MANN ITERATION CONVERGES FASTER THAN ISHIKAWA ITERATION FOR THE CLASS OF ZAMFIRESCU OPERATORS

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The purpose of this paper is to show that the Mann iteration converges faster than the Ishikawa iteration for the class of Zamfirescu operators of an arbitrary closed convex subset of a Banach space.

1. Introduction

Let E be a normed linear space, $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0,1]$ a sequence of real numbers. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0,1,2,\ldots, \tag{1.1}$$

is called the *Mann iteration* or *Mann iterative procedure.*

Let $y_0 \in E$ be arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in $[0,1]$. The sequence $\{y_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{aligned} y_{n+1} &= (1-\alpha_n)y_n + \alpha_n T z_n, \quad n = 0,1,2,\ldots, \\ z_n &= (1-\beta_n)y_n + \beta_n T y_n, \quad n = 0,1,2,\ldots, \end{aligned} \tag{1.2}$$

is called the *Ishikawa iteration* or *Ishikawa iteration procedure.*

Zamfirescu proved the following theorem.

Theorem 1.1 [5]. *Let (X,d) be a complete metric space, and $T : X \to X$ a map for which there exist real numbers a, b, and c satisfying* $0 < a < 1$, $0 < b,c < 1/2$ *such that for each pair x, y in X, at least one of the following is true:*

(z_1) $d(Tx,Ty) \le ad(x,y)$;

(z_2) $d(Tx,Ty) \le b[d(x,Tx)+d(y,Ty)]$;

(z_3) $d(Tx,Ty) \le c[d(x,Ty)+d(y,Tx)]$.

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Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0,1,2,\ldots, \tag{1.3}$$

converges to p, for any $x_0 \in X$.

An operator T which satisfies the contraction conditions (z_1)–(z_3) of Theorem 1.1 will be called a *Zamfirescu operator* [2].

Definition 1.2 [3]. Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{1.4}$$

If $l = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ *converges faster* to a than $\{b_n\}_{n=0}^{\infty}$ to b.

Definition 1.3 [3]. Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p with the error estimates

$$\begin{aligned} ||u_n - p|| \le a_n, \quad n = 0,1,2,\ldots, \\ ||v_n - p|| \le b_n, \quad n = 0,1,2,\ldots, \end{aligned} \tag{1.5}$$

where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then we say that $\{u_n\}_{n=0}^{\infty}$ *converges faster* than $\{v_n\}_{n=0}^{\infty}$ to p.

We use Definition 1.3 to prove our main results.

Based on Definition 1.3, Berinde [3] compared the Picard and Mann iterations of the class of Zamfirescu operators defined on a closed convex subset of a uniformly convex Banach space and concluded that the Picard iteration always converges faster than the Mann iteration, and these were observed empirically on some numerical tests in [1]. In fact, the uniform convexity of the space is not necessary to prove this conclusion, and hence the following theorem [3, Theorem 4] is established in arbitrary Banach spaces.

Theorem 1.4 [3]. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ be a Zamfirescu operator. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.1) and $x_0 \in K$, with $\{\alpha_n\} \subset [0,1]$ satisfying*

(i) $\alpha_0 = 1$,

(ii) $0 \le \alpha_n < 1$ *for* $n \ge 1$,

(iii) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T and, moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.3) for $x_0 \in K$, converges faster than the Mann iteration.

Some numerical tests have been performed with the aid of the software package fixed point [1] and raised the following *open problem* in [3]: *for the class of Zamfirescu operators, does the Mann iteration converge faster than the Ishikawa iteration?*

The aim of this paper is to answer this open problem affirmatively, that is, to show that the Mann iteration converges faster than the Ishikawa iteration.

For this purpose we use the following theorem of Berinde.

Theorem 1.5 [2]. *Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ be a Zamfirescu operator. Let $\{y_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (1.2) for $y_0 \in K$, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$ with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (iii).*

Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

2. Main result

Theorem 2.1. *Let E be an arbitrary Banach space, K be an arbitrary closed convex subset of E, and $T : K \to K$ be a Zamfirescu operator. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.1) for $x_0 \in K$, and $\{y_n\}_{n=0}^{\infty}$ be defined by (1.2) for $y_0 \in K$ with $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ real sequences satisfying* (a) $0 \le \alpha_n, \beta_n \le 1$ *and* (b) $\Sigma\alpha_n = \infty$. *Then $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ converge strongly to the unique fixed point of T, and moreover, the Mann iteration converges faster than the Ishikawa iteration to the fixed point of T.*

Proof. By [2, Theorem 1] (established in [4]), for $x_0 \in K$, the Mann iteration defined by (1.1) converges strongly to the unique fixed point of T.

By Theorem 1.5, for $y_0 \in K$, the Ishikawa iteration defined by (1.2) converges strongly to the unique fixed point of T. By the uniqueness of fixed point for Zamfirescu operators, the Mann and Ishikawa iterations must converge to the same unique fixed point, p (say) of T.

Since T is a Zamfirescu operator, it satisfies the inequalities

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|x - Tx\|, \tag{2.1}$$

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|y - Tx\| \tag{2.2}$$

for all $x, y \in K$, where $\delta = \max\{a, b/(1-b), c/(1-c)\}$, and $0 \le \delta < 1$, see [3].

Suppose that $x_0 \in K$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration associated with T, and $\{\alpha_n\}_{n=0}^{\infty}$. Now by using Mann iteration (1.1), we have

$$||x_{n+1} - p|| \le (1-\alpha_n)||x_n - p|| + \alpha_n||Tx_n - p||. \tag{2.3}$$

On using (2.1) with $x = p$ and $y = x_n$, we get

$$||Tx_n - p|| \le \delta||x_n - p||. \tag{2.4}$$

Therefore from (2.3),

$$||x_{n+1} - p|| \le (1-\alpha_n)||x_n - p|| + \alpha_n\delta||x_n - p|| = [1 - \alpha_n(1-\delta)]||x_n - p|| \tag{2.5}$$

and thus

$$||x_{n+1} - p|| \le \prod_{k=1}^{n}[1 - \alpha_k(1-\delta)] \cdot ||x_1 - p||, \quad n = 0,1,2,\ldots. \tag{2.6}$$

Here we observe that

$$1 - \alpha_k(1-\delta) > 0 \quad \forall k = 0,1,2,\ldots. \tag{2.7}$$

Now let $\{y_n\}_{n=0}^{\infty}$ be the sequence defined by Ishikawa iteration (1.2) for $y_0 \in K$. Then we have

$$||y_{n+1} - p|| \leq (1-\alpha_n)||y_n - p|| + \alpha_n||Tz_n - p||. \tag{2.8}$$

On using (2.2) with $x = p$ and $y = z_n$, we have

$$||Tz_n - p|| \leq \delta||z_n - p|| + 2\delta||z_n - p|| = 3\delta||z_n - p||. \tag{2.9}$$

Again using (2.2) with $x = p$ and $y = y_n$, we have

$$||Ty_n - p|| \leq \delta||y_n - p|| + 2\delta||y_n - p|| = 3\delta||y_n - p||. \tag{2.10}$$

Now

$$||z_n - p|| \leq (1-\beta_n)||y_n - p|| + \beta_n||Ty_n - p|| \tag{2.11}$$

and hence by (2.8)–(2.11), we obtain

$$\begin{aligned}
||y_{n+1} - p|| &\leq (1-\alpha_n)||y_n - p|| + 3\delta\alpha_n||z_n - p|| \\
&\leq (1-\alpha_n)||y_n - p|| + 3\delta\alpha_n\left[(1-\beta_n)||y_n - p|| + \beta_n||Ty_n - p||\right] \\
&= (1-\alpha_n)||y_n - p|| + 3\delta\alpha_n(1-\beta_n)||y_n - p|| + 3\delta\alpha_n\beta_n||Ty_n - p|| \\
&\leq (1-\alpha_n)||y_n - p|| + 3\delta\alpha_n(1-\beta_n)||y_n - p|| + 3\delta\alpha_n\beta_n 3\delta||y_n - p|| \\
&= \left[(1-\alpha_n) + 3\delta\alpha_n(1-\beta_n) + 9\alpha_n\beta_n\delta^2\right]\cdot||y_n - p|| \\
&= \left[1 - \alpha_n(1 - 3\delta + 3\beta_n\delta - 9\beta_n\delta^2)\right]\cdot||y_n - p|| \\
&= \left[1 - \alpha_n(1-3\delta)(1+3\beta_n\delta)\right]\cdot||y_n - p||.
\end{aligned} \tag{2.12}$$

Here we observe that

$$1 - \alpha_n(1-3\delta)(1+3\beta_n\delta) > 0 \quad \forall k = 0,1,2,\ldots. \tag{2.13}$$

We have the following two cases.

Case (i). Let $\delta \in (0, 1/3]$. In this case

$$1 - \alpha_n(1-3\delta)(1+3\beta_n\delta) \leq 1, \quad \forall n = 0,1,2,\ldots, \tag{2.14}$$

and hence the inequality (2.12) becomes

$$||y_{n+1} - p|| \leq ||y_n - p|| \quad \forall n \tag{2.15}$$

and thus,

$$||y_{n+1} - p|| \leq ||y_1 - p|| \quad \forall n. \tag{2.16}$$

We now compare the coefficients of the inequalities (2.6) and (2.16), using Definition 1.3, with

$$a_n = \prod_{k=1}^{n} [1 - \alpha_k(1-\delta)], \qquad b_n = 1, \tag{2.17}$$

by (b) we have $\lim_{n\to\infty}(a_n/b_n) = 0$.

Case (ii). Let $\delta \in (1/3, 1)$. In this case

$$1 < 1 - \alpha_n(1-3\delta)(1+3\beta_n\delta) \leq 1 - \alpha_n(1-9\delta^2) \tag{2.18}$$

so that the inequality (2.12) becomes

$$||y_{n+1} - p|| \leq [1 - \alpha_n(1-9\delta^2)]||y_n - p|| \quad \forall n. \tag{2.19}$$

Therefore

$$||y_{n+1} - p|| \leq \prod_{k=1}^{n} [1 - \alpha_k(1-9\delta^2)]||y_1 - p||. \tag{2.20}$$

We compare (2.6) and (2.20), using Definition 1.3 with

$$a_n = \prod_{k=1}^{n} [1 - \alpha_k(1-\delta)], \qquad b_n = \prod_{k=1}^{n} [1 - \alpha_k(1-\delta^2)]. \tag{2.21}$$

Here $a_n \geq 0$ and $b_n \geq 0$ for all n; and $b_n \geq 1$ for all n.

Thus $a_n/b_n \leq a_n$ and since $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty}(a_n/b_n) = 0$.

Hence, from Cases (i) and (ii), it follows that $\{a_n\}$ converges faster than $\{b_n\}$, so that the Mann iteration $\{x_n\}$ converges faster than the Ishikawa iteration to the fixed point p of T. □

Corollary 2.2. *Under the hypotheses of Theorem 2.1, the Picard iteration defined by (1.3) converges faster than the Ishikawa iteration defined by (1.2), to the fixed point of Zamfirescu operator.*

Proof. It follows from Theorems 1.4 and 2.1. □

Remark 2.3. The Ishikawa iteration (1.2) is depending upon the parameters $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ whereas the Mann iteration (1.1) is only on $\{\alpha_n\}_{n=0}^{\infty}$; and by Theorem 2.1, Mann iteration converges faster than the Ishikawa iteration. Now, the Picard iteration (1.3) is free from parameters and Theorem 1.4 says that the Picard iteration converges faster than the Mann iteration.

Perhaps, the reason for this phenomenon is due to increasing the number of parameters in the iteration may increase the damage of the fastness of the convergence of the iteration to the fixed point for the class of Zamfirescu operators.

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