FIXED POINTS, STABILITY, AND HARMLESS PERTURBATIONS

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Much has been written about systems in which each constant is a solution and each solution approaches a constant. It is a small step to conjecture that functions promoting such behavior constitute harmless perturbations of stable equations. That idea leads to a new way of avoiding delay terms in a functional-differential equation. In this paper we use fixed point theory to show that such a conjecture is valid for a set of classical equations.

1. Introduction

There is a large literature concerning equations typified by

$$x'(t) = g(x(t)) - g(x(t-L)) \tag{1.1}$$

(as well as distributed delays) where g is an arbitrary Lipschitz function and L is a positive constant. Under suitable conditions, three dominant properties emerge.

(i) Every constant function is a solution.
(ii) Every solution approaches a constant.
(iii) The differential equation has a first integral.

It is but a small step, then, to conjecture that such a pair of terms as those appearing in the right-hand side of (1.1) constitute a harmless perturbation of a stable equation. While this can be helpful in a given equation, there is a very important additional application. For if we have a difficult stability problem of the form

$$x'(t) = -g(x(t-L)), \tag{1.2}$$

then we can study

$$x'(t) = -g(x(t-L)) + g(x(t)) - g(x(t)), \tag{1.3}$$

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having the aforementioned harmless perturbation so that we need only to study

$$x'(t) = -g(x(t)). \tag{1.4}$$

The idea of ignoring the delay is ancient when the delay is small or when there is a considerable monotonicity; neither will be present in this discussion.

The thesis of this paper is that the conjecture is substantially correct and the solution is applied in a uniformly simple way using fixed point theory regardless of whether the delay is constant, variable, pointwise, distributed, finite, or infinite.

2. The conjecture

Cooke and Yorke [10] introduced a population model of the form of (1.1), where $g(x(t))$ is the birth rate and $g(x(t-L))$ is the death rate. They also introduced other models with distributed delays and they posed a number of questions. The unusual aspect of their study centered on the fact that g is an arbitrary Lipschitz function, laying to rest the controversy over just what the growth properties should be in a given population.

That paper generated a host of studies which continue to this day, as may be seen in [1, 2, 3, 13, 14, 15, 18, 19, 21], to mention just a few. Most of the subsequent studies asked that g should be monotone in some sense. Recently we noted [8] that every question raised in the Cooke-Yorke paper can be answered with two applications of the contraction mapping principle.

This paper begins a study of the conjecture that the right-hand side of (1.1) is a harmless perturbation. Thus, we list a number of classical delay equations of both first and second order to test the conjecture. Our aim is to continue using fixed point mappings to establish stability, as we have done in numerous earlier papers, including [6, 7, 8, 9].

In [6], we investigated the question of the relative effectiveness of fixed point theory versus Liapunov theory on stability problems. That question arises here again. It turns out that for the examples we consider, contraction mappings are very suitable for studying scalar delay equations, while Liapunov's direct method is perfectly suited for studying our second-order problems. Those are observations concerning these specific problems and not in the way of a general conjecture.

The problems we consider here using contractions are

$$x' = -\int_{t-L}^{t} p(s-t)g(x(s))\,ds, \tag{2.1}$$

$$x' = -\int_{0}^{t} e^{-a(t-s)}\sin(t-s)g(x(s))\,ds, \tag{2.2}$$

$$x' = -\int_{-\infty}^{t} q(s-t)g(x(s))\,ds, \tag{2.3}$$

$$x' = -a(t)g(x(q(t))). \tag{2.4}$$

We always have $xg(x) > 0$ for $x \neq 0$, so that each of these equations can be written as

$$x' = -g(x) + \text{ a harmless perturbation} \tag{2.5}$$

and then we use contraction mappings to show that the equation is stable. In this way we can show the fixed point technique working on a distributed bounded delay, a distributed unbounded delay, a distributed infinite delay, and a pointwise variable delay. Moreover, the fixed point arguments are simple and unified with promise of application to a very wide class of problems.

Our study will focus only on the above examples. But in a study to follow this one, we continue to test the idea by applying Liapunov theory to study

$$x'' + f(x)x' + g(x(t-L)) = 0, \tag{2.6}$$

$$x'' + f(x)x' + \int_{t-L}^{t} p(s-t)g(x(s))\,ds = 0, \tag{2.7}$$

$$x'' + f(x)x' + \int_{-\infty}^{t} q(s-t)g(x(s))\,ds = 0, \tag{2.8}$$

$$x'' + f(x)x' + a(t)g(x(q(t))) = 0, \tag{2.9}$$

$$x'' + a(t)g(x(q(t))) = 0, \quad a(t) \longrightarrow \infty. \tag{2.10}$$

In all of these, $f(x) \geq 0$ and $xg(x) \geq 0$. All of these problems bear out the conjecture that the functions in (1.1) represent a harmless perturbation.

All of these are important classical problems and are not merely contrived to make our point here. Equation (2.1) was studied by Volterra [25] in connection with a biological application, by Ergen [11] and Brownell and Ergen [4] in the study of a circulating-fuel nuclear reactor, by Levin and Nohel [22] in numerous contexts, and by Hale [16, page 458] in stability theory, all with convex kernels. We ask much less on the kernels here, but more on g. Equation (2.1) was studied by MacCamy and Wong [23, page 16] concerning positive kernel theory and they note that their methods fail to establish stability for that equation. Equation (2.3) has been studied by Hale [17, page 52] concerning limit sets when the kernel is convex. Equation (2.4) has been studied in many contexts, especially as a so-called 3/2-problem as may be seen in Graef et al. [12] and Krisztin [20], together with their many references. Equations (2.6)–(2.9) are all delayed Liénard equations about which there is much literature (see [26]). An early application was an automatic steering device for the large ship “The New Mexico” by Minorsky [24] or Burton [5, page 140]. Equation (2.10) has been an enduring problem when there is no delay. Investigators strive to give conditions ensuring that all solutions tend to zero.

Our study shows that the conjecture is valid for these nine classical problems. But the major focus of this investigation is on the fact that many papers have been written following the Cooke-Yorke model in which functions have been derived, which follow (i), (ii), and (iii) and, hence, may very well represent harmless perturbations. Here, we present an elementary technique available to a wide group of investigators, on a wide range of problems, and it illustrates the fact that fixed point theory is a viable stability tool. It remains to be seen if it will be successful on the problems in the aforementioned papers.

3. Stability by contraction mappings

In all of our examples in this section we will have a function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$|g(x) - g(y)| \le K|x - y| \tag{3.1}$$

for some $K > 0$ and all $x, y \in \mathbb{R}$,

$$\frac{g(x)}{x} \ge 0, \qquad \lim_{x\to 0} \frac{g(x)}{x} \quad \text{exists,} \tag{3.2}$$

and sometimes

$$\frac{g(x)}{x} \ge \beta \tag{3.3}$$

for some $\beta > 0$.

We have spoken in terms of stability. But each of our theorems claim that solutions are bounded and, with an additional assumption, that these solutions tend to zero. It is a simple, but lengthy, exercise to show that these statements could be extended to say that the zero solution is stable and globally asymptotically stable. One defines the mapping set in terms of the given $\epsilon > 0$ in the stability argument.

Existence theory is found in Burton's [5, Chapter 3], for example. Briefly, for the type of initial function which we will give, owing to the continuity and the Lipschitz condition, there will be a unique solution. Because of the Lipschitz growth condition, that solution can be continued for all future time.

Example 3.1. Consider the scalar equation

$$x' = -\int_{t-L}^{t} p(s-t)g(x(s))\,ds \tag{3.4}$$

with $L > 0$, p continuous,

$$\int_{-L}^{0} p(s)\,ds = 1, \tag{3.5}$$

and for the K of (3.1), let

$$2K\int_{-L}^{0} |p(v)v|\,dv =: \alpha < 1. \tag{3.6}$$

Theorem 3.2. *If (3.1), (3.2), (3.5), and (3.6) hold, then every solution of (3.4) is bounded. Moreover, if (3.3) also holds, then every solution tends to zero.*

Proof. Let $\psi : [-L, 0] \to \mathbb{R}$ be a given continuous initial function and let $x_1(t) := x(t, 0, \psi)$ be the unique resulting solution. By the growth condition on g, $x_1(t)$ exists on $[0, \infty)$. If we add and subtract $g(x)$, we can write the equation as

$$x' = -g(x) + \frac{d}{dt}\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))\,du\,ds. \tag{3.7}$$

Define a continuous nonnegative function $a:[0,\infty)\to[0,\infty)$ by

$$a(t) := \frac{g(x_1(t))}{x_1(t)}. \tag{3.8}$$

Since a is the quotient of continuous functions, it is continuous when assigned the limit at $x_1(t)=0$, if such a point exists.

Thus, for the fixed solution, our equation is

$$x' = -a(t)x + \frac{d}{dt}\int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))\,du\,ds \tag{3.9}$$

which, by the variation of parameters formula, followed by integration by parts, can then be written as

$$\begin{aligned}
x(t) &= \psi(0)e^{-\int_0^t a(s)ds} \\
&\quad + \int_0^t e^{-\int_v^t a(s)ds}\frac{d}{dv}\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))\,du\,ds\,dv \\
&= \psi(0)e^{-\int_0^t a(s)ds} + e^{-\int_v^t a(s)ds}\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))\,du\,ds\Big|_0^t \\
&\quad - \int_0^t a(v)e^{-\int_v^t a(s)ds}\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))\,du\,ds\,dv \\
&= \psi(0)e^{-\int_0^t a(s)ds} + \int_{-L}^{0} p(s)\int_{t+s}^{t} g(x(u))\,du\,ds \\
&\quad - e^{-\int_0^t a(s)ds}\int_{-L}^{0} p(s)\int_{s}^{0} g(\psi(u))\,du\,ds \\
&\quad - \int_0^t e^{-\int_v^t a(s)ds}a(v)\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))\,du\,ds\,dv.
\end{aligned} \tag{3.10}$$

Let

$$M = \{\phi : [-L,\infty) \longrightarrow \mathbb{R} \mid \phi_0 = \psi,\ \phi \in C, \phi \text{ bounded}\} \tag{3.11}$$

and define $P : M \to M$ using the above equation in $x(t)$. For $\phi \in M$ define $(P\phi)(t) = \psi(t)$ if $-L \le t \le 0$. If $t \ge 0$, then define

$$\begin{aligned}
(P\phi)(t) &= \psi(0)e^{-\int_0^t a(s)ds} + \int_{-L}^{0} p(s)\int_{t+s}^{t} g(\phi(u))\,du\,ds \\
&\quad - e^{-\int_0^t a(s)ds}\int_{-L}^{0} p(s)\int_{s}^{0} g(\psi(u))\,du\,ds \\
&\quad - \int_0^t e^{-\int_v^t a(s)ds}a(v)\int_{-L}^{0} p(s)\int_{v+s}^{v} g(\phi(u))\,du\,ds\,dv.
\end{aligned} \tag{3.12}$$

To see that P is a contraction, if $\phi,\eta \in M$, then

$$\begin{aligned} |(P\phi)(t)-(P\eta)(t)| &\le \int_{-L}^{0}|p(s)|\int_{t+s}^{t}|g(\phi(u))-g(\eta(u))|\,du\,ds \\ &\quad + \int_0^t e^{-\int_v^t a(s)ds}a(v)\int_{-L}^{0}|p(s)|\int_{v+s}^{v}|g(\phi(u))-g(\eta(u))|\,du\,ds\,dv \\ &\le 2K\|\phi-\eta\|\int_{-L}^{0}|p(s)s|\,ds \le \alpha\|\phi-\eta\|. \end{aligned} \tag{3.13}$$

Hence, there is a unique fixed point, a bounded solution.

If $g(x)/x \ge \beta > 0$, then add to M the condition that $\phi(t) \to 0$ as $t \to \infty$. We can show that $(P\phi)(t) \to 0$ whenever $\phi(t) \to 0$ and, hence, that the fixed point tends to zero. □

The next example concerns a problem of Halanay and later of MacCamy and Wong [23, page 16] in stability investigation using positive kernels. They mention that the positive kernel technique works on

$$x' = -\int_0^t e^{-a(t-s)}\cos(t-s)g(x(s))\,ds \tag{3.14}$$

but does not work on (2.2). The idea of adding and subtracting the same thing, which we use here, works in exactly the same way for both of them. It would even work if the right-hand side began with an unstable term such as $+\gamma g(x)$ where $\gamma < \beta$, although we do not take the space to show it.

Example 3.3. Consider the scalar equation (2.2) where $a > 0$, and for the K of (3.1), we have

$$\alpha := 2K\sup_{t\ge 0}\int_0^t\int_{t-u}^{\infty}e^{-av}|\sin v|\,dv\,du < 1. \tag{3.15}$$

Notice that

$$k := \int_0^{\infty}e^{-av}\sin v\,dv = \frac{1}{a^2+1}. \tag{3.16}$$

Because of the Lipschitz condition on g, we can show that for each $x(0)$ there is a unique solution $x(t,0,x(0))$ defined for all future t.

THEOREM 3.4. *If (3.1), (3.2), and (3.15) hold, then every solution of (2.2) is bounded. If, in addition, (3.3) holds, then every solution tends to zero as $t \to \infty$.*

Proof. Let $x(0) = x_0$ be given, resulting in a unique solution $x_1(t)$. With the k defined above, write the equation as

$$x' = -kg(x) + \frac{d}{dt}\int_0^t \int_{t-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds. \tag{3.17}$$

Define a function $c(t)$ by

$$c(t) := \frac{g(x_1(t))}{x_1(t)} \tag{3.18}$$

so that the equation can be written as

$$x' = -kc(t)x + \frac{d}{dt}\int_0^t \int_{t-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds, \tag{3.19}$$

which will still have the unique solution $x_1(t)$ for the given initial condition $x(0) = x_0$. We can then use the variation of parameters formula to write the solution as

$$\begin{aligned}
x(t) &= x_0 e^{-k\int_0^t c(s)ds} + \int_0^t e^{-k\int_u^t c(s)ds}\frac{d}{du}\int_0^u \int_{u-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds\,du \\
&= x_0 e^{-k\int_0^t c(s)ds} + e^{-k\int_u^t c(s)ds}\int_0^u \int_{u-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds\,\Big|_0^t \\
&\quad - \int_0^t e^{-k\int_u^t c(s)ds} kc(u)\int_0^u \int_{u-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds\,du \\
&= x_0 e^{-k\int_0^t c(s)ds} + \int_0^t \int_{t-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds \\
&\quad - \int_0^t e^{-k\int_u^t c(s)ds} kc(u)\int_0^u \int_{u-s}^{\infty} e^{-av}\sin v\,dv g(x(s))\,ds\,du.
\end{aligned} \tag{3.20}$$

Let M be defined as the set of bounded continuous $\phi : [0,\infty) \to \mathbb{R}$, $\phi(0) = x_0$, and define $P : M \to M$ using the above equation for $x(t)$, as we did in the proof of Theorem 3.2. To see that P is a contraction, if $\phi, \eta \in M$, then

$$\big|(P\phi)(t) - (P\eta)(t)\big| \le 2K\sup_{t\ge 0}\int_0^t \int_{t-u}^{\infty} e^{-av}|\sin v|\,dv\,du\|\phi - \eta\|. \tag{3.21}$$

Thus, P will have a unique fixed point, a bounded function satisfying the differential equation.

If $g(x)/x \ge \beta > 0$, then we can show that $(P\phi)(t) \to 0$ whenever $\phi(t) \to 0$, thereby concluding that all solutions tend to zero. □

Example 3.5. We next consider the equation

$$x'(t) = -\int_{-\infty}^{t} q(s-t)g(x(s))\,ds, \tag{3.22}$$

where

$$\int_{-\infty}^{0} q(s)ds = 1, \qquad \int_{-\infty}^{0}\int_{-\infty}^{v} |q(u)|\,du\,dv \quad \text{exists}, \tag{3.23}$$

and there is a positive number $\alpha < 1$ with

$$2K \sup_{t\geq 0} \int_0^t \int_{-\infty}^{s-t} |q(u)|\,du\,ds \leq \alpha, \tag{3.24}$$

where K is from (3.1).

Theorem 3.6. *Suppose that (3.1), (3.2), (3.23), and (3.24) hold. Then every solution of (3.22) with bounded continuous initial function $\psi : (-\infty, 0] \to \mathbb{R}$ is bounded. If, in addition, (3.3) holds, then those solutions tend to zero as $t \to \infty$.*

Proof. Write (3.22) as

$$x' = -g(x(t)) + \frac{d}{dt} \int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)du g(x(s))ds. \tag{3.25}$$

For a given bounded continuous initial function ψ, let $x_1(t)$ be the resulting unique solution which will be defined on $[0, \infty)$. Define a unique continuous function by

$$a(t) := \frac{g(x_1(t))}{x_1(t)} \tag{3.26}$$

and write the equation as

$$x' = -a(t)x(t) + \frac{d}{dt} \int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)du g(x(s))ds \tag{3.27}$$

which, for the same initial function, still has the unique solution $x_1(t)$. Use the variation of parameters formula to write the solution as the integral equation

$$\begin{aligned}
x(t) &= \psi(0)e^{-\int_0^t a(s)ds} + \int_0^t e^{-\int_v^t a(u)du} \frac{d}{dv} \int_{-\infty}^{v} \int_{-\infty}^{s-v} q(u)du g(x(s))ds\,dv \\
&= \psi(0)e^{-\int_0^t a(s)ds} + e^{-\int_v^t a(u)du} \int_{-\infty}^{v} \int_{-\infty}^{s-v} q(u)du g(x(s))ds \Big|_0^t \\
&\quad - \int_0^t a(v)e^{-\int_v^t a(s)ds} \int_{-\infty}^{v} \int_{-\infty}^{s-v} q(u)du g(x(s))ds\,dv \\
&= \psi(0)e^{-\int_0^t a(s)ds} + \int_{-\infty}^{t} \int_{-\infty}^{s-t} q(u)du g(x(s))ds \\
&\quad - e^{-\int_0^t a(u)du} \int_{-\infty}^{0} \int_{-\infty}^{s} q(u)du g(\psi(s))ds \\
&\quad - \int_0^t a(v)e^{-\int_v^t a(s)ds} \int_{-\infty}^{v} \int_{-\infty}^{s-v} q(u)du g(x(s))ds\,dv.
\end{aligned} \tag{3.28}$$

Let

$$M = \{\phi : \mathbb{R} \longrightarrow \mathbb{R} \mid \phi \in C,\ \phi(t) = \psi(t) \text{ for } t \leq 0, \phi \text{ bounded}\} \tag{3.29}$$

and define $P : M \to M$ by $\phi \in M$ implies that $(P\phi)(t) = \psi(t)$ for $t \leq 0$ and let $P\phi$ be defined from the last equation above for x with x replaced by ϕ, as we have done before.

To see that P is a contraction, if $\phi,\eta \in M$, then

$$\begin{aligned}
|(P\phi)(t)-(P\eta)(t)| &\le \int_{-\infty}^{t}\int_{-\infty}^{s-t}|q(u)|\,du\,|g(\phi(s))-g(\eta(s))|\,ds \\
&\quad + \int_0^t a(v)e^{-\int_v^t a(s)ds}\int_{-\infty}^{v}\int_{-\infty}^{s-v}|q(u)|\,du\,|g(\phi(s))-g(\eta(s))|\,ds\,dv \\
&\le \int_0^t\int_{-\infty}^{s-t}|q(u)|\,du\,ds K\|\phi-\eta\| \\
&\quad + \int_0^t a(v)e^{-\int_v^t a(s)ds}\int_0^v\int_{-\infty}^{s-v}|q(u)|\,du\,ds\,dv K\|\phi-\eta\| \\
&\le 2K\|\phi-\eta\|\sup_{t\ge 0}\int_0^t\int_{-\infty}^{s-t}|q(u)|\,du\,ds \le \alpha\|\phi-\eta\|.
\end{aligned} \tag{3.30}$$

If $g(x)/x \ge \beta > 0$, then we modify M to include $\phi(t)\to 0$ and we show that this means that $P\phi$ also tends to zero. This will complete the proof. □

Example 3.7. Finally, we consider a scalar equation

$$x'(t) = -a(t)g\big(x(q(t))\big), \tag{3.31}$$

where $q:[0,\infty)\to\mathbb{R}$ is continuous and strictly increasing, $q(t)<t$, q has the inverse function $h(t)$ so that $q(h(t))=t$, and $a:[0,\infty)\to[0,\infty)$ is continuous. We suppose that there is an $\alpha<1$ with

$$2K\sup_{t\ge 0}\int_t^{h(t)} a(u)\,du \le \alpha < 1, \tag{3.32}$$

where K is from (3.1).

Theorem 3.8. *Let (3.1), (3.2), and (3.32) hold. Then every solution of (3.31) is bounded. If, in addition, (3.3) holds and*

$$\int_0^t a(s)\,ds \longrightarrow \infty \quad \text{as } t\longrightarrow\infty, \tag{3.33}$$

then every solution of (3.31) tends to zero as $t\to\infty$.

Proof. Write (3.31) as

$$x'(t) = -a(h(t))h'(t)g(x(t)) - \frac{d}{dt}\int_{h(t)}^{t} a(s)g\big(x(q(s))\big)\,ds. \tag{3.34}$$

Given a continuous initial function $\psi:[q(0),0]\to\mathbb{R}$, let $x_1(t)$ denote the unique solution having that initial function and define a continuous function by

$$c(t) := \frac{a(h(t))h'(t)g(x_1(t))}{x_1(t)}. \tag{3.35}$$

For that fixed solution $x_1(t)$ and that initial function ψ, it follows that

$$x' = -c(t)x - \frac{d}{dt}\int_{h(t)}^{t} a(s)g(x(q(s)))\,ds \tag{3.36}$$

has the unique solution $x_1(t)$ and, by the Lipschitz condition, we can argue that it exists on $[0,\infty)$.

By the variation of parameters formula, we have

$$\begin{aligned}
x(t) &= \psi(0)e^{-\int_0^t c(s)ds} - \int_0^t e^{-\int_s^t c(u)du}\frac{d}{ds}\int_{h(s)}^{s} a(u)g(x(q(u)))\,du\,ds \\
&= \psi(0)e^{-\int_0^t c(s)ds} - e^{\int_s^t c(u)du}\int_{h(s)}^{s} a(u)g(x(q(u)))\,du\,\Big|_0^t \\
&\quad + \int_0^t c(s)e^{-\int_s^t c(u)du}\int_{h(s)}^{s} a(u)g(x(q(u)))\,du\,ds \\
&= \psi(0)e^{-\int_0^t c(s)ds} - \int_{h(t)}^{t} a(u)g(x(q(u)))\,du \\
&\quad + e^{-\int_0^t c(u)du}\int_{h(0)}^{0} a(u)g(x(q(u)))\,du \\
&\quad + \int_0^t c(s)e^{-\int_s^t c(u)du}\int_{h(s)}^{s} a(u)g(x(q(u)))\,du\,ds.
\end{aligned} \tag{3.37}$$

We would define the complete metric space of bounded continuous functions which agree with ψ and use the above equation for x to define a mapping. That mapping would be a contraction because of (3.32). We would complete the proof as before. □

Remark 3.9. This is a general method applied to four very different problems which have been studied closely by other methods for many years. Yet, new information is found in each case. In Theorem 3.2, far less is required on the kernel than in traditional approaches. Theorem 3.4 succeeds where the positive kernel method failed. Theorem 3.6 again requires less on the kernel than the traditional Liapunov functional did. Something intriguing occurs in Theorem 3.8. This problem has been studied intensively by many investigators for at least 54 years, using techniques devised specifically for it. Thus, it is unreasonable to expect a general technique to compare favorably with the special techniques. Yet, something new does occur. Our measure is in the integral with limits from t to $h(t)$, while traditional techniques measure from t to $t+r(t)$. It is known that $h(t)$ is smaller than $t+r(t)$ when $r(t)$ is decreasing. But the real value of the technique is that it is simple, can be applied to many problems with little difficulty, and indicates once more that fixed point theory is a viable stability tool.

Remark 3.10. This paper concerns stability by fixed point methods and condition (3.3) provides us with a simple way of showing that solutions tend to zero by endowing the mapping set with this property. But (3.3) can often be relaxed using an old technique from Liapunov theory called the annulus argument. See, for example, [5, page 231]. The idea works in all problems in which $x'(t)$ is bounded whenever $x(t)$ is a bounded function.

Instead of (3.3), strengthen (3.2) to

$$\frac{g(x)}{x} > 0 \quad \text{for } x \neq 0, \qquad \lim_{x \to 0} \frac{g(x)}{x} \quad \text{exists.} \tag{3.38}$$

Consider the proof of Theorem 3.2 at the point where we have shown that $x_1(t)$ is bounded, say $|x_1(t)| \leq H$, and suppose this means that $|x_1'(t)| \leq L$ for some $L > 0$. We will now show that $x_1(t) \to 0$. By way of contradiction, if it does not, then there is an $\epsilon > 0$ and sequence $\{t_n\} \uparrow \infty$ with $|x_1(t_n)| \geq \epsilon$. Now, from $g(x)/x > 0$ for $x \neq 0$, there is a $\gamma > 0$ with $g(x)/x \geq \gamma$ for $\epsilon/2 \leq |x| \leq H$. Moreover, since $|x'(t)| \leq L$, there is a $\mu > 0$ with $|x_1(t)| \geq \epsilon/2$ for $|t_n - t| \leq \mu$. Hence, $g(x_1(t))/x_1(t) \geq \gamma$ for $|t_n - t| \leq \mu$. Thus, $\int_0^\infty a(t)dt = \infty$. Now, add to M the condition that $\phi(t) \to 0$ as $t \to \infty$. It follows that $(P\phi)(t) \to 0$ as $t \to \infty$. Hence, the fixed point tends to zero.

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