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Research Article

Strong Convergence of Modified Halpern Iterations in CAT(0) Spaces

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Strong convergence theorems are established for the modified Halpern iterations of nonexpansive mappings in CAT(0) spaces. Our results extend and improve the recent ones announced by Kim and Xu (2005), Hu (2008), Song and Chen (2008), Saejung (2010), and many others.

1. Introduction

Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to C$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C. \tag{1.1}$$

A point $x \in C$ is called a fixed point of T if $x = Tx$. We will denote by $F(T)$ the set of fixed points of T. In 1967, Halpern [1] introduced an explicit iterative scheme for a nonexpansive mapping T on a subset C of a Hilbert space by taking any points $u, x_1 \in C$ and defined the iterative sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \text{for } n \geq 1, \tag{1.2}$$

where $\alpha_n \in [0, 1]$. He pointed out that the control conditions: (C1) $\lim_n \alpha_n = 0$ and (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the convergence of $\{x_n\}$ to a fixed point of T. Subsequently, many mathematicians worked on the Halpern iterations both in Hilbert and Banach spaces (see, e.g., [2–11] and the references therein). Among other things, Wittmann [7] proved strong convergence of the Halpern iteration under the control conditions (C1), (C2), and (C4) $\sum_{n=1}^{\infty} |\alpha_{n+1}-\alpha_n| < \infty$ in a Hilbert space. In 2005, Kim and Xu [12] generalized Wittmann's result by introducing a modified Halpern iteration in a Banach space as follows. Let C be a closed convex subset of a uniformly smooth Banach space X, and let $T : C \to C$ be a nonexpansive mapping. For any points $u, x_1 \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \beta_n u + (1-\beta_n) T(\alpha_n x_n + (1-\alpha_n) T x_n), \quad \text{for } n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. They proved under the following control conditions:

$$\begin{aligned} &(D1) \quad \lim_n \alpha_n = 0, \qquad \lim_n \beta_n = 0, \\ &(D2) \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} \beta_n = \infty, \\ &(D3) \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \qquad \sum_{n=1}^{\infty} \left|\beta_{n+1} - \beta_n\right| < \infty, \end{aligned} \tag{1.4}$$

that the sequence $\{x_n\}$ converges strongly to a fixed point of T.

The purpose of this paper is to extend Kim-Xu's result to a special kind of metric spaces, namely, CAT(0) spaces. We also prove a strong convergence theorem for another kind of modified Halpern iteration defined by Hu [13] in this setting.

2. CAT(0) Spaces

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [14]), $\mathbb{R}$-trees (see [15]), Euclidean buildings (see [16]), the complex Hilbert ball with a hyperbolic metric (see [17]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [14].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [18, 19]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [20–31] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology, and computer science (see, e.g., [15, 32–35]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\Delta\,(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points $x_1, x_2,$ and x_3 in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): let Δ be a geodesic triangle in X, and let $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Let $x, y \in X$, and by Lemma 2.1 (iv) of [23] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y), \qquad d(y, z) = (1 - t)d(x, y). \tag{2.2}$$

From now on, we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (2.2). We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.1. *Let X be a* CAT(0) *space. Then,*

(i) *(see [23, Lemma 2.4]) for each $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z), \tag{2.3}$$

(ii) *(see [21]) for each $x, y \in X$ and $t, s \in [0, 1]$, one has*

$$d((1 - t)x \oplus ty, (1 - s)x \oplus sy) = |t - s|d(x, y), \tag{2.4}$$

(iii) *(see [19, Lemma 3]) for each $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d((1 - t)z \oplus tx, (1 - t)z \oplus ty) \leq td(x, y), \tag{2.5}$$

(iv) *(see [23, Lemma 2.5]) for each $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2. \tag{2.6}$$

Recall that a continuous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a *Banach limit* if $\|\mu\| = \mu(1, 1, \ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 2.2 (see [8, Proposition 2]). *Let* $\{a_1, a_2, \ldots\} \in \ell_\infty$ *be such that* $\mu_n(a_n) \leq 0$ *for all Banach limits* μ *and* $\limsup_n(a_{n+1} - a_n) \leq 0$. *Then,* $\limsup_n a_n \leq 0$.

Lemma 2.3 (see [28, Lemma 2.1]). *Let* C *be a closed convex subset of a complete* CAT(0) *space* X, *and let* $T : C \to C$ *be a nonexpansive mapping. Let* $u \in C$ *be fixed. For each* $t \in (0,1)$, *the mapping* $S_t : C \to C$ *defined by*

$$S_t z = tu \oplus (1-t)Tz, \quad \text{for } z \in C \tag{2.7}$$

has a unique fixed point $z_t \in C$, *that is,*

$$z_t = S_t(z_t) = tu \oplus (1-t)T(z_t). \tag{2.8}$$

Lemma 2.4 (see [28, Lemma 2.2]). *Let* C *and* T *be as the preceding lemma. Then,* $F(T) \neq \emptyset$ *if and only if* $\{z_t\}$ *given by* (2.8) *remains bounded as* $t \to 0$. *In this case, the following statements hold:*

(1) $\{z_t\}$ *converges to the unique fixed point* z *of* T *which is nearest* u,

(2) $d^2(u,z) \leq \mu_n d^2(u,x_n)$ *for all Banach limits* μ *and all bounded sequences* $\{x_n\}$ *with* $\lim_n d(x_n, Tx_n) = 0$.

Lemma 2.5 (see [10, Lemma 2.1]). *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 1, \tag{2.9}$$

where $\{\gamma_n\}$ *and* $\{\sigma_n\}$ *are sequences of real numbers such that*

(1) $\{\gamma_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(2) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then, $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.6 (see [27, 36]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a* CAT(0) *space* X, *and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. *Suppose that* $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)x_n$ *for all* $n \in \mathbb{N}$ *and*

$$\limsup_{n\to\infty}\left(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)\right) \leq 0. \tag{2.10}$$

Then, $\lim_n d(x_n, y_n) = 0$.

3. Main Results

The following result is an analog of Theorem 1 of Kim and Xu [12]. They prove the theorem by using the concept of duality mapping, while we use the concept of Banach limit. We also observe that the condition $\sum_{n=1}^{\infty} \alpha_n = \infty$ in [12, Theorem 1] is superfluous.

Theorem 3.1. *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$, the following conditions are satisfied:*

(A1) $\lim_n \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(A2) $\lim_n \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ arbitrarily, and

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n)(\alpha_n x_n \oplus (1 - \alpha_n) T x_n), \quad \forall n \geq 1. \tag{3.1}$$

Then, $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest u.

Proof. For each $n \geq 1$, we let $y_n := \alpha_n x_n \oplus (1 - \alpha_n) T x_n$. We divide the proof into 3 steps. (i) We will show that $\{x_n\}$, $\{y_n\}$, and $\{Tx_n\}$ are bounded sequences. (ii) We show that $\lim_n d(x_n, Tx_n) = 0$. Finally, we show that (iii) $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest u.

(i) As in the first part of the proof of [12, Theorem 1], we can show that $\{x_n\}$ is bounded and so is $\{y_n\}$ and $\{Tx_n\}$. Notice also that

$$d(y_n, p) \leq d(x_n, p), \quad \forall p \in F(T). \tag{3.2}$$

(ii) It suffices to show that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{3.3}$$

Indeed, if (3.3) holds, we obtain

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + d(y_n, Tx_n) \\ &= d(x_n, x_{n+1}) + d(\beta_n u \oplus (1 - \beta_n) y_n, y_n) + d(\alpha_n x_n \oplus (1 - \alpha_n) T x_n, T x_n) \\ &\leq d(x_n, x_{n+1}) + \beta_n d(u, y_n) + \alpha_n d(x_n, Tx_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.4}$$

By using Lemma 2.1, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\beta_n u \oplus (1 - \beta_n) y_n, \beta_{n-1} u \oplus (1 - \beta_{n-1}) y_{n-1}) \\ &\leq d(\beta_n u \oplus (1 - \beta_n) y_n, \beta_n u \oplus (1 - \beta_n) y_{n-1}) \\ &\quad + d(\beta_n u \oplus (1 - \beta_n) y_{n-1}, \beta_{n-1} u \oplus (1 - \beta_{n-1}) y_{n-1}) \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\beta_n)d(y_n, y_{n-1}) + \left|\beta_n - \beta_{n-1}\right| d(u, y_{n-1}) \\
&= (1-\beta_n)d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}) \\
&\quad + \left|\beta_n - \beta_{n-1}\right| d(u, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1}) \\
&\leq (1-\beta_n)\left[d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, \alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_n)\right. \\
&\qquad + d(\alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_n, \alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_{n-1}) \\
&\qquad \left. + d(\alpha_n x_{n-1} \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1}x_{n-1} \oplus (1-\alpha_{n-1})Tx_{n-1})\right] \\
&\quad + \left|\beta_n - \beta_{n-1}\right| \left[\alpha_{n-1}d(u, x_{n-1}) + (1-\alpha_{n-1})d(u, Tx_{n-1})\right] \\
&\leq (1-\beta_n)\left[\alpha_n d(x_n, x_{n-1}) + (1-\alpha_n)d(Tx_n, Tx_{n-1}) + |\alpha_n - \alpha_{n-1}| d(x_{n-1}, Tx_{n-1})\right] \\
&\quad + \left|\beta_n - \beta_{n-1}\right| \left[\alpha_{n-1}d(u, x_{n-1}) + (1-\alpha_{n-1})d(u, Tx_{n-1})\right] \\
&= (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}| d(x_{n-1}, Tx_{n-1}) \\
&\quad + \left|\beta_n - \beta_{n-1}\right| \alpha_{n-1}d(u, x_{n-1}) + \left|\beta_n - \beta_{n-1}\right| (1-\alpha_{n-1})d(u, Tx_{n-1}) \\
&\leq (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}| d(x_{n-1}, Tx_{n-1}) \\
&\quad + \left|\beta_n - \beta_{n-1}\right| \alpha_{n-1}\left[d(u, Tx_{n-1}) + d(Tx_{n-1}, x_{n-1})\right] \\
&\quad + \left|\beta_n - \beta_{n-1}\right| d(u, Tx_{n-1}) - \left|\beta_n - \beta_{n-1}\right| \alpha_{n-1}d(u, Tx_{n-1}) \\
&= (1-\beta_n)d(x_n, x_{n-1}) + (1-\beta_n)|\alpha_n - \alpha_{n-1}| d(x_{n-1}, Tx_{n-1}) \\
&\quad + \left|\beta_n - \beta_{n-1}\right| \alpha_{n-1}d(x_{n-1}, Tx_{n-1}) + \left|\beta_n - \beta_{n-1}\right| d(u, Tx_{n-1}).
\end{aligned} \tag{3.5}
$$

Hence,

$$
d(x_{n+1}, x_n) \leq (1-\beta_n)d(x_n, x_{n-1}) + \gamma\left(|\alpha_n - \alpha_{n-1}| + 2\left|\beta_n - \beta_{n-1}\right|\right), \tag{3.6}
$$

where $\gamma > 0$ is a constant such that $\gamma \geq \max\{d(u, Tx_{n-1}), d(x_{n-1}, Tx_{n-1})\}$ for all $n \in \mathbb{N}$. By assumptions, we have

$$
\lim_{n\to\infty} \beta_n = 0, \qquad \sum_{n=1}^{\infty} \beta_n = \infty, \qquad \sum_{n=1}^{\infty} \left(|\alpha_n - \alpha_{n-1}| + 2\left|\beta_n - \beta_{n-1}\right|\right) < \infty. \tag{3.7}
$$

Hence, Lemma 2.5 is applicable to (3.6), and we obtain $\lim_n d(x_{n+1}, x_n) = 0$.

(iii) From Lemma 2.3, let $z = \lim_{t\to 0} z_t$, where z_t is given by (2.8). Then, z is the point of $F(T)$ which is nearest u. We observe that

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\beta_n u \oplus (1-\beta_n) y_n, z) \\ &\le \beta_n d^2(u,z) + (1-\beta_n) d^2(y_n, z) - \beta_n(1-\beta_n) d^2(u, y_n) \\ &\le \beta_n d^2(u,z) + (1-\beta_n) d^2(x_n, z) - \beta_n(1-\beta_n) d^2(u, y_n) \\ &= (1-\beta_n) d^2(x_n, z) + \beta_n \left[d^2(u,z) - (1-\beta_n) d^2(u, y_n) \right]. \end{aligned} \tag{3.8}$$

By Lemma 2.4, we have $\mu_n(d^2(u,z) - d^2(u,x_n)) \le 0$ for all Banach limit μ. Moreover, since $\lim_n d(x_{n+1}, x_n) = 0$,

$$\limsup_{n\to\infty} \left[\left(d^2(u,z) - d^2(u, x_{n+1}) \right) - \left(d^2(u,z) - d^2(u, x_n) \right) \right] = 0. \tag{3.9}$$

It follows from $\lim_n d(y_n, x_n) = 0$ and Lemma 2.2 that

$$\limsup_{n\to\infty} \left(d^2(u,z) - (1-\beta_n) d^2(u, y_n) \right) = \limsup_{n\to\infty} \left(d^2(u,z) - d^2(u, x_n) \right) \le 0. \tag{3.10}$$

Hence, the conclusion follows from Lemma 2.5. □

By using the similar technique as in the proof of Theorem 3.1, we can obtain a strong convergence theorem which is an analog of [13, Theorem 3.1] (see also [37, 38] for subsequence comments).

Theorem 3.2. *Let C be a nonempty closed and convex subset of a complete* CAT(0) *space X, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and an initial value $x_1 \in C$. The sequence $\{x_n\}$ is defined iteratively by*

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n)(\alpha_n u \oplus (1-\alpha_n) T x_n), \qquad n \ge 1. \tag{3.11}$$

Suppose that both $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying

(B1) $\lim_{n\to\infty} \alpha_n = 0$,
(B2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(B3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then, $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest u.

Proof. Let $y_n := \alpha_n u \oplus (1-\alpha_n)Tx_n$. We divide the proof into 3 steps.

Step 1. We show that $\{x_n\}$, $\{y_n\}$, and $\{Tx_n\}$ are bounded sequences. Let $p \in F(T)$, then we have

$$\begin{aligned} d(x_{n+1}, p) &= d(\beta_n x_n \oplus (1-\beta_n)(\alpha_n u \oplus (1-\alpha_n)Tx_n), p) \\ &\le \beta_n d(x_n, p) + (1-\beta_n) d(\alpha_n u \oplus (1-\alpha_n)Tx_n, p) \\ &\le \beta_n d(x_n, p) + (1-\beta_n)\alpha_n d(u,p) + (1-\beta_n)(1-\alpha_n) d(Tx_n, p) \\ &\le (\beta_n + (1-\beta_n)(1-\alpha_n)) d(x_n, p) + (1-\beta_n)\alpha_n d(u,p) \\ &= [1-(1-\beta_n)\alpha_n] d(x_n,p) + (1-\beta_n)\alpha_n d(u,p) \\ &\le \max\{d(x_n,p), d(u,p)\}. \end{aligned} \tag{3.12}$$

Now, an induction yields

$$d(x_{n+1}, p) \le \max\{d(x_1,p), d(u,p)\}, \quad n \ge 1. \tag{3.13}$$

Hence, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{Tx_n\}$.

Step 2. We show that $\lim_n d(x_n, Tx_n) = 0$. By using Lemma 2.1, we get

$$\begin{aligned} d(y_{n+1}, y_n) &= d(\alpha_{n+1} u \oplus (1-\alpha_{n+1})Tx_{n+1}, \alpha_n u \oplus (1-\alpha_n)Tx_n) \\ &\le \alpha_n d(\alpha_{n+1} u \oplus (1-\alpha_{n+1})Tx_{n+1}, u) \\ &\quad + (1-\alpha_n) d(\alpha_{n+1} u \oplus (1-\alpha_{n+1})Tx_{n+1}, Tx_n) \\ &\le \alpha_n (1-\alpha_{n+1}) d(Tx_{n+1}, u) + (1-\alpha_n)\alpha_{n+1} d(u, Tx_n) \\ &\quad + (1-\alpha_n)(1-\alpha_{n+1}) d(Tx_{n+1}, Tx_n) \\ &\le \alpha_n (1-\alpha_{n+1}) d(Tx_{n+1}, u) + (1-\alpha_n)\alpha_{n+1} d(u, Tx_n) \\ &\quad + (1-\alpha_n)(1-\alpha_{n+1}) d(x_{n+1}, x_n). \end{aligned} \tag{3.14}$$

This implies that

$$\begin{aligned} d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \le{}& \alpha_n (1-\alpha_{n+1}) d(Tx_{n+1}, u) + (1-\alpha_n)\alpha_{n+1} d(u, Tx_n) \\ &+ [\alpha_n \alpha_{n+1} - \alpha_n - \alpha_{n+1}] d(x_{n+1}, x_n). \end{aligned} \tag{3.15}$$

Since $\{x_n\}$ and $\{Tx_n\}$ are bounded and $\lim_{n\to\infty}\alpha_n = 0$, it follows that

$$\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0. \tag{3.16}$$

Hence, by Lemma 2.6, we get

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.17}$$

On the other hand,

$$d(y_n, Tx_n) = d(\alpha_n u \oplus (1-\alpha_n)Tx_n, Tx_n) \le \alpha_n d(u, Tx_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

Using (3.17) and (3.18), we get

$$d(x_n, Tx_n) \le d(x_n, y_n) + d(y_n, Tx_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.19}$$

Step 3. We show that $\{x_n\}$ converges to a fixed point of T. Let $z = \lim_{t\to 0} z_t$, where z_t is given by (2.8), then $z \in F(T)$. Finally, we show that $\lim_n x_n = z$

$$\begin{aligned}
d^2(x_{n+1}, z) &= d^2(\beta_n x_n \oplus (1-\beta_n)y_n, z) \\
&\le \beta_n d^2(x_n, z) + (1-\beta_n)d^2(y_n, z) - \beta_n(1-\beta_n)d^2(x_n, y_n) \\
&\le \beta_n d^2(x_n, z) + (1-\beta_n)d^2(\alpha_n u \oplus (1-\alpha_n)Tx_n, z) - \beta_n(1-\beta_n)d^2(x_n, y_n) \\
&\le (1-\beta_n)\left[\alpha_n d^2(u, z) + (1-\alpha_n)d^2(Tx_n, z) - \alpha_n(1-\alpha_n)d^2(u, Tx_n)\right] \\
&\quad - \beta_n(1-\beta_n)d^2(x_n, y_n) + \beta_n d^2(x_n, z) \\
&\le [\beta_n + (1-\beta_n)(1-\alpha_n)]d^2(x_n, z) + (1-\beta_n)\alpha_n\left[d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\right] \\
&= [1 - (1-\beta_n)\alpha_n]d^2(x_n, z) + (1-\beta_n)\alpha_n\left[d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\right].
\end{aligned} \tag{3.20}$$

By Lemma 2.4, we have $\mu_n(d^2(u, z) - d^2(u, x_n)) \le 0$ for all Banach limit μ. Moreover, since

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\beta_n x_n \oplus (1-\beta_n)y_n, x_n) \\
&\le (1-\beta_n)d(y_n, x_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \\
\limsup_{n\to\infty}&\left(d^2(u, z) + d^2(u, x_{n+1}) - d^2(u, z) - d^2(u, x_n)\right) = 0,
\end{aligned} \tag{3.21}$$

it follows from condition $(B1)$, $\lim_n d(x_n, Tx_n) = 0$ and Lemma 2.2 that

$$\limsup_{n\to\infty}\left(d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\right) = \limsup_{n\to\infty}\left(d^2(u, z) - d^2(u, x_n)\right) \le 0. \tag{3.22}$$

Hence, the conclusion follows by Lemma 2.5. □

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