u_n-u_{nk}\rangle,\end{aligned} \tag{62}$$

which implies that

$$\frac{1}{2k}\mu_n\left\{\|u_n-u_{nk}\|^2\right\} \geq \mu_n\{\langle b-u_{nk}+BW_nu_{nk},u_n-u_{nk}\rangle\}. \tag{63}$$

Since $\lim_k u_{nk} \to u^*$, independent of n, it follows that

$$\begin{aligned}0 &\geq \mu_n\{\langle b-u^*+Bu^*,u_n-u^*\rangle\} \\ &= \mu_n\{\langle b-(I-B)u^*,u_n-u^*\rangle\} \\ &= \mu_n\{\langle b-Au^*,u_n-u^*\rangle\}.\end{aligned} \tag{64}$$

From (12),

$$\lim |\langle b-u^*,u_{n+1}-u^*\rangle - \langle b-u^*,u_n-u^*\rangle| = 0. \tag{65}$$

We need the following result from [12]. If A is a real number and $\{a_1,a_2,\ldots\} \in \ell^\infty$ such that $\mu_n\{a_n\} \leq a$ for all Banach limits μ_n and $\limsup_n(a_{n+1}-a_n) \leq 0$, then $\limsup_n a_n \leq a$.

Consequently,

$$\limsup_n \langle b-u^*,u_n-u^*\rangle \leq 0. \tag{66}$$

Since $u^* \in F$,

$$\|W_nu_n-u^*\| = \|W_nu_n-W_nu^*\| \leq \|u_n-u^*\|. \tag{67}$$

From (11),

$$\begin{aligned}u_{n+1}-u^* &= \lambda_n(b-u^*)+(1-\lambda_n)(W_nu_n-u^*)+\lambda_nBW_nu_n \\ &= \lambda_n(b-u^*)+(1-\lambda_n+\lambda_nB)(W_nu_n-u^*)+\lambda_nBu^*\end{aligned} \tag{68}$$

or

$$(1-\lambda_n+\lambda_nB)(W_nu_n-u^*) = u_{n+1}-u^*-\lambda_n(b-u^*)-\lambda_nBu^*. \tag{69}$$

Therefore,

$$(1-\lambda_n+\lambda_n\|B\|)^2\|W_nu_n-u^*\|^2 \geq \|u_{n+1}-u^*\|^2 - 2\lambda_n\langle b-u^*+Bu^*,u_{n+1}-u^*\rangle, \tag{70}$$

which implies that

$$\begin{aligned}\|u_{n+1}-u^*\|^2 \le (1-\lambda_n+\lambda_n\|B\|)^2\|W_nu_n-u^*\|^2+2\lambda_n\langle b-u^*,u_{n+1}-u^*\rangle \\ +2\lambda_n\langle Bu^*,u_{n+1}-u^*\rangle.\end{aligned} \tag{71}$$

From (ii) and the boundedness of C_{u^*}, there exists a positive integer N such that, for all $n \ge N$,

$$\lambda_n\langle b-u^*,u_{n+1}-u^*\rangle \le \frac{\epsilon}{4}, \qquad \lambda_n\langle Bu^*,u_{n+1}-u^*\rangle \le \frac{\epsilon}{4}. \tag{72}$$

Therefore, for $n \ge N$,

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &\le (1-\lambda_n+\lambda_n\|B\|)^2\|u_n-u^*\|^2+\frac{\epsilon}{2}+\frac{\epsilon}{2}, \\ \|u_{n+m}-u^*\|^2 &\le \left(\prod_{i=m}^{n+m-1}(1-\lambda_i+\lambda_i\|B\|)^2\right)\|u_m-u^*\|^2 \\ &\quad +\epsilon\left(\prod_{i=m}^{n+m-1}(1-\lambda_i+\lambda_i\|B\|)\right)^2.\end{aligned} \tag{73}$$

Using (iii),

$$\limsup_n\|u_n-u^*\|^2=\limsup_n\|u_{n+m}-u^*\|^2 \le 0. \tag{74}$$

Thus, $\{u_n\}$ converges strongly to u^*.

Now let $u_0 \in \mathcal{H}$. Let $\{s_n\}$ be another sequence generated by (11) for some $s_0 \in C_{u^*}$. Then, by what we have just proved, $\lim s_n = u^*$. Since W_n is nonexpansive for each n,

$$\begin{aligned}\|u_{n+1}-s_{n+1}\| &= \|\lambda_nb+(1-\lambda_nA)W_nu_n-\lambda_nb-(1-\lambda_nA)W_ns_n\| \\ &\le \|(1-\lambda_nA)(W_nu_n-W_ns_n)\| \\ &\le (1-\lambda_n+\lambda_n\|B\|)\|W_nu_n-W_ns_n\| \\ &\le (1-\lambda_n+\lambda_n\|B\|)\|u_n-s_n\|.\end{aligned} \tag{75}$$

By induction,

$$\|u_n-s_n\| \le \|u_0-s_0\|\prod_{k=1}^{n}\{1-\lambda_k(1-\|B\|)\}. \tag{76}$$

Therefore, using (iii), $\lim\|u_n-s_n\|=0$ and $\|u_n-u^*\| \le \|u_n-s_n\|+\|s_n-u^*\|$ so that $\lim u_n = u^*$. □

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