C_q-COMMUTING MAPS AND INVARIANT APPROXIMATIONS

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We obtain common fixed point results for generalized I-nonexpansive C_q-commuting maps. As applications, various best approximation results for this class of maps are derived in the setup of certain metrizable topological vector spaces.

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1. Introduction and preliminaries

Let X be a linear space. A p-norm on X is a real-valued function on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$,

(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$,

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$,

for all $x, y \in X$ and all scalars α. The pair $(X, \|\cdot\|_p)$ is called a p-normed space. It is a metric linear space with a translation invariant metric d_p defined by $d_p(x,y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of the usual normed space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$ (see [7, 13] and references therein). The spaces l_p and L_p, $0 < p \leq 1$, are p-normed spaces. A p-normed space is not necessarily a locally convex space. Recall that dual space X^* (the dual of X) separates points of X if for each nonzero $x \in X$, there exists $f \in X^*$ such that $f(x) \neq 0$. In this case the weak topology on X is well defined and is Hausdorff. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0,1]$, $0 < p < 1$, then $X^* = \{0\}$ [17, pages 36–37]. However, there are some nonlocally convex spaces X (such as the p-normed spaces l_p, $0 < p < 1$) whose dual X^* separates the points of X. In the sequel, we will assume that X^* separates points of a p-normed space X whenever weak topology is under consideration.

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Let X be a metric linear space and M a nonempty subset of X. The set $P_M(u) = \{x \in M : d(x,u) = \operatorname{dist}(u,M)\}$ is called the set of best approximations to $u \in X$ out of M, where $\operatorname{dist}(u,M) = \inf\{d(y,u) : y \in M\}$. Let $f : M \to M$ be a mapping. A mapping $T : M \to M$ is called an f-contraction if there exists $0 \leq k < 1$ such that $d(Tx,Ty) \leq k\, d(fx,fy)$ for any $x,y \in M$. If $k = 1$, then T is called f-nonexpansive. The set of fixed points of T (resp., f) is denoted by $F(T)$ (resp., $F(f)$). A point $x \in M$ is a common fixed (coincidence) point of f and T if $x = fx = Tx$ ($fx = Tx$). The set of coincidence points of f and T is denoted by $C(f,T)$. A mapping $T : M \to M$ is called

(1) hemicompact if any sequence $\{x_n\}$ in M has a convergent subsequence whenever $d(x_n,Tx_n) \to 0$ as $n \to \infty$;

(2) completely continuous if $\{x_n\}$ converges weakly to x which implies that $\{Tx_n\}$ converges strongly to Tx;

(3) demiclosed at 0 if for every sequence $\{x_n\} \in M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$.

The pair $\{f,T\}$ is called

(4) commuting if $Tfx = fTx$ for all $x \in M$;

(5) R-weakly commuting if for all $x \in M$ there exists $R > 0$ such that $d(fTx,Tfx) \leq R\, d(fx,Tx)$. If $R = 1$, then the maps are called weakly commuting;

(6) compatible [10] if $\lim_n d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M;

(7) weakly compatible [2, 11] if they commute at their coincidence points, that is, if $fTx = Tfx$ whenever $fx = Tx$. The set M is called q-starshaped with $q \in M$ if the segment $[q,x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called

(8) R-subcommuting on M (see [19, 20]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx,Tfx) \leq (R/k)\, d((1-k)q + kTx, fx)$ for each $k \in (0,1]$;

(9) R-subweakly commuting on M (see [7, 21]) if for all $x \in M$, there exists a real number $R > 0$ such that $d(fTx,Tfx) \leq R \operatorname{dist}(fx,[q,Tx])$;

(10) C_q-commuting [2] if $fTx = Tfx$ for all $x \in C_q(f,T)$, where $C_q(f,T) = \cup\{C(f,T_k) : 0 \leq k \leq 1\}$ and $T_k x = (1-k)q + kTx$. Clearly, C_q-commuting maps are weakly compatible but not conversely in general. R-subcommuting and R-subweakly commuting maps are C_q-commuting but the converse does not hold in general [2].

Meinardus [14] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. Singh [22] proved the following extension of "Meinardus's" result.

Theorem 1.1. *Let T be a nonexpansive operator on a normed space X, M a T-invariant subset of X, and $u \in F(T)$. If $P_M(u)$ is nonempty compact and starshaped, then $P_M(u) \cap F(T) \neq \emptyset$.*

Sahab et al. [18] established an invariant approximation result which contains Theorem 1.1. Further generalizations of the result of Meinardus are obtained by Al-Thagafi [1], Shahzad [19–21], Hussain and Berinde [7], Rhoades and Saliga [16], and O'Regan and Shahzad [15].

The aim of this paper is to establish a general common fixed point theorem for C_q-commuting generalized I-nonexpansive maps in the setting of locally bounded topological vector spaces, locally convex topological vector spaces, and metric linear spaces. We apply a new theorem to derive some results on the existence of best approximations. Our results unify and extend the results of Al-Thagafi [1], Al-Thagafi and Shahzad [2], Dotson [3], Guseman and Peters [4], Habiniak [5], Hussain [6], Hussain and Berinde [7], Hussain and Khan [8], Hussain et al. [9], Jungck and Sessa [12], Khan and Khan [13], O'Regan and Shahzad [15], Rhoades and Saliga [16], Sahab et al. [18], Shahzad [19–21], and Singh [22].

2. Common fixed point and approximation results

The following result extends and improves [2, Theorem 2.1], [21, Theorem 2.1], and [15, Lemma 2.1].

Theorem 2.1. *Let M be a subset of a metric space (X,d), and let I and T be weakly compatible self-maps of M. Assume that* $\mathrm{cl}(T(M)) \subset I(M)$, $\mathrm{cl}(T(M))$ *is complete, and T and I satisfy for all $x, y \in M$ and $0 \le h < 1$,*

$$d(Tx,Ty) \le h\max\{d(Ix,Iy),d(Ix,Tx),d(Iy,Ty),d(Ix,Ty),d(Iy,Tx)\}. \tag{2.1}$$

Then $F(I) \cap F(T)$ is a singleton.

Proof. As $T(M) \subset I(M)$, one can choose x_n in M for $n \in N$, such that $Tx_n = Ix_{n+1}$. Then following the arguments in [15, Lemma 2.1], we infer that $\{Tx_n\}$ is a Cauchy sequence. It follows from the completeness of $\mathrm{cl}(T(M))$ that $Tx_n \to w$ for some $w \in M$ and hence $Ix_n \to w$ as $n \to \infty$. Consequently, $\lim_n Ix_n = \lim_n Tx_n = w \in \mathrm{cl}(T(M)) \subset I(M)$. Thus $w = Iy$ for some $y \in M$. Notice that for all $n \ge 1$, we have

$$\begin{aligned} d(w,Ty) &\le d(w,Tx_n) + d(Tx_n,Ty) \le d(w,Tx_n) \\ &\quad + h\max\{d(Ix_n,Iy),d(Tx_n,Ix_n),d(Ty,Iy),d(Ty,Ix_n),d(Tx_n,Iy)\}. \end{aligned} \tag{2.2}$$

Letting $n \to \infty$, we obtain $Iy = w = Ty$. We now show that Ty is a common fixed point of I and T. Since I and T are weakly compatible and $Iy = Ty$, we obtain by the definition of weak compatibility that $ITy = TIy$. Thus we have $T^2y = TIy = ITy$ and so by inequality (2.1),

$$\begin{aligned} d(TTy,Ty) &\le h\max\{d(ITy,Iy),d(ITy,TTy),d(Iy,Ty),d(ITy,Ty),d(Iy,TTy)\} \\ &\le hd(ITy,Ty). \end{aligned} \tag{2.3}$$

Hence $TTy = Ty$ as $h \in (0,1)$ and so $Ty = TTy = ITy$. This implies that Ty is a common fixed point of T and I. Inequality (2.1) further implies the uniqueness of the common fixed point Ty. Hence $F(I) \cap F(T)$ is a singleton. □

We can prove now the following.

Theorem 2.2. *Let I and T be self-maps on a q-starshaped subset M of a p-normed space X. Assume that $\mathrm{cl}(T(M)) \subset I(M)$, $q \in F(I)$, and I is affine. Suppose that T and I are C_q-commuting and satisfy*

$$\|Tx - Ty\|_p \le \max\left\{\begin{matrix} \|Ix - Iy\|_p,\ \mathrm{dist}\,(Ix,[Tx,q]),\ \mathrm{dist}\,(Iy,[Ty,q]), \\ \mathrm{dist}\,(Ix,[Ty,q]),\ \mathrm{dist}\,(Iy,[Tx,q]) \end{matrix}\right\} \tag{2.4}$$

for all $x, y \in M$. If T is continuous, then $F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds:

(i) $\mathrm{cl}(T(M))$ *is compact and I is continuous;*

(ii) *M is complete, $F(I)$ is bounded, and T is a compact map;*

(iii) *M is bounded, and complete, T is hemicompact and I is continuous;*

(iv) *X is complete, M is weakly compact, I is weakly continuous, and $I - T$ is demiclosed at* 0*;*

(v) *X is complete, M is weakly compact, T is completely continuous, and I is continuous.*

Proof. Define $T_n : M \to M$ by

$$T_n x = (1 - k_n)q + k_n Tx \tag{2.5}$$

for some q and all $x \in M$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1. Then, for each n, $\mathrm{cl}(T_n(M)) \subset I(M)$ as M is q-starshaped, $\mathrm{cl}(T(M)) \subset I(M)$, I is affine, and $Iq = q$. As I and T are C_q-commuting and I is affine with $Iq = q$, then for each $x \in C_q(I,T)$,

$$IT_n x = (1 - k_n)q + k_n ITx = (1 - k_n)q + k_n TIx = T_n Ix. \tag{2.6}$$

Thus $IT_n x = T_n Ix$ for each $x \in C(I,T_n) \subset C_q(I,T)$. Hence I and T_n are weakly compatible for all n. Also by (2.4),

$$\begin{aligned} \|T_n x - T_n y\|_p &= (k_n)^p \|Tx - Ty\|_p \\ &\le (k_n)^p \max\{\|Ix - Iy\|_p,\ \mathrm{dist}\,(Ix,[Tx,q]),\ \mathrm{dist}\,(Iy,[Ty,q]), \\ &\qquad \mathrm{dist}\,(Ix,[Ty,q]),\ \mathrm{dist}\,(Iy,[Tx,q])\} \\ &\le (k_n)^p \max\Big\{\|Ix - Iy\|_p,\ \|Ix - T_n x\|_p,\ \|Iy - T_n y\|_p, \\ &\qquad \|Ix - T_n y\|_p,\ \|Iy - T_n x\|_p\Big\}, \end{aligned} \tag{2.7}$$

for each $x, y \in M$.

(i) Since $\mathrm{cl}(T(M))$ is compact, $\mathrm{cl}(T_n(M))$ is also compact. By Theorem 2.1, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. The compactness of $\mathrm{cl}(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Then the definition of $T_m x_m$ implies $x_m \to y$, so by the continuity of T and I, we have $y \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(ii) As in (i), there is a unique $x_n \in M$ such that $x_n = T_n x_n = Ix_n$. As T is compact and $\{x_n\}$ being in $F(I)$ is bounded, so $\{Tx_n\}$ has a subsequence $\{Tx_m\}$ such that $\{Tx_m\} \to y$ as $m \to \infty$. Then the definition of $T_m x_m$ implies $x_m \to y$, so by the continuity of T and I, we have $y \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(iii) As in (i), there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$, and M is bounded, so $x_n - Tx_n = (1 - (k_n)^{-1})(x_n - q) \to 0$ as $n \to \infty$ and hence $d_p(x_n, Tx_n) \to 0$ as $n \to \infty$. The hemicompactness of T implies that $\{x_n\}$ has a subsequence $\{x_j\}$ which converges to some $z \in M$. By the continuity of T and I we have $z \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \emptyset$.

(iv) As in (i), there exists $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Since M is weakly compact, we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$ and as I is weakly continuous so $Iy = y$. By (iii) $Ix_m - Tx_m \to 0$ as $m \to \infty$. The demiclosedness of $I - T$ at 0 implies that $Iy = Ty$. Thus $F(T) \cap F(I) \neq \emptyset$.

(v) As in (iv), we can find a subsequence $\{x_m\}$ of $\{x_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$. Since T is completely continuous, $Tx_m \to Ty$ as $m \to \infty$. Since $k_n \to 1$, $x_m = T_m x_m = k_m T x_m + (1 - k_m) q \to Ty$ as $m \to \infty$. Thus $Tx_m \to T^2 y$ as $m \to \infty$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Ty$. Also, since $Ix_m = x_m \to Ty = w$, using the continuity of I and the uniqueness of the limit, we have $Iw = w$. Hence $F(T) \cap F(I) \neq \emptyset$. □

The following corollary improves and generalizes [2, Theorem 2.2] and [7, Theorem 2.2].

Corollary 2.3. *Let M be a q-starshaped subset of a p-normed space X, and I and T continuous self-maps of M. Suppose that I is affine with $q \in F(I)$,* $\mathrm{cl}(T(M)) \subset I(M)$, *and* $\mathrm{cl}(T(M))$ *is compact. If the pair $\{I, T\}$ is R-subweakly commuting and satisfies (2.4) for all $x, y \in M$, then $F(T) \cap F(I) \neq \emptyset$.*

Remark 2.4. Theorem 2.2 extends and improves Al-Thagafi's [1, Theorem 2.2], Dotson's [3, Theorem 1], Habiniak's [5, Theorem 4], Hussain and Berinde's [7, Theorem 2.2], O'Regan and Shahzad's [15, Theorem 2.2], Shahzad's [21, Theorem 2.2], and the main result of Rhoades and Saliga [16].

The following provides the conclusion of [13, Theorem 2] without the closedness of M.

Corollary 2.5. *Let M be a nonempty q-starshaped subset of a p-normed space X. If T is nonexpansive self-map of M and* $\mathrm{cl}(T(M))$ *is compact, then $F(T) \neq \emptyset$.*

The following result contains properly Theorem 1.1, [18, Theorem 3], and improves and extends [2, Theorem 3.1], [5, Theorem 8], [13, Theorem 4], and [19, Theorem 6].

THEOREM 2.6. *Let M be a subset of a p-normed space X and let $I,T : X \to X$ be mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Assume that $I(P_M(u)) = P_M(u)$ and the pair $\{I,T\}$ is C_q-commuting and continuous on $P_M(u)$ and satisfies for all $x \in P_M(u) \cup \{u\}$,*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p,\ \operatorname{dist}(Ix,[q,Tx]),\ \operatorname{dist}(Iy,[q,Ty]), & \\ \qquad \operatorname{dist}(Ix,[q,Ty]),\ \operatorname{dist}(Iy,[q,Tx])\} & \text{if } y \in P_M(u). \end{cases} \tag{2.8}$$

Suppose that $P_M(u)$ is closed, q-starshaped with $q \in F(I)$, I is affine, and $\operatorname{cl}(T(P_M(u)))$ is compact. Then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in P_M(u)$. Then $\|x - u\|_p = \operatorname{dist}(u,M)$. Note that for any $k \in (0,1)$, $\|ku + (1-k)x - u\|_p = (1-k)^p \|x - u\|_p < \operatorname{dist}(u,M)$.

It follows that the line segment $\{ku + (1-k)x : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Also since $Ix \in P_M(u)$, $u \in F(T) \cap F(I)$ and T, and I satisfy (2.8), we have

$$\|Tx - u\|_p = \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = \|Ix - u\|_p = \operatorname{dist}(u,M). \tag{2.9}$$

Thus $Tx \in P_M(u)$. Theorem 2.2(i) further guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$. □

Let $D = P_M(u) \cap C_M^I(u)$, where $C_M^I(u) = \{x \in M : Ix \in P_M(u)\}$.

The following result contains [1, Theorem 3.2], extends [2, Theorem 3.2], and provides a nonlocally convex space analogue of [8, Theorem 3.3] for more general class of maps.

THEOREM 2.7. *Let M be a subset of a p-normed space X, and I and $T : X \to X$ mappings such that $u \in F(T) \cap F(I)$ for some $u \in X$ and $T(\partial M \cap M) \subset M$. Suppose that D is closed q-starshaped with $q \in F(I)$, I is affine, $\operatorname{cl}(T(D))$ is compact, $I(D) = D$, and the pair $\{T,I\}$ is C_q-commuting and continuous on D and, for all $x \in D \cup \{u\}$, satisfies the following inequality:*

$$\|Tx - Ty\|_p \leq \begin{cases} \|Ix - Iu\|_p & \text{if } y = u, \\ \max\{\|Ix - Iy\|_p,\ \operatorname{dist}(Ix,[q,Tx]),\ \operatorname{dist}(Iy,[q,Ty]), & \\ \qquad \operatorname{dist}(Ix,[q,Ty]),\ \operatorname{dist}(Iy,[q,Tx])\} & \text{if } y \in D. \end{cases} \tag{2.10}$$

If I is nonexpansive on $P_M(u) \cup \{u\}$, then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$.

Proof. Let $x \in D$, then proceeding as in the proof of Theorem 2.6, we obtain $Tx \in P_M(u)$. Moreover, since I is nonexpansive on $P_M(u) \cup \{u\}$ and T satisfies (2.10), we obtain

$$\|ITx - u\|_p \leq \|Tx - Tu\|_p \leq \|Ix - Iu\|_p = \operatorname{dist}(u,M). \tag{2.11}$$

Thus $ITx \in P_M(u)$ and so $Tx \in C_M^I(u)$. Hence $Tx \in D$. Consequently, $\operatorname{cl}(T(D)) \subset D = I(D)$. Now Theorem 2.2(i) guarantees that $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$. □

Remark 2.8. Notice that approximation results similar to Theorems 2.6–2.7 can be obtained, using Theorem 2.2(ii)–(v).

3. Further remarks

(1) All results of the paper (Theorem 2.2–Remark 2.8) remain valid in the setup of a metrizable locally convex topological vector space (TVS) (X,d), where d is translation invariant and $d(\alpha x,\alpha y) \le \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x,y \in X$ (recall that d_p is translation invariant and satisfies $d_p(\alpha x,\alpha y) \le \alpha^p d_p(x,y)$ for any scalar $\alpha \ge 0$).

Consequently, Hussain and Khan's [8, Theorems 2.2–3.3] are improved and extended.

(2) Following the arguments as above, we can obtain all of the recent best approximation results due to Hussain and Berinde's [7, Theorem 3.2–Corollary 3.4] for more general class of C_q-commuting maps I and T.

(3) A subset M of a linear space X is said to have property (N) with respect to T [7, 9] if

(i) $T : M \to M$,

(ii) $(1-k_n)q + k_nTx \in M$, for some $q \in M$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1 and for each $x \in M$.

A mapping I is said to have property (C) on a set M with property (N) if $I((1-k_n)q + k_nTx) = (1-k_n)Iq + k_nITx$ for each $x \in M$ and $n \in N$.

All of the results of the paper (Theorem 2.2–Remark 2.8) remain valid, provided I is assumed to be surjective and the q-starshapedness of the set M and affineness of I are replaced by the property (N) and property (C), respectively, in the setup of p-normed spaces and metrizable locally convex topological vector spaces (TVS) (X,d) where d is translation invariant and $d(\alpha x,\alpha y) \le \alpha d(x,y)$, for each α with $0 < \alpha < 1$ and $x,y \in X$. Consequently, recent results due to Hussain [6], Hussain and Berinde [7], and Hussain et al. [9] are extended to a more general class of C_q-commuting maps.

(4) Let (X,d) be a metric linear space with a translation invariant metric d. We say that the metric d is strictly monotone [4] if $x \ne 0$ and $0 < t < 1$ imply $d(0,tx) < d(0,x)$. Each p-norm generates a translation invariant metric, which is strictly monotone [4, 7].

Using [10, Theorem 3.2], we establish the following generalization of Al-Thagafi and Shahzad's [2, Theorem 2.2], Dotson's [3, Theorem 1], Guseman and Peters's [4, Theorem 2], and Hussain and Berinde's [7, Theorem 3.6].

Theorem 3.1. *Let T and I be self-maps on a compact subset M of a metric linear space (X,d) with translation invariant and strictly monotone metric d. Assume that M is q-starshaped, $\mathrm{cl}(T(M)) \subset I(M)$, $q \in F(I)$, and I is affine (or M has the property (N) with $q \in F(I)$, I satisfies the condition (C), and $M = I(M)$). Suppose that T and I are continuous, C_q-commuting and satisfy*

$$d(Tx,Ty) \le \max\left\{\begin{matrix} d(Ix,Iy),\ \operatorname{dist}(Ix,[Tx,q]),\ \operatorname{dist}(Iy,[Ty,q]), \\ \frac{1}{2}[\operatorname{dist}(Ix,[Ty,q]) + \operatorname{dist}(Iy,[Tx,q])] \end{matrix}\right\} \tag{3.1}$$

for all $x,y \in M$. Then $F(T) \cap F(I) \ne \emptyset$.

Proof. Two continuous maps defined on a compact domain are compatible if and only if they are weakly compatible (cf. [10, Corollary 2.3]). To obtain the result, use an argument similar to that in Theorem 2.2(i) and apply [10, Theorem 3.2] instead of Theorem 2.1. □

(5) Similarly, all other results of Section 2 (Corollary 2.3–Theorem 2.7) hold in the setting of metric linear space (X,d) with translation invariant and strictly monotone metric d provided we replace compactness of $\mathrm{cl}(T(M))$ by compactness of M and using Theorem 3.1 instead of Theorem 2.2(i).

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