

DIAMETRICALLY CONTRACTIVE MAPS AND FIXED POINTS

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Contractive maps have nice properties concerning fixed points; a big amount of literature has been devoted to fixed points of nonexpansive maps. The class of shrinking (or strictly contractive) maps is slightly less popular: few specific results on them (not applicable to all nonexpansive maps) appear in the literature and some interesting problems remain open. As an attempt to fill this gap, a condition half way between shrinking and contractive maps has been studied recently. Here we continue the study of the latter notion, solving some open problems concerning these maps.

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1. Introduction

Let X be a Banach space and M a nonempty convex closed bounded subset of X . In the theory of fixed points, two classes of maps $T : M \rightarrow M$ are well known and deeply studied: the class of contractive maps

$$\forall x, y \text{ in } M, \quad \|Tx - Ty\| \leq \alpha \|x - y\|, \quad \alpha \in (0, 1), \quad (1.1)$$

and the class of nonexpansive maps

$$\forall x, y \text{ in } M, \quad \|Tx - Ty\| \leq \|x - y\|. \quad (1.2)$$

An intermediate class consists of the maps that satisfy the following condition:

$$\|Tx - Ty\| < \|x - y\| \quad \forall x \neq y, \text{ with } x, y \in M. \quad (S)$$

In the literature, these maps appear under different names, see for example [5] and the

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references therein; we will call them *shrinking*. We briefly recall some results and properties of maps in this class:

- (1) the fixed point, if it exists, is unique;
- (2) if M is a compact set (or more generally if \overline{TM} is compact), then T has a fixed point x^* , and moreover for each $x \in M$, $T^n x \rightarrow x^*$;
- (3) there is an example (see [5]) of a map on the unit ball of Hilbert spaces with fixed point x^* such that $T^n x$ does not converge to the fixed point for any $x \neq x^*$;
- (4) there are examples of maps without fixed points [4, 6, 9].

Not so much attention has been paid to shrinking maps; indeed the following questions are open. Let M be a weakly compact convex of a Banach space and let $T : M \rightarrow M$ be a shrinking mapping. Must T have a fixed point? If T has a fixed point x^* , is it true that $T^n x \rightarrow x^*$ for every x ?

Conditions stronger than (S) were considered, also in more general settings, see for example [3]. Another rather weak strengthening, which appeared probably for the first time in [2], is the one given by the following definition. T is *diametrically contractive* (DC) if $\delta(T(A)) < \delta(A)$ for every closed, convex, bounded non singleton subset A of M , where $\delta(A)$ is the diameter of A .

Such a notion was studied in details in [10]. We collect some relations between the previous classes of mappings:

- (1) diametrically contractive maps are shrinking;
- (2) if M is a compact set and T is shrinking, then it is diametrically contractive;
- (3) there are examples of shrinking maps that are not diametrically contractive [4, 10].

A most important result is the following, see [10, Theorem 2.3].

THEOREM 1.1. *Let M be a weakly compact subset of a Banach space X and let $T : M \rightarrow M$ be diametrically contractive, then T has a fixed point.*

The proof of this theorem appeared probably for the first time in [7, Theorem 2] and in the case of reflexive spaces can be found in [1, 8].

The following problems appear to be open (see [10]).

Problem 1.2. Can we substitute weakly compact subset with closed convex bounded one in Theorem 1.1?

Problem 1.3. If T is diametrically contractive and x^* is the fixed point of T , do we have $T^n x \rightarrow x^*$ for all (or at least for some) $x \in M$?

In this paper, we solve in the negative both problems: the first example (Section 2) solves Problem 1.2; the second example (Section 3) solves Problem 1.3.

2. First example

Now we give an example of a fixed point free DC self-map of a closed convex bounded set.

Consider the vector space of all continuous real functions on the closed unit interval, with the norm (equivalent to the classical one)

$$\|f\| = \|f\|_\infty + \|f\|_1 = \max_{0 \leq x \leq 1} |f(x)| + \int_0^1 |f(x)| dx. \quad (2.1)$$

Let $M = \{f \in X : f(0) = 0; f(1) = 1; 0 \leq f(x) \leq x; f \text{ is monotone nondecreasing}\}$.

Define $T : M \rightarrow M$ in the following way:

$$Tf(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2}, \\ (2x-1)f(2x-1) & \frac{1}{2} \leq x \leq 1. \end{cases} \quad (2.2)$$

Claim 2.1. The map T is fixed point free.

Proof. Suppose that $f \in M$ is such that $Tf = f$. Clearly $f(x) = 0$ for every $x \in [0; 1/2]$. If $x \in [1/2; 1]$, then $(2x-1)f(2x-1) = f(x)$ implies that $f(x) = 0$ for every $x \in [0; 3/4]$. By iterating the reasoning, we can easily prove that $f(x) = 0$ for all $x \in [0; 1 - 1/2^n]$ and all $n \in \mathbb{N}$. Since f is continuous and $f(1) = 1$, this is a contradiction proving the claim. \square

Claim 2.2. The map T is shrinking.

Proof. Let be $f, g \in M$ with $f \neq g$. Then

$$\begin{aligned} \|Tf - Tg\| &= \max_{0 \leq x \leq 1} |Tf(x) - Tg(x)| + \int_0^1 |Tf(x) - Tg(x)| dx \\ &= \max_{1/2 \leq x \leq 1} (2x-1) |(f(2x-1) - g(2x-1))| \\ &\quad + \int_{1/2}^1 (2x-1) |f(2x-1) - g(2x-1)| dx \\ &= \max_{0 \leq x \leq 1} |x(f(x) - g(x))| + \frac{1}{2} \int_0^1 x |f(x) - g(x)| dx \\ &< \|f - g\|_\infty + \frac{1}{2} \|f - g\|_1 \leq \|f - g\|. \end{aligned} \quad (2.3) \quad \square$$

Claim 2.3. The map T is diametrically contractive.

Proof. Let A be a closed subset of M such that $\delta(A) > 0$. We have, for two suitable subsequences f_n, g_n ,

$$\begin{aligned} \delta(T(A)) &= \lim_{n \rightarrow \infty} \|Tf_n - Tg_n\| = \lim_{n \rightarrow \infty} \left(\|Tf_n - Tg_n\|_\infty + \|Tf_n - Tg_n\|_1 \right) \\ &\leq \lim_{n \rightarrow \infty} \left(\|f_n - g_n\|_\infty + \frac{1}{2} \|f_n - g_n\|_1 \right) \leq \lim_{n \rightarrow \infty} \|f_n - g_n\| \leq \delta(A). \end{aligned} \quad (2.4)$$

So, if we assume that $\delta(T(A)) = \delta(A)$, then (by passing again if necessary to a subsequence) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|f_n - g_n\|_1 &= \lim_{n \rightarrow \infty} \|Tf_n - Tg_n\|_1 = 0, \\ \lim_{n \rightarrow \infty} \|f_n - g_n\|_\infty &= \lim_{n \rightarrow \infty} \|Tf_n - Tg_n\|_\infty = \delta(T(A)) = \delta(A). \end{aligned} \quad (2.5)$$

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But we can choose a sequence (x_n) such that $\|Tf_n - Tg_n\|_\infty = x_n |f_n(x_n) - g_n(x_n)|$. By considering eventually a subsequence, we may assume that $x_n \rightarrow x_0 \in [0; 1]$. Then

$$\delta(A) = \lim_{n \rightarrow \infty} x_n |f_n(x_n) - g_n(x_n)| \leq \lim_{n \rightarrow \infty} x_n \|f_n - g_n\|_\infty = x_0 \delta(A), \quad (2.6)$$

thus $x_0 = 1$.

By considering subsequences, and by exchanging eventually the sequences, we may assume that

$$f_n(x_n) \rightarrow l, \quad g_n(x_n) \rightarrow L \quad (2.7)$$

with $L \leq l \leq 1$.

Therefore (2.6) implies that

$$l - L = \delta(A), \quad (2.8)$$

so

$$f_n(x_n) \rightarrow l, \quad g_n(x_n) \rightarrow l - \delta(A). \quad (2.9)$$

Now take any $f \in A$; since $\lim_{n \rightarrow \infty} x_n = 1$, we have

$$\delta(A) \geq |f(x_n) - g_n(x_n)| \xrightarrow{n \rightarrow \infty} |1 - l + \delta(A)| \geq \delta(A). \quad (2.10)$$

Thus we have $l = 1$; $\lim_{n \rightarrow \infty} |f(x_n) - g_n(x_n)| = \delta(A)$ for every $f \in A$, and then

$$\lim_{n \rightarrow \infty} \|f - g_n\|_\infty = \delta(A). \quad (2.11)$$

Now take $\epsilon \in (0, \delta(A))$, then there exists $\eta > 0$ such that for every $x \in [1 - \eta, 1]$, we have $1 - \epsilon \leq f(x) \leq 1$. For n large, $x_n > 1 - \eta$; therefore, by using also the monotonicity assumption for the functions, we have (for suitable points c_n)

$$\begin{aligned} \int_0^1 |f(x) - g_n(x)| dx &\geq \int_{1-\eta}^{x_n} |f(x) - g_n(x)| dx = (x_n - 1 + \eta) |f(c_n) - g_n(c_n)| \\ &\geq (x_n - 1 + \eta) (1 - \epsilon - g_n(x_n)); \end{aligned} \quad (2.12)$$

also, since $\lim_{n \rightarrow \infty} g_n(x_n) = 1 - \delta(A)$,

$$\lim_{n \rightarrow \infty} (x_n - 1 + \eta) (1 - \epsilon - g_n(x_n)) = \eta(\delta(A) - \epsilon). \quad (2.13)$$

Thus we obtain

$$\liminf_{n \rightarrow \infty} \|f - g_n\|_1 \geq \eta(\delta(A) - \epsilon) \quad (2.14)$$

and this implies that

$$\liminf_{n \rightarrow \infty} \|f - g_n\| \geq \lim_{n \rightarrow \infty} \|f - g_n\|_\infty + \liminf_{n \rightarrow \infty} \|f - g_n\|_1 \geq \delta(A) + \eta(\delta(A) - \epsilon). \quad (2.15)$$

This is a contradiction, proving the claim and thus the result. \square

3. Second example

The next example shows that for a DC self-map of a bounded closed convex set M , the existence of a fixed point does not imply the convergence of iterates $T^n x$ to the fixed point.

Consider the vector space c_0 , endowed with the following norm (equivalent to the usual one):

$$\|x\| = \|x\|_\infty + \sum_{n=1}^{\infty} \frac{|x_n|}{2^n}. \quad (3.1)$$

We denote by B^+ the intersection of the positive cone with the unit closed ball. Define $T : B^+ \rightarrow B^+$ in this way:

$$(Tx)_1 = 0, \quad \text{for } n \geq 2, \quad (Tx)_n = a_{n-1}x_{n-1}, \quad (3.2)$$

where (a_n) , $n \geq 1$, is a strictly positive and strictly increasing sequence such that $\prod_{n=1}^{\infty} a_n = \alpha > 0$. Clearly T is linear and its unique fixed point is the null vector.

The map T is shrinking: in fact, for $x \neq y$,

$$\begin{aligned} \|Tx - Ty\| &= \|(0, a_1(x_1 - y_1), a_2(x_2 - y_2), \dots)\| \\ &< \|(0, (x_1 - y_1), (x_2 - y_2), \dots)\| < \|x - y\|. \end{aligned} \quad (3.3)$$

Consider now the orbit of non-null elements in B^+ . Take x and let for example $x_k \neq 0$. We have

$$\|T^n x\| \geq |(T^n x)_{k+n}| = a_k a_{k+1} \cdots a_{k+n-1} x_k \xrightarrow{n \rightarrow \infty} \left(\prod_{n=k}^{\infty} a_n \right) x_k \neq 0. \quad (3.4)$$

Now we will prove that our map T is diametrically contractive.

Consider a bounded closed convex set A contained in B^+ . Let us suppose that

$$\delta(A) = \delta(T(A)) > 0. \quad (3.5)$$

Consider two sequences $x^{(n)}$ and $y^{(n)}$ such that

$$\lim_{n \rightarrow \infty} \|Tx^{(n)} - Ty^{(n)}\| = \delta(T(A)). \quad (3.6)$$

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Since T is shrinking, this implies that $\lim_{n \rightarrow \infty} \|x^{(n)} - y^{(n)}\| = \delta(A)$. We have

$$\begin{aligned}
 \delta(T(A)) &= \lim_{n \rightarrow \infty} \left(\|T(x^{(n)} - y^{(n)})\|_{\infty} + \sum_{k=1}^{\infty} \frac{|T((x^{(n)} - y^{(n)})_k)|}{2^k} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\max_{k \geq 2} |a_{k-1} (x_{k-1}^{(n)} - y_{k-1}^{(n)})| + \sum_{k=2}^{\infty} \frac{a_{k-1} |x_{k-1}^{(n)} - y_{k-1}^{(n)}|}{2^k} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\max_{k \geq 1} a_k |x_k^{(n)} - y_k^{(n)}| + \sum_{k=1}^{\infty} \frac{a_k |x_k^{(n)} - y_k^{(n)}|}{2^{k+1}} \right) \\
 &\leq \limsup_{n \rightarrow \infty} \left(\|x^{(n)} - y^{(n)}\|_{\infty} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{|x_k^{(n)} - y_k^{(n)}|}{2^k} \right) \leq \delta(A).
 \end{aligned} \tag{3.7}$$

From this, we obtain

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \|x^{(n)} - y^{(n)}\|_{\infty} &= \delta(A), \\
 \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{|x_k^{(n)} - y_k^{(n)}|}{2^k} &= 0.
 \end{aligned} \tag{3.8}$$

For every n , there exists $k(n)$ such that $\|x^{(n)} - y^{(n)}\|_{\infty} = |x_{k(n)}^{(n)} - y_{k(n)}^{(n)}|$, so

$$\lim_{n \rightarrow \infty} |x_{k(n)}^{(n)} - y_{k(n)}^{(n)}| = \delta(A). \tag{3.9}$$

Set $K = \{k(n); n \in \mathbb{N}\}$. If K is finite, then $k(n) = k_0$ for infinitely many n , so

$$\sum_{k=1}^{\infty} \frac{|x_k^{(n)} - y_k^{(n)}|}{2^k} \geq \frac{|x_{k_0}^{(n)} - y_{k_0}^{(n)}|}{2^{k_0}} \xrightarrow{n \rightarrow \infty} \frac{\delta(A)}{2^{k_0}} \neq 0, \tag{3.10}$$

which is an absurdity since we have proved that the left-hand side tends to 0. Thus K is infinite. Take a subsequence of $k(n)$ tending to infinity, that we still call $k(n)$, such that $x_{k(n)}^{(n)} \rightarrow \delta(A) + l$ and $y_{k(n)}^{(n)} \rightarrow l$ ($l \geq 0$).

Now let $x \in A$; we have

$$\delta(A) + l = \lim_{n \rightarrow \infty} |x_{k(n)} - x_{k(n)}^{(n)}| \leq \lim_{n \rightarrow \infty} \|x - x^{(n)}\|_{\infty} \leq \lim_{n \rightarrow \infty} \|x - x^{(n)}\| \leq \delta(A). \tag{3.11}$$

This implies that $l = 0$.

Therefore, for every $x \in A$, $\lim_{n \rightarrow \infty} \|x - x^{(n)}\| = \delta(A)$. So

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{|x_k - x_k^{(n)}|}{2^k} = 0 \tag{3.12}$$

which implies, for every k , that

$$\lim_{n \rightarrow \infty} x_k^{(n)} = x_k, \quad (3.13)$$

(remember that this should be true for every $x \in A$) so A cannot contain two or more elements. This would imply $\delta(A) = 0$, against the assumption. This contradiction proves the assertion.

4. Final remarks

After discussing Problems 1.2 and 1.3, another rather awkward condition, stronger than DC, was introduced in [10].

Given a set M , say that $T : M \rightarrow M$ is *asymptotically diametrically contractive* ADC if for all nested sequences (A_n) of closed bounded subsets of M with $\lim_{n \rightarrow \infty} \delta(A_n) = \delta > 0$, we have $\lim_{n \rightarrow \infty} \delta(T(A_n)) < \delta$.

We try to clarify its position among other simpler conditions.

Clearly, ADC maps are DC; as proved in [10, Theorem 2.6], the following result holds. If $T : M \rightarrow M$ is an ADC map and T has a bounded orbit for some $x_0 \in M$, then T has a unique fixed point ξ , and for every $x \in M : T^n(x) \rightarrow \xi$. In particular, this fact is true whenever M is bounded.

If M is compact, then (S) implies DC and DC implies ADC. But there are (S) maps on compact sets which are not contractive; thus ADC does not imply contractiveness, also when the map is defined on a compact set. An example of a map, on an unbounded set, which is ADC but not contractive, was given in [10, Remark 2.7].

An example of a map satisfying (S), but which is not DC, was given in [10]; according to the previous result, our first and second examples (Sections 2 and 3) show that DC maps are not in general ADC.

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