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Research Article

Hybrid Algorithms of Common Solutions of Generalized Mixed Equilibrium Problems and the Common Variational Inequality Problems with Applications

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We introduce new iterative algorithms by hybrid method for finding a common element of the set of solutions of fixed points of infinite family of nonexpansive mappings, the set of common solutions of generalized mixed equilibrium problems, and the set of common solutions of the variational inequality with inverse-strongly monotone mappings in a real Hilbert space. We prove the strong convergence of the proposed iterative method under some suitable conditions. Finally, we apply our results to complementarity problems and optimization problems. Our results improve and extend the results announced by many others.

1. Introduction

Throughout this paper, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. The set of *fixed points* of T denoted by $F(T)$; that is, $F(T) = \{x \in C : Tx = x\}$. If $C \subset H$ is bounded, closed, and convex and T is a nonexpansive mapping of C into itself, then $F(T) \neq \emptyset$; see, for instance, [1]. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $A : C \to H$ a mapping, and $\varphi : C \to \mathbb{R}$ a real-valued function. The *generalized mixed equilibrium problem* is for finding $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{GMEP}(F, \varphi, A)$; that is,

$$\mathrm{GMEP}(F,\varphi,A) = \{x \in C : F(x,y) + \langle Ax, y-x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in C\}. \tag{1.2}$$

The generalized mixed equilibrium problem include fixed point problems, optimization problems, variational inequalities problems, Nash equilibrium problems, noncooperative games, economics, and the equilibrium problems as special cases [2–7].

In particular, if $A \equiv 0$, the problem (1.1) is reduced into the *mixed equilibrium problem* [8] for finding $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{MEP}(F, \varphi)$.

If $\varphi \equiv 0$, (1.1) is reduced into the *generalized equilibrium problem* [9] for finding $x \in C$ such that

$$F(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $\mathrm{GEP}(F, A)$, which this problem was studied by S. Takahashi and W. Takahashi [10].

If $A \equiv 0$ and $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.1) becomes the following *equilibrium problem* which is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $\mathrm{EP}(F)$. Many problems in applied sciences, such as numerous problems in physics, optimization, and economics reduce into finding a solution of (1.5). Some methods have been proposed to solve the generalized mixed equilibrium problems, equilibrium problems, and fixed point problems ([2, 6, 11–29]) and references therein. If $F \equiv 0$ and $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.1) becomes the following *variational inequality problem*, denoted by $\mathrm{VI}(C, A)$, is to find $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The variational inequality problem has been extensively studied in the literature. See, for example [30, 31] and the references therein. A mapping A of C into H is called *monotone* if

$$\langle Ax - Ay, x-y\rangle \geq 0, \quad \forall x, y \in C. \tag{1.7}$$

A is called an *α-inverse-strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x-y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.8}$$

In 2008, Takahashi et al. [32] introduced an iterative method for finding the set of fixed point by Hybrid method in Hilbert spaces. Starting with $C_1 = C$, $x_1 = P_{C_1}x_0$, define sequence $\{x_n\}$, $\{y_n\}$ as follows:

$$\begin{gathered} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 1, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1, \end{gathered} \tag{1.9}$$

where P_C is a metric projection of H onto C and T is a nonexpansive mapping of C into itself. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generated by (1.9) converges strongly to $P_{F(T)}x_0$. In 2009, Kumam [20] introduced an iterative method for finding a common element of the set of common fixed points of nonexpansive mapping, the set of solutions of a variational inequality problem, and the set of solutions of an equilibrium problem in Hilbert spaces. Starting with an arbitrary $C_1 = C$, $x_1 = P_{C_1}x_0$, define sequence $\{x_n\}$, $\{z_n\}$ as follows:

$$\begin{gathered} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)TP_C(z_n - \lambda_n B z_n), \quad n \geq 1, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1, \end{gathered} \tag{1.10}$$

where T is a nonexpansive mapping of C into itself and B is a β-inverse-strongly monotone mapping of C into H. He proved that if the sequences $\{\alpha_n\}$, $\{r_n\}$, and $\{\lambda_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generated by (1.10) converges strongly to $P_{F(T)\cap \mathrm{EP}(F)\cap \mathrm{VI}(C,B)}x_0$. In 2010, Kangtunyakarn [33] introduced a new method for a common of generalized equilibrium problems, common of variational inequality problems, and fixed point problems by using S-mapping generated by a finite family of nonexpansive mappings and real numbers in Hilbert spaces. Starting with an arbitrary x_1, u, v in C, define the sequences $\{x_n\}$, $\{y_n\}$ as follows:

$$\begin{gathered} F(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ G(v_n, v) + \langle Bx_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \delta_n P_C(u_n - \lambda_n A u_n) + (1-\delta_n)P_C(v_n - \eta_n B v_n), \quad n \geq 1, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n y_n, \quad \forall n \geq 1, \end{gathered} \tag{1.11}$$

where S_n is the S-mapping and A, B are α, β-inverse-strongly monotone mappings of C into H, respectively. He proved that if the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{r_n\}$, $\{s_n\}$, $\{\eta_n\}$, and $\{\lambda_n\}$ of parameters satisfies appropriate conditions, then $\{x_n\}$ generated by (1.11) converges strongly to $P_{\mathfrak{I}:=\cap_{n=1}^{\infty}F(S_n)\cap \mathrm{MEP}(F,A)\cap \mathrm{MEP}(G,B)\cap \mathrm{VI}(C,A)\cap \mathrm{VI}(C,B)}x_1$.

Recently, Shehu [34] motivated Chantarangsi et al. [35] who studied the problem of approximating a common element of the set of fixed points of an infinite family of nonexpansive mapping, the set of common solutions of generalized mixed equilibrium problems, and the set of solutions to a variational inequality problem in a real Hilbert spaces.

In this paper, motivated by the above results, we present a new hybrid iterative scheme for finding a common element of the set of solutions of a common of generalized mixed equilibrium problems, the common solutions of the variational inequality for inverse-strongly monotone mapping, and the set of fixed points of infinite family of nonexpansive mappings in the set of Hilbert spaces. Then, we prove strong convergence theorems under some mild conditions. Finally, we give some applications of our results. The results presented in this paper generalize, extend, and improve the results of Takahashi et al. [32], Kumam [20], Kangtunyakarn [33], and many authors.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and let C be a closed convex subset of H. When $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ means $\{x_n\}$ converges weakly to x, and $x_n \rightarrow x$ means $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\begin{gathered}\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,\\ \|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,\end{gathered} \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x-y\|, \quad \forall y \in C. \tag{2.2}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.5}$$

for all $x \in H$, $y \in C$. It is also known that H satisfies the Opial condition; for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.6}$$

holds for every $y \in H$ with $y \neq x$.

It is obvious that any α-inverse-strongly monotone mapping A is $(1/\alpha)$-Lipschitz monotone and continuous mapping. We also have that for all $x, y \in H$ and $\lambda > 0$,

$$\begin{aligned}\|(I - rA)x - (I - rA)y\|^2 &= \|(x - y) - r(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r\langle x - y, Ax - Ay\rangle + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2.\end{aligned} \tag{2.7}$$

So, if $r \leq 2\alpha$, then $I - rA$ is a nonexpansive mapping of C into H.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$, the nonlinear mapping $A : C \to H$ is continuous monotone, $\varphi : C \to \mathbb{R}$ is convex, and lower semicontinuous satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone; that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$,

(A3) F is upper-hemicontinuous; that is, for each $u, x, y \in C$,

$$\limsup_{t \to 0^+} F(tu + (1 - t)x, y) \leq F(x, y), \tag{2.8}$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$,

(B1) For each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \operatorname{dom}(\varphi)$ such that for any $u \in C \setminus D_x$,

$$F(u, y_x) + \langle Au, y_x - u\rangle + \varphi(y_x) + \frac{1}{r}\langle y_x - u, u - x\rangle < \varphi(u), \tag{2.9}$$

(B2) C is a bounded set.

The following lemma appears implicitly in [2]. We need the following lemmas for proving our main result.

Lemma 2.1 (see [2]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $u \in C$ such that*

$$F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

The following lemma was also given in [36].

Lemma 2.2 (see [36]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $K_r : H \to C$ as follows:*

$$K_r(x) = \left\{u \in C : F(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0,\ \forall y \in C\right\}, \tag{2.11}$$

for all $x \in H$. Then, the following hold:

(1) K_r *is single-valued,*

(2) K_r *is firmly nonexpansive, that is, for any* $x, y \in H$, $\|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y\rangle$,

(3) $F(K_r) = \mathrm{EP}(F)$,

(4) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.3 (see [37]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction mapping satisfies (A1)–(A4), and let $\varphi : C \to \mathbb{R}$ be convex and lower semicontinuous such that $C \cap \operatorname{dom} \varphi \neq \emptyset$. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, there exists $u \in C$ such that*

$$F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle. \tag{2.12}$$

Define a mapping $K_r : H \to C$ as follows:

$$T_r^{(F,\varphi)}(x) = \left\{u \in C : F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0,\ \forall y \in C\right\}, \tag{2.13}$$

for all $x \in H$. Then, the following hold:

(1) $T_r^{(F,\varphi)}$ *is single-valued,*

(2) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, that is, for any* $x, y \in H$, $\|T_r^{(F,\varphi)} x - T_r^{(F,\varphi)} y\|^2 \leq \langle T_r^{(F,\varphi)} x - T_r^{(F,\varphi)} y, x - y\rangle$,

(3) $F(T_r^{(F,\varphi)}) = \mathrm{MEP}(F, \varphi)$,

(4) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

Lemma 2.4 (see [38]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0, \tag{2.14}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of solutions of a common of generalized mixed equilibrium problems, the common solutions of the variational inequality for inverse-strongly monotone mapping, and the set of fixed points of infinite family of nonexpansive mappings in the set of Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A, B, D, and E be α, β, δ, and η-inverse-strongly monotone mapping of C into H, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite nonexpansive mapping such that $\Theta := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap \mathrm{VI}(C, D) \cap \mathrm{VI}(C, E) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, $C_{1,i} = C$, $C_1 = \cap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$ and*

$$\begin{aligned}
t_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
u_n &= T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n), \\
w_n &= \xi_n P_C(u_n - \lambda_n D u_n) + (1 - \xi_n) P_C(t_n - \mu_n E t_n), \\
y_{n,i} &= \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) T_i w_n, \\
C_{n+1,i} &= \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i} \left(\|x_0\|^2 + 2\langle w_n - x_0, z\rangle \right) \right\}, \\
C_{n+1} &= \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} &= P_{C_{n+1}} x_0,
\end{aligned} \tag{3.1}$$

for every $n \geq 0$, where $\{r_n\}, \{s_n\} \subset (0, \infty)$, $\lambda_n \in (0, 2\delta)$ and $\mu_n \in (0, 2\eta)$ satisfying the following conditions:

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $0 < c \leq s_n \leq d < 2\beta$,

(iii) $\lim_{n \to \infty} \alpha_{n,i} = 0$,

(iv) $\lim_{n \to \infty} \xi_n = \xi \in (0, 1)$,

(v) $0 < e \leq \lambda_n \leq f < 2\delta$,

(vi) $0 < g \leq \mu_n \leq j < 2\eta$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

Proof. Let $p \in \Theta$, then $p = T_{r_n}^{(F_1,\varphi_1)}(p - r_n Ap)$, $p = T_{s_n}^{(F_2,\varphi_2)}(p - s_n Bp)$, $p = P_C(p - \lambda_n Dp)$, and $p = P_C(p - \mu_n Ep)$. By nonexpansiveness of $P_C, T_{r_n}^{(F_1,\varphi_1)}$, and $T_{s_n}^{(F_2,\varphi_2)}$, we have

$$\begin{aligned}
&\|w_n - p\|^2 \\
&\quad = \left\|\xi_n P_C(u_n - \lambda_n Du_n) + (1-\xi_n)P_C(t_n - \mu_n Et_n) - \xi_n P_C(p - \lambda_n Dp) - (1-\xi_n)P_C(p - \mu_n Ep)\right\|^2 \\
&\quad = \left\|\xi_n\{P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\} + (1-\xi_n)\{P_C(t_n - \mu_n Et_n) - P_C(p - \mu_n Ep)\}\right\|^2 \\
&\quad \le \xi_n \left\|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\right\|^2 + (1-\xi_n)\left\|(t_n - \mu_n Et_n) - (p - \mu_n Ep)\right\|^2 \\
&\quad = \xi_n \left\|(u_n - p) - \lambda_n(Du_n - Dp)\right\|^2 + (1-\xi_n)\left\|(t_n - p) - \mu_n(Et_n - Ep)\right\|^2 \\
&\quad = \xi_n\left\{\|u_n - p\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\right\} \\
&\qquad + (1-\xi_n)\left\{\|t_n - p\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\right\} \\
&\quad \le \xi_n\left\{\left\|T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n Bx_n) - T_{s_n}^{(F_2,\varphi_2)}(p - s_n Bp)\right\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\right\} \\
&\qquad + (1-\xi_n)\left\{\left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n Ax_n) - T_{r_n}^{(F_1,\varphi_1)}(p - r_n Ap)\right\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\right\} \\
&\quad \le \xi_n\left\{\|(x_n - s_n Bx_n) - (p - s_n Bp)\|^2\right\} + (1-\xi_n)\left\{\|(x_n - r_n Ax_n) - (p - r_n Ap)\|^2\right\} \\
&\quad \le \xi_n\|x_n - p\|^2 + (1-\xi_n)\|x_n - p\|^2 \\
&\quad \le \|x_n - p\|^2.
\end{aligned} \tag{3.2}$$

Since both $I - r_n A$ and $I - s_n B$ are nonexpansive for each $n \ge 1$ and (2.7), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \left\|T_{s_n}^{(F_2,\varphi_2)}(I - s_n B)x_n - T_{s_n}^{(F_2,\varphi_2)}(I - s_n B)p\right\|^2 \\
&\le \|(I - s_n B)x_n - (I - s_n B)p\|^2 \\
&\le \|x_n - p\|^2 + s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 \\
&\le \|x_n - p\|^2, \\
\|t_n - p\|^2 &= \left\|T_{r_n}^{(F_1,\varphi_1)}(I - r_n A)x_n - T_{r_n}^{(F_1,\varphi_1)}(I - r_n A)p\right\|^2 \\
&\le \|(I - r_n A)x_n - (I - r_n A)p\|^2 \\
&\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned} \tag{3.3}$$

Therefore, we obtain $\|u_n - p\| \le \|x_n - p\|$ and $\|t_n - p\| \le \|x_n - p\|$.

Next, we will divide the proof into four steps.

Step 1. We show that $\{x_n\}$ is well defined. Let $n = 1$, then $C_{1,i} = C$ is closed and convex for each $i \geq 1$. Suppose that $C_{n,i}$ is closed convex for some $n > 1$. Then, by definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for $n \geq 1$. Hence, $C_{n,i}$ is closed convex for $n \geq 1$ and for each $i \geq 1$. This implies that C_n is closed convex for $n \geq 1$. Moreover, we show that $\Theta \subset C_n$. For $n = 1$, $\Theta \subset C = C_{1,i}$. For $n \geq 2$, let $p \in \Theta$. Then,

$$\begin{aligned}
\left\|y_{n,i} - p\right\|^2 &= \left\|\alpha_{n,i}(x_0 - p)^2 + (1-\alpha_{n,i})(T_i w_n - p)\right\|^2 \\
&\leq \alpha_{n,i}\left\|x_0 - p\right\|^2 + (1-\alpha_{n,i})\left\|w_n - p\right\|^2 \\
&= \left\|w_n - p\right\|^2 + \alpha_{n,i}\left(\left\|x_0 - p\right\|^2 - \left\|w_n - p\right\|^2\right) \\
&\leq \left\|x_n - p\right\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle w_n - x_0, p\rangle\right),
\end{aligned} \tag{3.4}$$

which shows that $p \in C_{n,i}$, for all $n \geq 2$, for all $i \geq 1$. So, $\Theta \subset C_{n,i}$, for all $n \geq 1$, for all $i \geq 1$. Therefore, it follows that $\emptyset \neq \Theta \subset C_n$, for all $n \geq 1$. This implies that $\{x_n\}$ is well defined.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|y_{n,i} - x_n\| = 0$.

From $x_n = P_{C_n}x_0$, we get

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \tag{3.5}$$

for each $y \in C_n$. Since $\Theta \subset C_n$, we have

$$\langle x_0 - x_n, x_n - p\rangle \geq 0 \quad \text{for each } p \in \Theta,\ n \in \mathbb{N}. \tag{3.6}$$

Hence, for $p \in \Theta$, we obtain

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - p\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\
&= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - p\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\left\|x_0 - p\right\|.
\end{aligned} \tag{3.7}$$

It follows that

$$\|x_0 - x_n\| \leq \left\|x_0 - p\right\|, \quad \forall p \in \Theta,\ n \in \mathbb{N}. \tag{3.8}$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \tag{3.9}$$

For $n \in \mathbb{N}$, we compute

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \end{aligned} \tag{3.10}$$

and then

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Thus, the sequence $\{\|x_n - x_0\|\}$ is a bounded and nondecreasing sequence, so $\lim_{n\to\infty}\|x_n - x_0\|$ exists. That is, there exists m such that

$$m = \lim_{n\to\infty} \|x_n - x_0\|. \tag{3.12}$$

Hence, $\{x_n\}$ is bounded and so are $\{Ax_n\}$, $\{Bx_n\}$, $\{u_n\}$, $\{Du_n\}$, $\{t_n\}$, $\{Et_n\}$, $\{w_n\}$, $\{T_i w_n\}$, and $\{y_{n,i}\}$ for $i = 1, 2, \ldots$, and $n \geq 1$. From (3.9), we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_n - x_0 \rangle + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned} \tag{3.13}$$

By (3.12), we obtain

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.14}$$

Since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\left\| y_{n,i} - x_{n+1} \right\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_{n,i} \Big(\|x_0\|^2 + 2\langle w_n - x_0, x_{n+1} \rangle \Big). \tag{3.15}$$

By (iii) and (3.14), we get

$$\lim_{n\to\infty} \left\| y_{n,i} - x_{n+1} \right\| = 0. \tag{3.16}$$

It follows that

$$\left\|y_{n,i}-x_n\right\| \le \left\|y_{n,i}-x_{n+1}\right\| + \|x_n-x_{n+1}\|. \tag{3.17}$$

By (3.14) and (3.16), we have

$$\lim_{n\to\infty}\|y_{n,i}-x_n\| = 0, \quad i=1,2,\ldots. \tag{3.18}$$

Step 3. We claim that the following statements hold:

(S1) $\lim_{n\to\infty}\|x_n-u_n\|=0$,

(S2) $\lim_{n\to\infty}\|x_n-t_n\|=0$,

(S3) $\lim_{n\to\infty}\|w_n-x_n\|=0$.

For (3.2), we note that

$$\begin{aligned}
\left\|y_{n,i}-p\right\|^2 &\le \alpha_{n,i}\left\|x_0-p\right\|^2 + (1-\alpha_{n,i})\left\|T_iw_n-p\right\|^2 \\
&= \alpha_{n,i}\left\|x_0-p\right\|^2 + (1-\alpha_{n,i})\left\|w_n-p\right\|^2 \\
&\le \alpha_{n,i}\left\|x_0-p\right\|^2 + (1-\alpha_{n,i}) \\
&\quad \times \left\{\xi_n\left\|(x_n-s_nBx_n)-(p-s_nBp)\right\|^2 + (1-\xi_n)\left\|(x_n-r_nAx_n)-(p-r_nAp)\right\|^2\right\} \\
&\le \alpha_{n,i}\left\|x_0-p\right\|^2 + (1-\alpha_{n,i}) \\
&\quad \times \left\{\xi_n\left(\left\|x_n-p\right\|^2 + s_n(s_n-2\beta)\left\|Bx_n-Bp\right\|^2\right)\right. \\
&\qquad \left. +(1-\xi_n)\left(\left\|x_n-p\right\|^2 + r_n(r_n-2\alpha)\left\|Ax_n-Ap\right\|^2\right)\right\} \\
&= \alpha_{n,i}\left\|x_0-p\right\|^2 + (1-\alpha_{n,i}) \\
&\quad \times \left\{\left\|x_n-p\right\|^2 + \xi_ns_n(s_n-2\beta)\left\|Bx_n-Bp\right\|^2 + (1-\xi_n)r_n(r_n-2\alpha)\left\|Ax_n-Ap\right\|^2\right\} \\
&= \alpha_{n,i}\left\|x_0-p\right\|^2 + \left\|x_n-p\right\|^2 + (1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\left\|Bx_n-Bp\right\|^2 \\
&\quad + (1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\left\|Ax_n-Ap\right\|^2 \\
&= \alpha_{n,i}\left\|x_0-p\right\|^2 + \left\|x_n-p\right\|^2 + (1-\alpha_{n,i})\xi_ns_n(s_n-2\beta)\left\|Bx_n-Bp\right\|^2.
\end{aligned} \tag{3.19}$$

Since $0<c\le s_n\le d\le 2\beta$, $0\le k_i\le \alpha_{n,i}\le h_i<1$, we have

$$\begin{aligned}
(1-h_i)\xi c(2\beta-d)\left\|Bx_n-Bp\right\|^2 &\le \alpha_{n,i}\left\|x_0-p\right\|^2 + \left\|x_n-p\right\|^2 - \left\|y_{n,i}-p\right\|^2 \\
&\le \alpha_{n,i}\left\|x_0-p\right\|^2 + \left\|y_{n,i}-x_n\right\|\left(\left\|x_n-p\right\| + \left\|y_{n,i}-p\right\|\right).
\end{aligned} \tag{3.20}$$

By condition (iii) and (3.18), $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$ by using the same method with (3.20). Hence, from (3.19), since $0 < a \le r_n \le b \le 2\alpha$, $0 \le k_i \le \alpha_{n,i} \le h_i < 1$, we have

$$\begin{aligned}(1-h_i)(1-\xi)a(2\alpha-b)\left\|Ax_n - Ap\right\|^2 &\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - p\right\|^2 - \left\|y_{n,i} - p\right\|^2 \\ &\le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|y_{n,i} - x_n\right\|\left(\left\|x_n - p\right\| + \left\|y_{n,i} - p\right\|\right).\end{aligned} \tag{3.21}$$

By condition (iii) and (3.18), then we have $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$. On the other hand, we compute

$$\begin{aligned}\left\|u_n - p\right\|^2 &= \left\|T_{s_n}^{(F_2,\varphi_2)}(I - s_nB)x_n - T_{s_n}^{(F_2,\varphi_2)}(I - s_nB)p\right\|^2 \\ &\le \left\langle (x_n - s_nBx_n) - (p - s_nBp), u_n - p\right\rangle \\ &= \frac{1}{2}\Big\{\left\|(x_n - s_nBx_n) - (p - s_nBp)\right\|^2 + \left\|u_n - p\right\|^2 \\ &\quad -\left\|(x_n - s_nBx_n) - (p - s_nBp) - (u_n - p)\right\|^2\Big\} \\ &\le \frac{1}{2}\Big\{\left\|x_n - p\right\|^2 + \left\|u_n - p\right\|^2 - \left\|(x_n - s_nBx_n) - (p - s_nBp) - (u_n - p)\right\|^2\Big\} \\ &= \frac{1}{2}\Big\{\left\|x_n - p\right\|^2 + \left\|u_n - p\right\|^2 - \|u_n - x_n\|^2 + 2s_n\left\langle x_n - u_n, Bx_n - Bp\right\rangle \\ &\quad -s_n^2\left\|Bx_n - Bp\right\|^2\Big\},\end{aligned} \tag{3.22}$$

and hence,

$$\begin{aligned}\left\|u_n - p\right\|^2 &\le \left\|x_n - p\right\|^2 - \|u_n - x_n\|^2 + 2s_n\left\langle x_n - u_n, Bx_n - Bp\right\rangle - s_n^2\left\|Bx_n - Bp\right\|^2 \\ &\le \left\|x_n - p\right\|^2 - \|u_n - x_n\|^2 + 2s_n\|x_n - u_n\|\left\|Bx_n - Bp\right\|.\end{aligned} \tag{3.23}$$

By using the same method as (3.23), we also have

$$\begin{aligned}\left\|t_n - p\right\|^2 &\le \left\|x_n - p\right\|^2 - \|t_n - x_n\|^2 + 2r_n\left\langle x_n - t_n, Ax_n - Ap\right\rangle - r_n^2\left\|Ax_n - Ap\right\|^2 \\ &\le \left\|x_n - p\right\|^2 - \|t_n - x_n\|^2 + 2r_n\|x_n - t_n\|\left\|Ax_n - Ap\right\|.\end{aligned} \tag{3.24}$$

Furthermore, we observe that

$$
\begin{aligned}
&\left\|y_{n,i}-p\right\|^2 \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i})\left\|T_i w_n-p\right\|^2 \\
&\quad = \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i})\left\|w_n-p\right\|^2 \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i})\left\{\left\|\xi_n P_C(u_n-\lambda_n D u_n)+(1-\xi_n)P_C(t_n-\mu_n E t_n)-p\right\|^2\right\} \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i}) \\
&\qquad \times\left\{\xi_n\left(\left\|u_n-p\right\|^2-\lambda_n(2\delta-\lambda_n)\left\|Du_n-Dp\right\|^2\right)\right. \\
&\qquad\quad \left.+(1-\xi_n)\left(\left\|t_n-p\right\|^2-\mu_n(2\eta-\mu_n)\left\|Et_n-Ep\right\|^2\right)\right\} \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i})\left\{\xi_n\left\|u_n-p\right\|^2+(1-\xi_n)\left\|t_n-p\right\|^2\right\} \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+(1-\alpha_{n,i}) \\
&\qquad \times\left\{\xi_n\left(\left\|x_n-p\right\|^2-\|u_n-x_n\|^2+2s_n\|x_n-u_n\|\left\|Bx_n-Bp\right\|\right)\right. \\
&\qquad\quad \left.+(1-\xi_n)\left(\left\|x_n-p\right\|^2-\|t_n-x_n\|^2+2r_n\|x_n-t_n\|\left\|Ax_n-Ap\right\|\right)\right\} \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-p\right\|^2-(1-\alpha_{n,i})\xi_n\|u_n-x_n\|^2 \\
&\qquad +(1-\alpha_{n,i})\xi_n 2s_n\|x_n-u_n\|\left\|Bx_n-Bp\right\|-(1-\alpha_{n,i})(1-\xi_n)\|t_n-x_n\|^2 \\
&\qquad +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\left\|Ax_n-Ap\right\| \\
&\quad \le \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-p\right\|^2-(1-\alpha_{n,i})\xi_n\|u_n-x_n\|^2+(1-\alpha_{n,i})\xi_n 2s_n\|x_n-u_n\|\left\|Bx_n-Bp\right\| \\
&\qquad +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\left\|Ax_n-Ap\right\|.
\end{aligned}
\tag{3.25}
$$

By condition (i)–(iv), (3.18), $\lim_{n\to\infty}\|Ax_n-Ap\|=0$ and $\lim_{n\to\infty}\|Bx_n-Bp\|=0$, then we get

$$
\begin{aligned}
(1-\alpha_{n,i})\xi_n\|u_n-x_n\|^2 &\le \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-p\right\|^2-\left\|y_{n,i}-p\right\|^2 \\
&\quad +(1-\alpha_{n,i})\xi_n 2s_n\|x_n-u_n\|\left\|Bx_n-Bp\right\| \\
&\quad +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\left\|Ax_n-Ap\right\| \\
&\le \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-y_{n,i}\right\|\left(\left\|x_n-p\right\|+\left\|y_{n,i}-p\right\|\right) \\
&\quad +(1-\alpha_{n,i})\xi_n 2s_n\|x_n-u_n\|\left\|Bx_n-Bp\right\| \\
&\quad +(1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n-t_n\|\left\|Ax_n-Ap\right\|.
\end{aligned}
\tag{3.26}
$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.27}$$

Similar to (3.26), from (3.25) by conditions (i)–(iv), (3.18), $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$, and $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$, we get

$$\begin{aligned}
(1-\alpha_{n,i})(1-\xi_n)\|t_n - x_n\|^2 &\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - \|y_{n,i} - p\|^2 \\
&\quad + (1-\alpha_{n,i})\xi_n 2s_n\|x_n - u_n\|\|Bx_n - Bp\| \\
&\quad + (1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n - t_n\|\|Ax_n - Ap\| \\
&\le \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - y_{n,i}\|(\|x_n - p\| + \|y_{n,i} - p\|) \\
&\quad + (1-\alpha_{n,i})\xi_n 2s_n\|x_n - u_n\|\|Bx_n - Bp\| \\
&\quad + (1-\alpha_{n,i})(1-\xi_n)2r_n\|x_n - t_n\|\|Ax_n - Ap\|.
\end{aligned} \tag{3.28}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.29}$$

From (3.1), (3.3), we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|\xi_n P_C(u_n - \lambda_n Du_n) + (1-\xi_n)P_C(t_n - \mu_n Et_n) \\
&\qquad -\xi_n P_C(p - \lambda_n Dp) - (1-\xi_n)P_C(p - \mu_n Ep)\|^2 \\
&= \xi_n\|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|^2 \\
&\quad + (1-\xi_n)\|P_C(t_n - \mu_n Et_n) - P_C(p - \mu_n Ep)\|^2 \\
&\le \xi_n\left\{\|u_n - p\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\right\} \\
&\quad + (1-\xi_n)\left\{\|t_n - p\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\right\} \\
&\le \xi_n\left\{\|x_n - p\|^2 + s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 - \lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2\right\} \\
&\quad + (1-\xi_n)\left\{\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 - \mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2\right\} \\
&\le \|x_n - p\|^2 + \xi_n s_n(s_n - 2\beta)\|Bx_n - Bp\|^2 - \xi_n\lambda_n(2\delta - \lambda_n)\|Du_n - Dp\|^2 \\
&\quad + (1-\xi_n)r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 - (1-\xi_n)\mu_n(2\eta - \mu_n)\|Et_n - Ep\|^2.
\end{aligned} \tag{3.30}$$

Furthermore, we observe that

$$\begin{aligned}
\|y_{n,i}-p\|^2 &\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|T_iw_n-p\|^2\\
&= \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\|w_n-p\|^2\\
&\le \alpha_{n,i}\|x_0-p\|^2+(1-\alpha_{n,i})\\
&\quad\times\Big\{\|x_n-p\|^2+\xi_n s_n(s_n-2\beta)\|Bx_n-Bp\|^2-\xi_n\lambda_n(2\delta-\lambda_n)\|Du_n-Dp\|^2\\
&\qquad+(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2-(1-\xi_n)\mu_n(2\eta-\mu_n)\|Et_n-Ep\|^2\Big\}\\
&\le \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2+(1-\alpha_{n,i})\xi_n s_n(s_n-2\beta)\|Bx_n-Bp\|^2-(1-\alpha_{n,i})\xi_n\lambda_n\\
&\quad\times(2\delta-\lambda_n)\|Du_n-Dp\|^2+(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\quad-(1-\alpha_{n,i})(1-\xi_n)\mu_n(2\eta-\mu_n)\|Et_n-Ep\|^2\\
&\le \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2+(1-\alpha_{n,i})\xi_n s_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad-(1-\alpha_{n,i})\xi_n\lambda_n(2\delta-\lambda_n)\|Du_n-Dp\|^2\\
&\quad+(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2.
\end{aligned}\tag{3.31}$$

Since $0<e\le\lambda_n\le f<2\delta$, $0\le k_i\le\alpha_{n,i}\le h_i<1$, we have

$$\begin{aligned}
(1-h_i)\xi e(2\delta-f)\|Du_n-Dp\|^2 &\le \alpha_{n,i}\|x_0-p\|^2+\|x_n-p\|^2-\|y_{n,i}-p\|^2\\
&\quad+(1-\alpha_{n,i})\xi_n s_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad+(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2\\
&\le \alpha_{n,i}\|x_0-p\|^2+\|y_{n,i}-x_n\|(\|x_n-p\|-\|y_{n,i}-p\|)\\
&\quad+(1-\alpha_{n,i})\xi_n s_n(s_n-2\beta)\|Bx_n-Bp\|^2\\
&\quad+(1-\alpha_{n,i})(1-\xi_n)r_n(r_n-2\alpha)\|Ax_n-Ap\|^2.
\end{aligned}\tag{3.32}$$

By conditions (i)–(v), (3.18), $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$, and $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$, then $\lim_{n\to\infty}\|Du_n - Dp\| = 0$. By using the same method with (3.32). Hence, from (3.31), and since $0 < g \le \mu_n \le j \le 2\eta$, $0 \le k_i \le \alpha_{n,i} \le h_i \le 1$, we have

$$
\begin{aligned}
&(1-h_i)(1-\xi)g(2\eta - j)\left\|Et_n - Ep\right\|^2 \\
&\quad \le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|x_n - p\right\|^2 - \left\|y_{n,i} - p\right\|^2 \\
&\qquad + (1-\alpha_{n,i})\xi_n s_n\left(s_n - 2\beta\right)\left\|Bx_n - Bp\right\|^2 \\
&\qquad + (1-\alpha_{n,i})(1-\xi_n)r_n(r_n - 2\alpha)\left\|Ax_n - Ap\right\|^2 \\
&\quad \le \alpha_{n,i}\left\|x_0 - p\right\|^2 + \left\|y_{n,i} - x_n\right\|\left(\left\|x_n - p\right\| - \left\|y_{n,i} - p\right\|\right) \\
&\qquad + (1-\alpha_{n,i})\xi_n s_n\left(s_n - 2\beta\right)\left\|Bx_n - Bp\right\|^2 \\
&\qquad + (1-\alpha_{n,i})(1-\xi_n)r_n(r_n - 2\alpha)\left\|Ax_n - Ap\right\|^2.
\end{aligned}
\tag{3.33}
$$

By conditions (i)–(iv), (vi), (3.18), $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$, and $\lim_{n\to\infty}\|Bx_n - Bp\| = 0$, then $\lim_{n\to\infty}\|Et_n - Ep\| = 0$. From (3.1), we have

$$
\begin{aligned}
\left\|w_n - p\right\|^2 &\le \Big\|\xi_n\left\{P_C(u_n - \lambda_n Du_n) - P_C\left(p - \lambda_n Dp\right)\right\} \\
&\qquad + (1-\xi_n)\left\{P_C\left(t_n - \mu_n Et_n\right) - P_C\left(p - \mu_n Ep\right)\right\}\Big\|^2 \\
&\le \xi_n\left\|u_n' - p\right\|^2 + (1-\xi_n)\left\|t_n' - p\right\|^2.
\end{aligned}
\tag{3.34}
$$

Assume that $u_n' = P_C(u_n - \lambda_n Du_n)$ and $t_n' = P_C(t_n - \mu_n Et_n)$. By nonexpansiveness of $I - \lambda_n D$ and $I - \mu_n E$, we also have

$$
\begin{aligned}
\left\|u_n' - p\right\|^2 &\le \left\|P_C(I-\lambda_n D)u_n - P_C(I-\lambda_n D)p\right\|^2 \\
&\le \left\langle (u_n - \lambda_n Du_n) - (p - \lambda_n Dp), u_n' - p\right\rangle \\
&= \frac{1}{2}\Big\{\left\|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\right\|^2 + \left\|u_n' - p\right\|^2 \\
&\qquad - \left\|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp) - (u_n' - p)\right\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\left\|u_n - p\right\|^2 + \left\|u_n' - p\right\|^2 - \left\|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp) - (u_n' - p)\right\|^2\Big\} \\
&= \frac{1}{2}\Big\{\left\|T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n Bx_n) - T_{s_n}^{(F_2,\varphi_2)}(p - s_n Bp)\right\|^2 + \left\|u_n' - p\right\|^2 - \left\|u_n - u_n'\right\|^2 \\
&\qquad + 2\lambda_n\left\langle u_n - u_n', Du_n - Dp\right\rangle - \lambda_n^2\left\|Du_n - Dp\right\|^2\Big\} \\
&\le \frac{1}{2}\Big\{\left\|x_n - p\right\|^2 + \left\|u_n' - p\right\|^2 - \left\|u_n - u_n'\right\|^2 + \lambda_n(\lambda_n - 2\delta)\left\|Du_n - Dp\right\|^2\Big\}.
\end{aligned}
\tag{3.35}
$$

It follows that

$$\|u_n' - p\|^2 \le \|x_n - p\|^2 - \|u_n - u_n'\|^2 + \lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2. \tag{3.36}$$

Similar to (3.36), we obtain

$$\|t_n' - p\|^2 \le \|x_n - p\|^2 - \|t_n - t_n'\|^2 + \mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2. \tag{3.37}$$

Substituting (3.36), (3.37) into (3.34), we have

$$\begin{aligned}
\|w_n - p\|^2 &\le \xi_n\|u_n' - p\|^2 + (1-\xi_n)\|t_n' - p\|^2 \\
&\le \xi_n\left\{\|x_n - p\|^2 - \|u_n - u_n'\|^2 + \lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2\right\} \\
&\quad + (1-\xi_n)\left\{\|x_n - p\|^2 - \|t_n - t_n'\|^2 + \mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2\right\} \\
&\le \|x_n - p\|^2 - \xi_n\|u_n - u_n'\|^2 + \xi_n\lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2 \\
&\quad - (1-\xi_n)\|t_n - t_n'\|^2 + (1-\xi_n)\mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2.
\end{aligned} \tag{3.38}$$

By (3.38), we have

$$\begin{aligned}
&\|y_{n,i} - p\|^2 \\
&\quad \le \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|T_i w_n - p\|^2 \\
&\quad = \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i})\|w_n - p\|^2 \\
&\quad = \alpha_{n,i}\|x_0 - p\|^2 + (1-\alpha_{n,i}) \\
&\qquad \times \left\{\|x_n - p\|^2 - \xi_n\|u_n - u_n'\|^2 + \xi_n\lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2\right. \\
&\qquad\quad \left. -(1-\xi_n)\|t_n - t_n'\|^2 + (1-\xi_n)\mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2\right\} \\
&\quad = \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - (1-\alpha_{n,i})\xi_n\|u_n - u_n'\|^2 \\
&\qquad + (1-\alpha_{n,i})\xi_n\lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2 - (1-\alpha_{n,i})(1-\xi_n)\|t_n - t_n'\|^2 \\
&\qquad + (1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2 \\
&\quad = \alpha_{n,i}\|x_0 - p\|^2 + \|x_n - p\|^2 - (1-\alpha_{n,i})\xi_n\|u_n - u_n'\|^2 \\
&\qquad + (1-\alpha_{n,i})\xi_n\lambda_n(\lambda_n - 2\delta)\|Du_n - Dp\|^2 + (1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n - 2\eta)\|Et_n - Ep\|^2.
\end{aligned} \tag{3.39}$$

It follows that

$$\begin{aligned}(1-\alpha_{n,i})\xi_n\left\|u_n-u_n'\right\|^2 &\leq \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-p\right\|^2-\left\|y_{n,i}-p\right\|^2+(1-\alpha_{n,i})\xi_n\lambda_n\\ &\quad\times(\lambda_n-2\delta)\left\|Du_n-Dp\right\|^2+(1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n-2\eta)\left\|Et_n-Ep\right\|^2\\ &\leq \alpha_{n,i}\left\|x_0-p\right\|^2+\left\|x_n-y_{n,i}\right\|\left(\left\|x_n-p\right\|+\left\|y_{n,i}-p\right\|\right)+(1-\alpha_{n,i})\xi_n\lambda_n\\ &\quad\times(\lambda_n-2\delta)\left\|Du_n-Dp\right\|^2+(1-\alpha_{n,i})(1-\xi_n)\mu_n(\mu_n-2\eta)\left\|Et_n-Ep\right\|^2.\end{aligned} \tag{3.40}$$

By conditions (iii)–(vi), (3.18), $\lim_{n\to\infty}\|Du_n-Dp\|=0$ and $\lim_{n\to\infty}\|Et_n-Ep\|=0$, we get

$$\lim_{n\to\infty}\left\|u_n-u_n'\right\|=0. \tag{3.41}$$

By using (3.41), we can prove that

$$\lim_{n\to\infty}\left\|t_n-t_n'\right\|=0. \tag{3.42}$$

Applying (3.27) and (3.41), we also have

$$\lim_{n\to\infty}\left\|x_n-u_n'\right\|=0. \tag{3.43}$$

From (3.29) and (3.42), we obtain

$$\lim_{n\to\infty}\left\|x_n-t_n'\right\|=0. \tag{3.44}$$

Since $u_n'=P_C(u_n-\lambda_nDu_n)$ and $t_n'=P_C(t_n-\mu_nEt_n)$, we have

$$w_n-x_n=\xi_n\left(u_n'-x_n\right)+(1-\xi_n)\left(t_n'-x_n\right). \tag{3.45}$$

By (3.43) and (3.44), we obtain

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{3.46}$$

By condition (iii), we have $y_{n,i}=\alpha_{n,i}x_0+(1-\alpha_{n,i})T_iw_n$, which implies that

$$\left\|y_{n,i}-T_iw_n\right\|=\alpha_{n,i}\|x_0-T_iw_n\|\longrightarrow 0,\quad n\longrightarrow\infty,\ \forall i\geq 1. \tag{3.47}$$

From (3.18) and $\lim_{n\to\infty}\|y_{n,i}-T_iw_n\|=0$, we have

$$\|x_n-T_iw_n\|\leq\left\|y_{n,i}-T_iw_n\right\|+\left\|y_{n,i}-x_n\right\|\longrightarrow 0,\quad n\longrightarrow\infty,\ \forall i\geq 1. \tag{3.48}$$

Since

$$\|w_n - T_i w_n\| \leq \|w_n - x_n\| + \|x_n - T_i w_n\|. \tag{3.49}$$

By (3.46) and (3.48), we have that $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0$, for all $i = 1, 2, \ldots$.

Step 4. We show that $z \in \Theta := (\bigcap_{i=1}^{\infty} F(T_i)) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap \mathrm{VI}(C, D) \cap \mathrm{VI}(C, E)$.

First, we show that $z \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $z \notin \bigcap_{i=1}^{\infty} F(T_i)$. Since $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z\| = 0$, we have that $\lim_{n\to\infty}\|w_n - z\| = 0$. By $\lim_{n\to\infty}\|w_n - z\| = 0$ and $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0$, $i = 1, 2, \ldots$, from Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|w_{n_i} - z\| &< \liminf_{i\to\infty}\|w_{n_i} - T_i z\| \\
&\leq \liminf_{i\to\infty}\left(\|w_{n_i} - T_i w_{n_i}\| + \|T_i w_{n_i} - T_i z\|\right) \\
&\leq \liminf_{i\to\infty}\|w_{n_i} - z\|,
\end{aligned} \tag{3.50}$$

which is a contradiction. Thus, we obtain $z \in \bigcap_{i=1}^{\infty} F(T_i)$.

Next, we show that $z \in \mathrm{GMEP}(F_1, \varphi, A)$. Since $t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n)$, $n \geq 1$, we have for any $y \in C$ that

$$F_1(t_n, y) + \varphi_1(y) - \varphi_1(t_n) + \langle Ax_n, y - t_n\rangle + \frac{1}{r_n}\langle y - t_n, t_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.51}$$

From (A2), we also have

$$\varphi_1(y) - \varphi_1(t_n) + \langle Ax_n, y - t_n\rangle + \frac{1}{r_n}\langle y - t_n, t_n - x_n\rangle \geq F_1(y, t_n), \quad \forall y \in C. \tag{3.52}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$. Then, we have

$$\begin{aligned}
\langle y_t - t_{n_i}, Ay_t\rangle &\geq \langle y_t - t_{n_i}, Ay_t\rangle - \varphi_1(y_t) + \varphi_1(t_{n_i}) - \langle y_t - t_{n_i}, Ax_{n_i}\rangle \\
&\quad - \left\langle y_t - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F_1(y_t, t_{n_i}) \\
&= \langle y_t - t_{n_i}, Ay_t - At_{n_i}\rangle + \langle y_t - t_{n_i}, At_{n_i} - Ax_{n_i}\rangle - \varphi_1(y_t) + \varphi_1(t_{n_i}) \\
&\quad - \left\langle y_t - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle + F_1(y_t, t_{n_i}).
\end{aligned} \tag{3.53}$$

Since $\|t_{n_i} - x_{n_i}\| \to 0$, we have $\|At_{n_i} - Ax_{n_i}\| \to 0$. Further, from an inverse-strongly monotonicity of A, we have $\langle y_t - t_{n_i}, Ay_t - At_{n_i}\rangle \geq 0$. So, from (A4) and the weak lower semicontinuity of φ_1, $(t_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $t_{n_i} \rightharpoonup z$, we have at the limit

$$\langle y_t - z, Ay_t\rangle \geq -\varphi_1(y_t) + \varphi_1(z) + F_1(y_t, z), \tag{3.54}$$

as $i \to \infty$. From (A1), (A4), and (3.54), we also get

$$\begin{aligned}
0 &= F_1(y_t, y_t) + \varphi_1(y_t) - \varphi_1(y_t)\\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, z) + t\varphi_1(y) - (1-t)\varphi_1(z) - \varphi(y_t)\\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)[F_1(y_t, z) + \varphi_1(z) - \varphi_1(y_t)]\\
&\leq t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)\langle y_t - z, Ay_t\rangle\\
&= t[F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t)] + (1-t)t\langle y - z, Ay_t\rangle,\\
0 &\leq F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t) + (1-t)\langle y - z, Ay_t\rangle.
\end{aligned} \tag{3.55}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F_1(z, y) + \varphi_1(y) - \varphi_1(z) + \langle y - z, Az\rangle \geq 0. \tag{3.56}$$

This implies that $z \in \mathrm{GMEP}(F_1, \varphi_1, A)$. By the same arguments, we can show that $z \in \mathrm{GMEP}(F_2, \varphi_2, B)$.

Lastly, by the same proof of [39, Theorem 3.1, pages 346-347], we can show that $z \in \mathrm{VI}(C, D)$ and $z \in \mathrm{VI}(C, E)$. Therefore, $z \in (\bigcap_{i=1}^{\infty} F(T_i)) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap \mathrm{VI}(C, D) \cap \mathrm{VI}(C, E)$; that is, $z \in \Theta$.

Noting that since $x_n = P_{C_n}x_0$, by(2.4), we have

$$\langle x_0 - x_n, y - x_n\rangle \leq 0, \quad \forall y \in C_n. \tag{3.57}$$

Since $\Theta \subset C_n$ and by the continuity of inner product, we obtain from the above inequality that

$$\langle x_0 - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{3.58}$$

By (2.4), again, we conclude that $z = P_{\Theta}x_0$. This completes the proof. □

Using Theorem 3.1, we obtain the following corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A, B, D, and E be α, β, δ, and η-inverse-strongly monotone mapping of C into H, respectively. Let T be nonexpansive mapping such that*

$\Theta := F(T) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap \mathrm{VI}(C, D) \cap \mathrm{VI}(C, E) \neq \emptyset$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$, $x_1 = P_{C_1}x_0$ *and*

$$\begin{aligned}
t_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n),\\
u_n &= T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n),\\
w_n &= \xi_n P_C(u_n - \lambda_n D u_n) + (1-\xi_n) P_C\big(t_n - \mu_n E t_n\big),\\
y_n &= \alpha_n x_0 + (1-\alpha_n) T w_n,\\
C_{n+1} &= \left\{ z \in C_n : \left\| y_n - z \right\|^2 \le \|x_n - z\|^2 + \alpha_n \left(\|x_0\|^2 + 2\langle w_n - x_0, z\rangle \right) \right\},\\
x_{n+1} &= P_{C_{n+1}} x_0,
\end{aligned} \tag{3.59}$$

for every $n \ge 0$, *where* $\{r_n\}, \{s_n\} \subset (0, \infty)$, $\lambda_n \in (0, 2\delta)$, *and* $\mu_n \in (0, 2\eta)$ *satisfy the following conditions:*

(i) $0 < a \le r_n \le b < 2\alpha$,

(ii) $0 < c \le s_n \le d < 2\beta$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$,

(iv) $\lim_{n\to\infty} \xi_n = \xi \in (0, 1)$,

(v) $0 < e \le \lambda_n \le f < 2\delta$,

(vi) $0 < g \le \mu_n \le j < 2\eta$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. Taking $T_i = T$ for $i = 1, 2, \ldots$, to be nonexpansive mappings, in Theorem 3.1, we can conclude the desired conclusion easily. This completes the proof. □

A mapping $T : C \to C$ is said to be a κ-*strict pseudocontraction* [40] if there exists a constant $0 \le \kappa < 1$ such that

$$\left\| Tx - Ty \right\|^2 \le \left\| x - y \right\|^2 + \kappa \left\| (I-T)x - (I-T)y \right\|^2, \quad \forall x, y \in C, \tag{3.60}$$

where I denotes the identity operator on C.

Lemma 3.3 (see [41]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *and let* $T : C \to C$ *be a* κ-*strict pseudocontraction. Define* $Sx : C \to C$ *by* $Sx = \alpha x + (1-\alpha)Tx$ *for each* $x \in C$. *Then, as* $\alpha \in [\kappa, 1)$ S *is nonexpansive such that* $F(S) = F(T)$.

Using Theorem 3.1, we obtain the following result.

Theorem 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F_1, F_2 *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A4), and let* $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A, B, D, *and* E *be* α, β, δ, *and* η-*inverse-strongly monotone mapping of* C *into* H, *respectively. Let* $T_1, T_2, \ldots, T_N$ *be a finite family of* κ_i-*psuedocontractions such that* $\Theta := \bigcap_{i=1}^N F(T_i) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap \mathrm{VI}(C, D) \cap \mathrm{VI}(C, E) \neq \emptyset$. *Define a mapping* T_{κ_i} *by* $T_{\kappa_i} = \kappa_i x + (1-\kappa_i) T_i x$ *for all* $x \in C$, $i \in \{1, 2, \ldots, N\}$. *Let*

S_n be the S-mappings generated by $T_{\kappa_1}, T_{\kappa_2}, \ldots, T_{\kappa_N}$ and $\alpha_{n,i}^1, \alpha_{n,i}^2, \ldots, \alpha_{n,i}^N$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1}x_0$ and

$$\begin{gathered}
t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
u_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n), \\
w_n = \xi_n P_C(u_n - \lambda_n D u_n) + (1-\xi_n) P_C(t_n - \mu_n E t_n), \\
y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) T_i w_n, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle w_n - x_0, z\rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered} \tag{3.61}$$

for every $n \ge 0$, where $\{r_n\}, \{s_n\} \subset (0,\infty)$, $\lambda_n \in (0, 2\delta)$, and $\mu_n \in (0, 2\eta)$ satisfy the following conditions:

(i) $0 < a \le r_n \le b < 2\alpha$,

(ii) $0 < c \le s_n \le d < 2\beta$,

(iii) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(iv) $\lim_{n\to\infty} \xi_n = \xi \in (0,1)$,

(v) $0 < e \le \lambda_n \le f < 2\delta$,

(vi) $0 < g \le \mu_n \le j < 2\eta$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

Proof. From Theorem 3.1, $\{T_i\}_{i=1}^N$ is a finite family of κ_i-strict pseudocontraction. By Lemma 3.3, we have that T_{κ_i} is nonexpansive mappings. The conclusion of Theorem 3.4 can be obtained from Theorem 3.1 immediately. □

4. Some Applications

4.1. Complementarity Problem

Let C be a nonempty closed and convex cone in H, and let E be an operator of C into H. We define the *polar* of C in H to be the set

$$K^* := \{ y^* \in H : \langle x, y^* \rangle \ge 0, \ \forall x \in C \}. \tag{4.1}$$

Then, the element $u \in C$ is called a solution of the *complementarity problem* if

$$Eu \in K^*, \quad \langle u, Eu \rangle = 0. \tag{4.2}$$

The set of solution of the complementarity problem is denoted by $C'(C,D)$, $C'(C,E)$. We will assume that D, E satisfies the following conditions:

(E1) D, E is a δ, η-inverse-strongly monotone mapping, respectively,

(E2) $C'(C,D),\ C'(C,E) \neq \emptyset$.

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C \cap \operatorname{dom}(\varphi)$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z), \tag{4.3}$$

(B2) C is a bounded set.

Corollary 4.1. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A, B, D, and E be α, β, δ, and η-inverse-strongly monotone mapping of C into H, respectively. Let $T_1, T_2, \ldots$ be infinite nonexpansive mapping such that $\Theta := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, \varphi_1, A) \cap \mathrm{GMEP}(F_2, \varphi_2, B) \cap C'(C,D) \cap C'(C,E) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$ and*

$$\begin{gathered}
t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
u_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n), \\
w_n = \xi_n P_C(u_n - \lambda_n D u_n) + (1 - \xi_n) P_C(t_n - \mu_n E t_n), \\
y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) T_i w_n, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle w_n - x_0, z\rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered} \tag{4.4}$$

for every $n \ge 0$, where $\{r_n\}, \{s_n\} \subset (0,\infty)$, $\lambda_n \in (0, 2\delta)$, and $\mu_n \in (0, 2\eta)$ satisfy the following conditions:

(i) $0 < a \le r_n \le b < 2\alpha$,

(ii) $0 < c \le s_n \le d < 2\beta$,

(iii) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(iv) $\lim_{n\to\infty} \xi_n = \xi \in (0,1)$,

(v) $0 < e \le \lambda_n \le f < 2\delta$,

(vi) $0 < g \le \mu_n \le j < 2\eta$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

Proof. Using Lemma 7.1.1 of [42], we have that $\mathrm{VI}(C,D) = C'(C,D)$ and $\mathrm{VI}(C,E) = C'(C,E)$. Hence, by Corollary 4.1, we can conclude the desired conclusion easily. This completes the proof. □

4.2. Optimization Problem

In this section, we study a kind of multiobjective optimization problem by using the result of this paper. We will give an iterative algorithm of solution for the following *optimization problem* with nonempty set of solutions:

$$\begin{aligned} &\min h_1(x), \\ &\min h_2(x), \\ &x \in C, \end{aligned} \tag{4.5}$$

where $h(x)$ is a convex and lower semicontinuous functional, and define C as a closed convex subset of a real Hilbert space H. We denote the set of solutions of (4.5) by $M(h_1)$ and $M(h_2)$. Let $F_i : C \times C \to \mathbb{R}$ be a bifunction defined by $F_i(x,y) = h_i(y) - h_i(x)$. We consider the equilibrium problem, it is obvious that $\mathrm{EP}(F_i) = M(h_i)$, $i = 1,2$. Therefore, from Theorem 3.1, we obtained the following corollary.

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F_1, F_2 be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A, B, D, and E be α, β, δ, and η-inverse-strongly monotone mapping of C into H, respectively. Let $T_1, T_2, \ldots$ be an infinite nonexpansive mapping such that $\Theta := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{MEP}(F_1,\varphi_1) \cap \mathrm{MEP}(F_2,\varphi_2) \cap \mathrm{VI}(C,D) \cap \mathrm{VI}(C,E) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$, and*

$$\begin{gathered} h_1(t) - h_1(t_n) + \frac{1}{r_n}\langle t - t_n, t_n - x_n\rangle \geq 0, \quad \forall t \in C, \\ h_2(u) - h_2(u_n) + \frac{1}{s_n}\langle u - u_n, u_n - t_n\rangle \geq 0, \quad \forall u \in C, \\ w_n = \xi_n P_C(u_n - \lambda_n D u_n) + (1-\xi_n) P_C(t_n - \mu_n E t_n), \\ y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) T_i w_n, \\ C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle w_n - x_0, z\rangle \right) \right\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{gathered} \tag{4.6}$$

for every $n \geq 0$, where $\{r_n\}, \{s_n\} \subset (0, \infty)$, $\lambda_n \in (0, 2\delta)$, and $\mu_n \in (0, 2\eta)$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(ii) $\lim_{n\to\infty} \xi_n = \xi \in (0, 1)$,

(iii) $0 < e \leq \lambda_n \leq f < 2\delta$,

(iv) $0 < g \leq \mu_n \leq j < 2\eta$.

Then, $\{x_n\}$ converges strongly to $P_\Theta x_0$.

Proof. From Theorem 3.1, put $F_1(t_n, t) = h_1(t) - h_1(t_n)$, $F_2(u_n, u) = h_2(u) - h_2(u_n)$, and $A, B, \varphi_1, \varphi_2 \equiv 0$. The conclusion of Corollary 4.2 can be obtained from Theorem 3.1 immediately. □

4.3. Minimization Problem

In this section, we study the problem for finding a minimizer of a continuously Frèchet differentiable convex functional in a Hilbert space.

First, we use the following lemma in our result.

Lemma 4.3 (see [43]). *Let E be a Banach space, let f be a continuously Frèchet differentiable convex functional on E, and let ∇f be the gradient of f. If ∇f is $(1/\alpha)$-Lipschitz continuous, then ∇f is an α-inverse-strongly monotone.*

Let f_1, f_2 be functionals on H which satisfies the following conditions:

(C1) let, f_1, f_2 be a continuously Frèchet differentiable convex functional on H, and let, ∇f_1, ∇f_2 be $(1/\delta)$, $(1/\eta)$-Lipschitz continuous, respectively,

(C2) $(\nabla f_1)^{-1}0 = \{z_1 \in H : f_1(z_1) = \min_{y_1 \in H} f_1(y_1)\} \neq \emptyset$ and $(\nabla f_2)^{-1}0 = \{z_2 \in H : f_2(z_2) = \min_{y_2 \in H} f_2(y_2)\} \neq \emptyset$.

Corollary 4.4. *Let H be a real Hilbert Space. Let F_1, F_2 be a bifunction of $H \times H$ into real numbers $\mathbb{R}$ satisfying (A1)–(A4), and let $\varphi_1, \varphi_2 : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A, B be α, β-inverse-strongly monotone mapping of H into H, respectively. Let $T_1, T_2, \ldots$ be infinite nonexpansive mappings. Let f_1, f_2 be functionals on H which satisfies the conditions*

(C1) and (C2). Suppose that $\Theta := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, \varphi_1) \cap \mathrm{GMEP}(F_2, \varphi_2) \cap (\nabla f_1)^{-1}0 \cap (\nabla f_2)^{-1}0 \neq \emptyset$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$, *and*

$$\begin{gathered}
t_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n), \\
u_n = T_{s_n}^{(F_2,\varphi_2)}(x_n - s_n B x_n), \\
w_n = \xi_n\big(u_n - \lambda_n \nabla f_1(u_n)\big) + (1-\xi_n)\big(t_n - \mu_n \nabla f_2(t_n)\big), \\
y_{n,i} = \alpha_{n,i} x_0 + (1-\alpha_{n,i}) T_i w_n, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}\Big(\|x_0\|^2 + 2\langle w_n - x_0, z\rangle \Big) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0,
\end{gathered} \tag{4.7}$$

for every $n \geq 0$, *where* $\{r_n\}, \{s_n\} \subset (0, \infty)$, $\lambda_n \in (0, 2\delta)$, *and* $\mu_n \in (0, 2\eta)$ *satisfying the following conditions:*

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $0 < c \leq s_n \leq d < 2\beta$,

(iii) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(iv) $\lim_{n\to\infty} \xi_n = \xi \in (0,1)$,

(v) $0 < e \leq \lambda_n \leq f < 2\delta$,

(vi) $0 < g \leq \mu_n \leq j < 2\eta$.

Then, $\{x_n\}$ *converges strongly to* $P_\Theta x_0$.

Proof. We know form condition (C1) and Lemma 4.3 that ∇f_1, ∇f_2 is a δ, η-inverse-strongly monotone operator from H in to itself, respectively. The conclusion of Corollary 4.4 can be obtained from Theorem 3.1 immediately. □

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