FIXED POINTS AS NASH EQUILIBRIA

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The existence of fixed points for single or multivalued mappings is obtained as a corollary of Nash equilibrium existence in finitely many players games.

1. Introduction

In game theory, the existence of equilibrium was uniformly obtained by the application of a fixed point theorem. In fact, Nash [3, 4] shows the existence of equilibria for noncooperative static games as a direct consequence of Brouwer [1] or Kakutani [2] theorems. More precisely, under some regularity conditions, given a game, there always exists a correspondence whose fixed points coincide with the equilibrium points of the game.

However, it is natural to ask whether fixed points arguments are in fact necessary tools to guarantee the Nash equilibrium existence. (In this direction, Zhao [5] shows the equivalence between Nash equilibrium existence theorem and Kakutani (or Brouwer) fixed point theorem in an *indirect way*. However, as he points out, a constructive proof is preferable. In fact, any pair of logical sentences A and B that are true will be equivalent (in an indirect way). For instance, to show that A implies B it is sufficient to repeat the proof of B.) For this reason, we study conditions to assure that fixed points of a continuous function, or of a closed-graph correspondence, can be attained as Nash equilibria of a noncooperative game.

2. Definitions

Let $Y \subset \mathbb{R}^n$ be a convex set. A function $v : Y \to \mathbb{R}$ is *quasiconcave* if, for each $\lambda \in (0,1)$, we have $v(\lambda y_1 + (1-\lambda)y_2) \geq \min\{v(y_1), v(y_2)\}$, for all $(y_1, y_2) \in Y \times Y$. Moreover, if for each pair $(y_1, y_2) \in Y \times Y$ such that $y_1 \neq y_2$ the inequality above is strict, independently of the value of $\lambda \in (0,1)$, we say that v is *strictly quasiconcave*.

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A mapping $f : X \subset \mathbb{R}^m \to X$ has a *fixed point* if there is $\overline{x} \in X$ such that $f(\overline{x}) = \overline{x}$. A vector $\overline{x} \in X$ is a fixed point of a correspondence $\Phi : X \twoheadrightarrow X$ if $\overline{x} \in \Phi(\overline{x})$.

Given a game $\mathcal{G} = \{I, S_i, v^i\}$, in which each player $i \in I = \{1, 2, \dots, n\}$ is characterized by a set of strategies $S_i \subset \mathbb{R}^{n_i}$, and by an objective function $v^i : \prod_{j=1}^n S_j \to \mathbb{R}$, a *Nash equilibrium* is a vector $\overline{s} = (\overline{s}_1, \overline{s}_2, \dots, \overline{s}_n) \in \Pi_{i=1}^n S_i$, such that $v^i(\overline{s}) \geq v^i(s_i, \overline{s}_{-i})$, for all $s_i \in S_i$, for all $i \in I$, where $\overline{s}_{-i} = (\overline{s}_1, \dots, \overline{s}_{i-1}, \overline{s}_{i+1}, \dots, \overline{s}_n)$.

Finally, let $\mathcal{H} = \{S : \exists n \in \mathbb{N},\ S \subset \mathbb{R}^n \text{ is nonempty, convex, and compact}\}$.

3. Main Results

Consider the following statements.

[Nash-1]. Given $\mathcal{G} = \{I, S_i, v^i\}$, suppose that each set $S_i \in \mathcal{H}$ and that objective functions are continuous in its domains and *strictly quasiconcave* in its own strategy. Then there is a Nash equilibrium for $\mathcal{G}$.

[Nash-2]. Given $\mathcal{G} = \{I, S_i, v^i\}$, suppose that each set $S_i \in \mathcal{H}$ and that objective functions are continuous in its domains and *quasiconcave* in its own strategy. Then there is a Nash equilibrium for $\mathcal{G}$.

[Brouwer]. Given $X \in \mathcal{H}$, every continuous function $f : X \to X$ has a fixed point.

[Kakutani].* Given $X \in \mathcal{H}$, every closed-graph correspondence $\Phi : X \twoheadrightarrow X$, with $\Phi(x) \in \mathcal{H}$ for all $x \in X$, has a fixed point, provided that $\Phi(x) = \prod_{j=1}^m \pi_j^m(\Phi(x))$ for each $x \in X \subset \mathbb{R}^m$. (For each $j \in \{1, \dots, m\}$, the projections $\pi_j^m : \mathbb{R}^m \to \mathbb{R}$ are defined by $\pi_j^m(x) = x_j$, where $x = (x_1, \dots, x_m) \in \mathbb{R}^m$.) (The last property, $\Phi(x) = \prod_{j=1}^m \pi_j^m(\Phi(x))$, is not necessary to assure the existence of a fixed point, provided that the other assumptions hold. However, when objective functions are quasiconcave, [Kakutani*] is sufficient to assure the existence of a Nash equilibrium.)

Our results are [Nash-1] → [Brouwer].

Proof. Given a nonempty, convex, and compact set $X \subset \mathbb{R}^m$ and a continuous function $f : X \to X$, consider a game $\mathcal{G}$ with two players $I = \{A, B\}$, which are characterized by the sets of strategies $S_A = S_B = X$ and by the objective functions: $v^A(x_A, x_B) = -\|x_A - x_B\|^2$ and $v^B(x_A, x_B) = -\|f(x_A) - x_B\|^2$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^m$.

As objective functions are continuous in $S_A \times S_B$ and strictly quasiconcave in own strategy, there exists a Nash equilibrium $(\overline{x}_A, \overline{x}_B)$. Moreover, $\overline{x}_A = \overline{x}_B$ and $\overline{x}_B = f(\overline{x}_A)$, that assure the existence of a fixed point of $f : X \to X$.

In fact, if $\overline{x}^A \neq \overline{x}^B$, then $v^A(\overline{x}^A, \overline{x}^B) < 0$. Thus, player A can improve his gains choosing a response $x^A = \overline{x}^B \in X$, as $v^A(\overline{x}^B, \overline{x}^B) = 0$, a contradiction. Analogous arguments prove that $\overline{x}^B = f(\overline{x}^A)$ because $f(\overline{x}^A) \in X$.

We have [Nash-2] → [Kakutani*]. □

Proof. Fix a set $X \subset \mathcal{H}$ and a correspondence $\Phi : X \twoheadrightarrow X$ that satisfies the assumptions of [Kakutani*]. Define, for each $1 \leq i \leq m$, the functions $\kappa_i^m : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}$ by $\kappa_i^m(x, y) = (x, y_i)$, where $y = (y_1, \dots, y_m) \in \mathbb{R}^m$.

Consider a game $\mathcal{G}$ with $m+1$ players, $\{0,1,\ldots,m\}$, characterized by the sets of strategies $(S_0,(S_i;\ i>0)) := (X,(\pi_i^m(X);\ i>0))$ and by the objective functions: $v^0(x_0,x_{-0}) = -\|x_0 - x_{-0}\|^2$, $v^i(x_0,x_1,\ldots,x_m) = -\min_{(r,s_i)\in\kappa_i^m(Gr[\Phi])}\|(x_0,x_i)-(r,s_i)\|_{\max}$, where $Gr[\Phi]$ denotes the graph of Φ, $\|\cdot\|_{\max}$ is the max-norm, and $x_{-0} := (x_1,\ldots,x_m)$.

As hypotheses of [Nash-2] hold (see the appendix), there is an equilibrium $(\overline{x}_0; (\overline{x}_1,\ldots,\overline{x}_m))$ for $\mathcal{G}$. It follows that $\overline{x}_i \in \pi_i^m(\Phi(\overline{x}_0))$(If $\overline{x}_i \notin \pi_i^m(\Phi(\overline{x}_0))$ then, by definition, $(\overline{x}_0,\overline{x}_i) \notin \kappa_i^m(Gr[\Phi])$. Thus, player i's utility, $v^i(\overline{x}_0,\overline{x}_1,\ldots,\overline{x}_m) < 0$. However, choosing any $x_i \in \pi_i^m(\Phi(\overline{x}_0)) \neq \emptyset$, the player i can improve his gains, as $(\overline{x}_0,x_i) \in \kappa_i^m(Gr[\Phi])$ and, therefore, his utility will be equal to zero, a contradiction.) Therefore, by Assumption [Kakutani*], $(\overline{x}_1,\ldots,\overline{x}_m) \in \Phi(\overline{x}_0)$. Finally, $\overline{x}_0 = (\overline{x}_1,\ldots,\overline{x}_m) \in \Phi(\overline{x}_0)$(If $\overline{x}_0 \neq \overline{x}_{-0} := (\overline{x}_1,\ldots,\overline{x}_n)$, we have that $v^0(\overline{x}_0,\overline{x}_{-0}) := -\|\overline{x}_0 - \overline{x}_{-0}\|^2 < 0$. Thus, player 0 can improve his position choosing $x_0 = \overline{x}_{-0} \in X$.) That concludes the proof. □

Appendix

It follows from definitions above that the sets of strategies satisfy the assumptions of [Nash-2], objective functions are continuous and v^0 is quasiconcave in it own strategy. Thus, rest to assure that functions v^i, with $i \geq 1$, are quasiconcave in its own strategy. This will be a direct consequence of the following lemma, taking $Z = \kappa_i^m(Gr[\Phi])$.

Given $x \in \mathbb{R}^{m+1}$ and a nonempty set $Z \subset \mathbb{R}^{m+1}$, the distance from x to Z is $d(x,Z) = \inf_{z\in Z}\|x-z\|_{\max}$. Note that the function $x \mapsto d(x,Z)$ is continuous, because $|d(x_1,Z) - d(x_2,Z)| \leq \|x_1 - x_2\|_{\max}$. For convenience of notations, let $\pi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ be the projection defined by $\pi(x,y) = x$.

Lemma A.1. *Suppose that $Z \subset \mathbb{R}^{m+1}$ is a nonempty and compact set such that, for each $x \in \mathbb{R}^m$, both $\pi(Z)$ and $Z \cap \pi^{-1}(x)$ are convex sets. Then the function $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,t) = d((x,t),Z)$ is quasiconvex in t.*

Proof. Fix a vector $x_0 \in \mathbb{R}^m$. We have to show that $L_c = \{t \in \mathbb{R};\ d((x_0,t),Z) \leq c\}$ is a convex set for every $c \geq 0$. Assume by contradiction that, given $c \geq 0$, there are scalars $t_1 < t_* < t_2$ such that $t_1,t_2 \in L_c$ and $t_* \notin L_c$. Let $A := \{x \in \pi(Z);\ \|x - x_0\|_{\max} \leq c\}$ and consider the following sets:

$$\begin{aligned} A_1 &= \{x \in A;\ \exists t \in \mathbb{R} \text{ s.t. } (x,t) \in Z,\ t \leq t_* - c\}; \\ A_2 &= \{x \in A;\ \exists t \in \mathbb{R} \text{ s.t. } (x,t) \in Z,\ t \geq t_* + c\}. \end{aligned} \tag{A.1}$$

Since $d((x_0,t_*),Z) > c$, we have $A = A_1 \cup A_2$. Moreover, $A_1 \cap A_2 = \emptyset$. (If there exists a vector $x \in A_1 \cap A_2$, then $\|x - x_0\|_{\max} \leq c$ and, by the convexity of $Z \cap \pi^{-1}(x)$, $(x,t_*) \in Z$, contradicting $d((x_0,t_*),Z) > c$.) Since Z is compact, A_1 and A_2 are compact as well. And since $\pi(Z)$ is convex, so is A. In particular, A is connected. Therefore, $A_1 = \emptyset$ or $A_2 = \emptyset$.

On the other hand, $d((x_0,t_1),Z) \leq c$, so there exists a point $(x',t') \in Z$ c-close to (x_0,t_1). Then $\|x' - x_0\|_{\max} \leq c$ and $t' \leq t_1 + c < t_* + c$, therefore $x' \in A \setminus A_2$, proving that $A_1 \neq \emptyset$. Analogously, it follows from $d((x_0,t_2),Z) \leq c$ that $A_2 \neq \emptyset$. We have obtained a contradiction. □

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